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Big Picard's theorems for holomorphic mappings of several complex variables into $P^N(C)$ with moving hyperplanes ☆

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Abstract

Motivated by the accomplishment of the second main theorem with moving targets, many authors studied the moving target problems in value distribution theory and related topics. In this paper, we prove some big Picard's theorems for holomorphic mappings of several complex variables into $P^N(C)$ with moving hyperplanes, related to Nochka's little Picard-type theorems. As its application, we give a new quasi-normal criterion for families of meromorphic mappings of several complex variables into $P^N(C)$ with moving hyperplanes. The new results in this paper greatly improve some earlier related results.

Keywords: Complex projective spaces; Meromorphic mappings; Moving hyperplanes; Picard's theorems; Value distribution theory

1. Introduction

Picard proved the following theorems for meromorphic functions in one complex variable.

Theorem 1.A *(Little Picard theorem). Let $f(z)$ be a meromorphic function on the complex plane. If there exist three mutually distinct points w_1, w_2 and w_3 on the Riemann sphere such that $f(z) - w_i$ $(i = 1, 2, 3)$ has no zero on the complex plane, then f is a constant.*

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Theorem 1.B *(Big Picard theorem). Let $f(z)$ be a meromorphic function on $\Delta^* = \{z \in C;\ 0 < |z| < R\}$ $(R > 0)$. If there exist three mutually distinct points w_1, w_2 and w_3 on the Riemann sphere such that $f(z) - w_i$ $(i = 1, 2, 3)$ has no zero on Δ^*, then f does not have an essential singularity at the origin.*

In the case of higher dimension, Fujimoto [5] and Green [7] established the following little Picard-type theorem for holomorphic mappings of C^n into $P^N(C)$, the complex N-dimensional projective space, in 1972.

Theorem 1.C. *Suppose that f is a non-constant holomorphic mapping from C^n into $P^N(C)$. Then f cannot omit $2N + 1$ hyperplanes in $P^N(C)$ located in general position.*

In 1983, Nochka [11] improved Theorem 1.C and got the following little Picard-type result.

Theorem 1.D. *Suppose that $H_1, \ldots, H_q$ are q $(\geqslant 2N + 1)$ hyperplanes given in general position in $P^N(C)$, along with q positive integers $m_1, \ldots, m_q$ (some of them may be ∞) such that $\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q-(N+1)}{N}$. Then any non-constant holomorphic mapping from C^n into $P^N(C)$ cannot intersect H_j with multiplicity at least m_j $(j = 1, \ldots, q)$.*

These results turned out to be part of a rather deep and elegant theory, known as value distribution theory or, after its founder, Nevanlinna theory. Motivated by the accomplishment of the second main theorem of value distribution theory in higher dimension for moving targets (e.g., Ru and Stoll [14]), many authors studied the moving target problems in value distribution theory and related topics (see, e.g., Ru [13], Ru and Vojta [15] and Tu [19]). In particular, Wang [21] extended Theorem 1.C to the case of the holomorphic mappings of C into $P^N(C)$ with the moving hyperplanes, where Wang does not make any restriction on the growth of the coefficients of the linear forms defining the moving hyperplanes. Roughly speaking, Wang [21] proved the following little Picard-type theorem: For given $2N + 1$ moving hyperplanes in pointwise general position and any holomorphic mapping f from C into a N-dimensional complex projective space omitting those $2N + 1$ moving hyperplanes, then there exist finitely many $(N + 1) \times (N + 1)$ everywhere invertible matrices with entire functions as entries such that the holomorphic mapping f multiplied by one of the matrices becomes constant. Inspired by the idea in Tu [17] and Wang [21], Tu and Li [20] recently have given some normality criteria for families of meromorphic mappings in several complex variables into $P^N(C)$ with moving targets, related to little Picard-type Theorems 1.C and 1.D.

Thereafter up to the present, all of researches about big Picard-type theorems for holomorphic mappings of several complex variables into $P^N(C)$ have been restricted to the fixed target case. It seems clear that the idea in Tu and Li [20] and Wang [21] suggests some insights into big Picard-type theorems for holomorphic mappings of several complex variables into $P^N(C)$ with moving hyperplanes. Following this line, in this paper we prove some big Picard-type theorems for holomorphic mappings of several complex variables into $P^N(C)$ with moving hyperplanes, related to Nochka's little Picard-type theorem. In the end, we will apply our big Picard-type theorems to prove a new quasi-normal criterion for families of meromorphic mappings of several complex variables into $P^N(C)$ with moving hyperplanes. The new results in this paper greatly improve earlier related results.

