NEW UNIVALENCE CRITERIA

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Abstract. In this paper we obtain sufficient conditions for univalence, which generalize some well known univalence criteria for analytic functions in the unit disk of the complex plane.

1. **Introduction**

Let A be the class of analytic functions f in the unit disk

$$U = \{z \in \mathbf{C} : |z| < 1\}$$

of the form

$$f(z) = z + a_2 z^2 + \dots \quad , z \in U. \tag{1}$$

In order to prove our results a brief summary of Loewner parametric method is needed.

A family of functions $L(z,t),\ z \in U,\ t \in [0,\infty)$ is a Loewner chain if $L(z,t)$ is analytic and univalent in U and $L(z,t)$ is subordinate to $L(z,s)$ for all $0 \le s < t$.

Theorem 1. *[4] Let r be a real number such that $r \in (0,1]$ and let $L(z,t) = a_1(t)z + \dots$ be an analytic function in $U_r = \{z \in \mathbf{C} : |z| < r\}$, for all $t \ge 0$. If*

i) *$L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniformly with respect to U_r;*
ii) *$a_1 \neq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r;*

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iii) $z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}$, $z \in U_r$, *a.e.* $t \in [0,\infty)$ *where* $p(z,t)$ *is analytic in* U *and satisfies* $Re\, p(z,t) > 0$, *for all* $z \in U$, $t \in [0,\infty)$,

then $L(z,t)$ *has an analytic and univalent extension to the whole unit disk* U.

2. Main results

Theorem 2. *Let* α *be a complex number such that* $Re\,\alpha > \frac{1}{2}$ *and let* $f \in A$. *Let* g *and* h *be two analytic functions in* U, $g(z) = 1 + b_1 z + \ldots$, $h(z) = c_0 + c_1 z + \ldots$. *If the following inequalities are satisfied*

$$\left|\frac{1}{\alpha}\cdot\frac{f'(z)}{g(z)} - 1\right| < 1,\ z \in U \tag{2}$$

$$\left|\left(\frac{1}{\alpha}\cdot\frac{f'(z)}{g(z)} - 1\right)|z|^4 + \right. \tag{3}$$

$$(1-|z|^2)|z|^2 z\left[\left(\frac{1}{\alpha}-1\right)\frac{f'(z)}{f(z)} + \frac{2}{\alpha}\cdot\frac{f'(z)h(z)}{g(z)} + \frac{g'(z)}{g(z)}\right] + (1-|z|^2)^2 z^2\cdot$$

$$\left.\left[\left(\frac{1}{\alpha}-1\right)\frac{f'(z)h(z)}{f(z)} + \frac{1}{\alpha}\cdot\frac{f'(z)h^2(z)}{g(z)} + \frac{g'(z)h(z)}{g(z)} - h'(z)\right]\right| \le |z|^2$$

for all $z \in U$, *then the function* f *is univalent in* U.

Proof. We define

$$L(z,t) = f^{1-\alpha}(e^{-t}z)\left[f(e^{-t}z) + \frac{(e^t - e^{-t})zg(e^{-t}z)}{1+(e^t-e^{-t})zh(e^{-t}z)}\right]^{\alpha}$$

and we will prove that $L(z,t)$ satisfies theorem 1.

From the analyticity of the functions f, g and h it follows that $L(z,t)$ is analytic in a neighborhood U_r, $r \in (0,1]$ of $z = 0$.

Elementary calculation shows that

$$L(z,t) = a_1(t)z + \ldots \ \text{ where } \ a_1(t) = e^{(2\alpha-1)t}.$$

We have $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$.

Since $L(z,t)$ is an analytic function in U_r, for all $t \in (0,\infty)$ we obtain that there exist a number $0 < r_1 < r$ and a constant $k = k(r_1)$ such that

$$\left|\frac{L(z,t)}{a_1(t)}\right| < k,\ z \in U_{r_1},\ t \in (0,\infty)$$

and hence $\{L(z,t)/a_1(t)\}$ forms a normal family in U_{r_1}.

It can be easy see that $\frac{\partial L(z,t)}{\partial t}$ is an analytic function in U_{r_1} and hence we obtain the absolute continuity requirements of theorem 1.

We define

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t}$$

and we will prove that the function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \geq 0$.