2. Statement of main results

For the general reference of this paper, see Fujimoto [6], Tu and Li [20] and Wang [21].

Let f be a meromorphic mapping of a domain D in C^n into $P^N(C)$. Then for any $z \in D$, f always has a reduced representation on some neighborhood of z in D. We denote by $I(f)$ the set of all points of indetermination of f on D. Then $I(f)$ is an analytic set in D with $\dim I(f) \leqslant n-2$. Obviously a meromorphic mapping from D into $P^N(C)$ is a holomorphic mapping from D into $P^N(C)$ if and only if $I(f) = \emptyset$.

Let F be a family of holomorphic mappings of a domain D in C^n into a compact complex manifold M. F is said to be a normal family on D if any sequence in F contains a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D into M.

A sequence $\{f^{(p)}(z)\}$ of meromorphic mappings from a domain D in C^n into $P^N(C)$ is said to be quasi-regular on D if and only if any $z \in D$ has a neighborhood U with the property that $\{f^{(p)}(z)\}$ converges compactly on U outside a nowhere dense analytic subset S of U, i.e., for any domain $G \Subset U\backslash S$ (the closure $\overline{G}$ of G in $U\backslash S$ is a compact subset of $U\backslash S$), there is some p_0 such that $I(f^{(p)}) \cap G = \emptyset$ $(p \geqslant p_0)$ and $\{f^{(p)}|_G;\ p \geqslant p_0\}$ converges uniformly on G to a holomorphic mapping of G into $P^N(C)$.

Let F be a family of meromorphic mappings of a domain D in C^n into $P^N(C)$. F is said to be a quasi-normal family on D if any sequence in F has a subsequence so as to be quasi-regular on D. For the detailed discussion about quasi-normal family, see Fujimoto [6].

A subset H of $P^N(C)$ is called a hyperplane if there is a N-dimensional linear subspace $\widetilde{H}$ of C^{N+1} with $\rho(\widetilde{H}\backslash\{0\}) = H$, where $\rho : C^{N+1}\backslash\{0\} \to P^N(C)$ is the standard projective mapping. Let $H_1, \ldots, H_q$ $(q \geqslant N+1)$ be hyperplanes in $P^N(C)$. Write $(C^{N+1})^*$ for the dual space of C^{N+1}. Then there exists $\alpha_i \in (C^{N+1})^*\backslash\{0\}$ such that $H_i = \rho(\ker(\alpha_i)\backslash\{0\})$ $(i = 1, \ldots, q)$. We say $\{H_i\}_{i=1}^q$ to be in general position if for any choice of indices $1 \leqslant j_1 < \cdots < j_{N+1} \leqslant q$, $\dim\langle\alpha_{j_1}, \ldots, \alpha_{j_{N+1}}\rangle = N+1$, where $\langle\alpha_{j_1}, \ldots, \alpha_{j_{N+1}}\rangle$ denotes the linear subspace of $(C^{N+1})^*$ spanned by $\alpha_{j_1}, \ldots, \alpha_{j_{N+1}}$.

Let D be a domain in C^n. For $z \in D$, define $L_i(z)(Z) := \sum_{j=1}^{N+1} a_{ij}(z)Z_j$, $Z = (Z_1, \ldots, Z_{N+1}) \in C^{N+1}$, where $a_{ij}(z)$ $(1 \leqslant j \leqslant N+1)$ are holomorphic functions on D without common zeroes. For any fixed point $z \in D$, denote by $\ker(L_i(z))$ $(\subset C^{N+1})$ the kernel of the surjective linear function $L_i(z) : C^{N+1} \to C$, and let $H_i(z) := \rho(\ker(L_i(z))\backslash\{0\})$ $(\subset P^N(C))$ be the moving hyperplane corresponding to the linear form $L_i(z)$. The set $\{H_i(z);\ 1 \leqslant i \leqslant q\}$ is said to be located in $P^N(C)$ in pointwise general position on D if the determinant of the $(N+1) \times (N+1)$ matrix formed by the coefficients of any $N+1$ linear forms among $L_i(z)$ $(1 \leqslant i \leqslant q)$ does not equal to 0 for any $z \in D$, i.e., the set $\{H_i(z);\ 1 \leqslant i \leqslant q\}$ is located in $P^N(C)$ in pointwise general position on D iff $\{H_i(z);\ 1 \leqslant i \leqslant q\}$ is in general position for any $z \in D$.

For a set $E \subset C^n$, the set $\{H_i(z);\ 1 \leqslant i \leqslant q\}$ is said to be located in $P^N(C)$ in pointwise general position on E iff $\{H_i(z);\ 1 \leqslant i \leqslant q\}$ is in pointwise general position on some neighborhood of E in C^n.

Let $f(z) \not\equiv 0$ be a holomorphic function on the connected open neighborhood D of $a \in C^n$. Then $f(z) = \sum_{m=0}^{\infty} p_m(z-a)$, where the series converges uniformly to f on an open neighborhood of $a \in C^n$ and the term p_m is either identically zero or a homogeneous polynomial of degree m. The number $\nu_f(a) := \min\{m;\ p_m \not\equiv 0\}$ is said to be the zero-multiplicity of f at a. By definition, a non-negative divisor on a domain D in C^n is a non-negative integer-valued function ν on D such that for every $a \in D$ there exists a holomorphic function $f(z) \not\equiv 0$ on a neighborhood U of a with $\nu(z)=\nu_f(z)$ on U. Furthermore we define the support $\operatorname{supp}\nu$ of the divisor ν on D by $\operatorname{supp}\nu := \{z\in D;\ \nu(z)\neq 0\}$.

Let f be a meromorphic mapping from a domain D in C^n into $P^N(C)$. Take a moving hyperplane $H(z)$ in $P^N(C)$ defined by the linear form

$$L(z)(Z)=a_1(z)Z_1+\cdots+a_{N+1}(z)Z_{N+1},$$

where $Z=(Z_1,\dots,Z_{N+1})\in C^{N+1}$, $z\in D$, and $a_j(z)$ $(1\leqslant j\leqslant N+1)$ are holomorphic functions on D without common zeroes. For $a\in D$, let f has a reduced representation

$$\tilde f(z)=\bigl(f_1(z),\dots,f_{N+1}(z)\bigr)$$

on a neighborhood U of a. We consider the holomorphic function

$$F(z):=a_1(z)f_1(z)+\cdots+a_{N+1}(z)f_{N+1}(z).$$

Then the divisor $\nu(f,H)(z):=\nu_F(z)$ $(z\in U)$ is determined independently of a choice of reduced representations and hence is well defined on the totality of D and obviously $\operatorname{Supp}\nu(f,H)$ is either empty or a pure $(n-1)$-dimensional analytic set in D if $F(z)\not\equiv 0$ on U (i.e., $f(z)\in P^N(C)\backslash H(z)$ for some $z\in U$). We define $\nu(f,H):=\infty$ on D and $\operatorname{supp}\nu(f,H)=D$ if $f(z)\in H(z)$ for all $z\in D$. Sometimes we identify $f^{-1}(H)$ with the divisor $\nu(f,H)$ on D. Rewrite $\nu(f,H)$ as the formal sum $\nu(f,H)=\sum_{\lambda\in\Lambda}n_\lambda X_\lambda$, where X_λ are the irreducible components of $\operatorname{supp}\nu(f,H)$ and n_λ are the constant $\nu(f,H)(z)$ on $X_\lambda\cap\operatorname{Reg}(\operatorname{supp}\nu(f,H))$. For any positive integer or infinite m, the closure

$$\overline{\bigl\{z\in\operatorname{supp}\nu(f,H);\ \nu(f,H)(z)<m\bigr\}}=\bigcup_{\lambda:\ n_\lambda<m}X_\lambda$$

is either empty or a pure $(n-1)$-dimensional analytic set in D and the $2(n-1)$-dimensional Lebesgue areas of the two sets

$$\bigl\{z\in\operatorname{supp}\nu(f,H);\ \nu(f,H)(z)<m\bigr\}\quad\text{and}\quad\overline{\bigl\{z\in\operatorname{supp}\nu(f,H);\ \nu(f,H)(z)<m\bigr\}}$$

coincide.