Let $W(z,t)$ be the function defined by

$$W(z,t) = \frac{p(z,t)-1}{p(z,t)+1}.$$

Elementary calculation shows that

$$W(z,t) = \left[\frac{1}{\alpha}\frac{f'(e^{-t}z)}{g(e^{-t}z)} - 1\right]e^{-2t} + \tag{4}$$

$$(1-e^{-2t})e^{-t}z\cdot\left[\left(\frac{1}{\alpha}-1\right)\frac{f'(e^{-t}z)}{f(e^{-t}z)} + \frac{2}{\alpha}\frac{f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} + \frac{g'(e^{-t}z)}{g(e^{-t}z)}\right] + (1-e^{-2t})^2z^2\cdot$$

$$\left[\left(\frac{1}{\alpha}-1\right)\frac{f'(e^{-t}z)h(e^{-t}z)}{f(e^{-t}z)} + \frac{1}{\alpha}\frac{f'(e^{-t}z)h^2(e^{-t}z)}{g(e^{-t}z)} + \frac{g'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} - h'(e^{-t}z)\right]$$

We have

$$|W(z,0)| = \left|\frac{1}{\alpha}\cdot\frac{f'(z)}{g(z)} - 1\right|$$

and

$$|W(0,t)| = \left|\left(\frac{1}{\alpha}\cdot\frac{f'(0)}{g(0)} - 1\right)e^{-2t} + \left(\frac{1}{\alpha}-1\right)(1-e^{-2t})\right| = \left|\frac{1}{\alpha}-1\right|$$

From (2) and since $\operatorname{Re}\alpha > \frac{1}{2}$ we obtain that

$$|W(z,0)| < 1 \text{ and also } |W(0,t)| < 1. \tag{5}$$

Let t be a fixed positive number. Since $|e^{-tz}| \leq e^{-t} < 1$ for all $z \in \bar{U} = \{z \in \mathbf{C} : |z| \leq 1\}$ it results that the function $W(z,t)$ is analytic in $\bar{U}$. By using the maximum principle we obtain

$$|W(z,t)| < \max_{|\xi|=1}|W(\xi,t)| = |W(e^{i\theta,t})|, \tag{6}$$

where $\theta = \theta(t) \in \mathbf{R}$.

We denote $u = e^{-t} \cdot e^{i\theta}$. Then $|u| = e^{-t} < 1$ and from (4) it results

$$|W(e^{i\theta}, t)| = \left| \left(\frac{1}{\alpha} \cdot \frac{f'(u)}{g(u)} - 1 \right) |u|^2 + \right.$$

$$(1 - |u|^2) u \left[\left(\frac{1}{\alpha} - 1 \right) \frac{f'(u)}{f(u)} + \frac{2}{\alpha} \cdot \frac{f'(u)h(u)}{g(u)} + \frac{g'(u)}{g(u)} \right] +$$

$$\left. \frac{(1 - |u|^2)^2}{|u|^2} u^2 \left[\left(\frac{1}{\alpha} - 1 \right) \frac{f'(u)h(u)}{f(u)} + \frac{1}{\alpha} \cdot \frac{f'(u)h^2(u)}{g(u)} + \frac{g'(u)h(u)}{g(u)} - h'(u) \right] \right|$$

The inequality (3) implies $|W(e^{i\theta}, t)| \leq 1$ and by using (6) we obtain $|W(z, t)| < 1$ for all $z \in U$ and $t > 0$. From (5) it follows that $|W(z, t)| < 1$ for all $z \in U$ and $t \geq 0$. Hence the requirements for the function $p(z, t)$ are satisfied.

Finally, from Theorem 1 we obtain that the function $L(z, t)$ has an analytic and univalent extension to the whole unit disk U. For $t = 0$ we have $L(z, t) = f(z)$, $z \in U$ and thus the function f is univalent in U. □

Remark 1. The univalence criterion which results from Theorem 2 when $\alpha = 1$ is due to H. Ovesea-Tudor [2].

Specific choices for the functions g and h in Theorem 2 gives us various univalence criteria, between them being the very well known Nehari's criterion [1] and also Ozaki's criterion [3].

Corollary 1. *Let α be a complex number, with Re $\alpha > \frac{1}{2}$ and let $f \in A$. Suppose there exists an analytic function h in U, $h(z) = c_0 + c_1 z + \dots$ such as*

$$\left| \left(\frac{1}{\alpha} - 1 \right) |z|^4 + (1 - |z|^2)|z|^2 z \left[\left(\frac{1}{\alpha} - 1 \right) \frac{f'(z)}{f(z)} + \frac{2}{\alpha} \cdot h(z) + \frac{f''(z)}{f'(z)} \right] + \right.$$

$$\left. +(1 - |z|^2)^2 z^2 \left[\left(\frac{1}{\alpha} - 1 \right) \frac{f'(z)h(z)}{f(z)} + \frac{1}{\alpha} \cdot h^2(z) + \frac{f''(z)h(z)}{f'(z)} - h'(z) \right] \right|$$

$$\leq |z|^2, \ z \in U \tag{7}$$

then the function f is univalent in U.

Proof. It results from Theorem 2 with $g = f'$. □

If we choose $g = f'$ and $h = -\frac{1}{2} \cdot \frac{f''}{f'}$ in Theorem 2 we obtain the following univalence criterion.

Corollary 2. **[5]***Let α be a complex number with Re $\alpha > \frac{1}{2}$ and let $f \in A$. Suppose*

$$\left|\left(\frac{1}{\alpha}-1\right)|z|^4+(1-|z|^2)|z|^2\left(\frac{1}{\alpha}-1\right)\left(\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right)+\right.$$
$$\left.+(1-|z|^2)^2\left\{\frac{1}{2}z^2\{f;z\}+\frac{1}{2}\left(\frac{1}{\alpha}-1\right)\left[\frac{1}{2}\left(\frac{zf''(z)}{f'(z)}\right)^2-\frac{z^2f''(z)}{f(z)}\right]\right\}\right|$$
$$\leq |z|^2,\ z \in U \tag{8}$$

then f is univalent in the unit disk U.

Remark 2. If we consider $\alpha = 1$ in Corollary 2 we obtain the univalence criterion due to Nehari [1].

Corollary 3. *Let α be a complex number with Re $\alpha > \frac{1}{2}$ and let $f \in A$. If there exists a function $h : U \to \mathbf{C}$ $h(z) = b_0 + b_1 z + \ldots$ such as*

$$\left|\frac{1}{\alpha}\cdot\frac{z^2f'(z)}{f^2(z)}-1\right|<1,\ z\in U \tag{9}$$

$$\left|\left(\frac{1}{\alpha}\cdot\frac{z^2f'(z)}{f^2(z)}-1\right)|z|^4+\right. \tag{10}$$
$$(1-|z|^2)|z|^2z\left[\frac{\alpha+1}{\alpha}\cdot\frac{f'(z)}{f(z)}+\frac{2}{\alpha}\cdot\frac{z^2f'(z)h(z)}{f^2(z)}-\frac{2}{z}\right]+$$
$$\left.+(1-|z|^2)z^2\left[\frac{\alpha+1}{\alpha}\cdot\frac{f'(z)h(z)}{f(z)}+\frac{1}{\alpha}\cdot\frac{z^2f'(z)h^2(z)}{f^2(z)}-\frac{2h(z)}{z}-h'(z)\right]\right|\leq|z|^2,$$

for all $z \in U$, then f is univalent in U.

Proof. It results from Theorem 2 with $g(z) = \left(\frac{f(z)}{z}\right)^2$. □

Finally, if we choose $g(z) = \left(\frac{f(z)}{z}\right)^2$ and $h(z) = \frac{1}{z} - \frac{f(z)}{z^2}$ in Theorem 2 we obtain the following corollary.

Corollary 4. **[6]***Let α be a complex number with Re $\alpha > \frac{1}{2}$ and let $f \in A$. Suppose*

$$\left|\frac{1}{\alpha}\cdot\frac{z^2f'(z)}{f^2(z)}-1\right|<1,\ z\in U \tag{11}$$

$$\left|\frac{1}{\alpha}\cdot\frac{z^2f'(z)}{f^2(z)}-1+\frac{\alpha-1}{\alpha}(1-|z|^2)\frac{zf'(z)}{f(z)}\right|<|z|^2,\ z\in U \tag{12}$$

then the function f is univalent in the unit disk.

Remark 3. If we consider $\alpha = 1$ in Corollary 4 we obtain the univalence criterion due to Ozaki [3].

References

[1] Nehari, C., *The Schwartzian derivative and Schlicht functions*, Bull. Amer. Math. Soc. **55**(1949), 545-551.

[2] Ovesea-Tudor, H., Owa, S., *An extension of the univalence criteria of Nehari and Ozaki*, Hokkaido Math. J (to appear).

[3] Ozaki, S., Nokawa, M., *The Schwartzian derivative and univalent functions*, Proc. Amer. Math. Soc. 33(2), 1972, 392-394.

[4] Pommerenke, Ch., *Univalent function*, Vandenhoech Ruprecht in Göttingen, 1975.

[5] Răducanu, D., *On a univalence criterion*, Mathematica 37(60), 1-2, 1995, 227-231.

[6] Răducanu, D., *On a generalization of Ozaki's univalence criterion*, (to appear).

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