Remark, for any given moving hyperplane $H(z)$ $(z\in D)$ in $P^N(C)$, $I(f)\subset\operatorname{supp}\nu(f,H)$ always holds and $I(f)=\emptyset$ if $f(z)\in P^N(C)\backslash H(z)$ for all $z\in D$. We say that a meromorphic mapping $f(z)$ intersects $H(z)$ on D with multiplicity at least m if $\nu(f,H)(z)\geqslant m$ for all $z\in\operatorname{supp}\nu(f,H)$, and in particular, f intersects H on D with multiplicity ∞ if $f(z)\in H(z)$ for all $z\in D$ or $f(z)\in P^N(C)\backslash H(z)$ for all $z\in D$.

Motivated by the accomplishment of the second main theorem of value distribution theory for moving targets (e.g., Ru and Stoll [14]), Wang [21] recently has given a generalization of Theorem 1.C to the moving hyperplane case, where Wang does not make any restriction on the growth of the coefficients of the linear forms defining the moving hyperplanes. From the reasoning in Wang [21], it might appear that the little Picard-type theorem with moving hyperplanes is not as strong as the little Picard-type theorem with fixed hyperplanes (e.g., Theorem 1.C). Inspired by the idea in Tu [17] and Wang [21], Tu and Li [20] gave some normality criteria for a family of meromorphic mappings of a domain D in C^n into $P^N(C)$ with moving targets as follows in 2005.

Theorem 2.A. (Tu and Li [20]) *Let F be a family of holomorphic mappings of a domain D in C^n into $P^N(C)$ and let $H_1(z),\dots,H_q(z)$ $(z\in D)$ be q $(\geqslant 2N+1)$ moving hyperplanes in $P^N(C)$ located in pointwise general position such that each $f(z)$ in F intersects $H_j(z)$ on D with multiplicity at least m_j $(j=1,\dots,q)$, where $m_1,\dots,m_q$ are fixed positive integers and may be ∞, with $\sum_{j=1}^q\frac{1}{m_j}<\frac{q-(N+1)}{N}$. Then F is a normal family on D.*

Theorem 2.B. (Tu and Li [20]) *Let F be a family of meromorphic mappings of a domain D in C^n into $P^N(C)$ and let $H_1(z), \dots, H_{2N+1}(z)$ $(z \in D)$ be $2N+1$ moving hyperplanes in $P^N(C)$ located in pointwise general position such that for any fixed compact subset K of D, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_j) \cap K$ $(j = 1, \dots, 2N+1)$ with counting multiplicities for all f in F are bounded above. Then F is a meromorphically normal family on D.*

For the detailed discussion about the concept of meromorphically normal family, see Fujimoto [6].

Thereafter up to the present, all of researches about big Picard-type theorems for holomorphic mappings of several complex variables into $P^N(C)$ have been restricted to the fixed target case. It seems clear that the idea in Tu and Li [20] and Wang [21] suggests some insights into big Picard-type theorems for holomorphic mappings of several complex variables into $P^N(C)$ with moving hyperplanes. Following this line, we prove some big Picard-type theorems for holomorphic mappings of several complex variables into $P^N(C)$ with moving hyperplanes, related to Nochka's little Picard-type theorem, as follows.

Theorem 2.1. *Let S be an analytic subset of a domain D in C^n with $\dim_C S \leqslant n-2$. Let f be a holomorphic mapping from $D \backslash S$ into $P^N(C)$. Let $H_1(z), \dots, H_q(z)$ $(z \in D)$ be q $(\geqslant 2N+1)$ moving hyperplanes in $P^N(C)$ located in pointwise general position such that $f(z)$ intersects $H_j(z)$ on $D \backslash S$ with multiplicity at least m_j $(j = 1, \dots, q)$, where $m_1, \dots, m_q$ are positive integers and may be ∞, with $\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q-(N+1)}{N}$. Then the holomorphic mapping f from $D \backslash S$ into $P^N(C)$ extends to a holomorphic mapping from D into $P^N(C)$.*

Theorem 2.2. *Let S be an analytic subset of a domain D in C^n with codimension one, whose singularities are normal crossings. Let f be a holomorphic mapping from $D \backslash S$ into $P^N(C)$. Let $H_1(z), \dots, H_q(z)$ $(z \in D)$ be q $(\geqslant 2N+1)$ moving hyperplanes in $P^N(C)$ located in pointwise general position such that $f(z)$ intersects $H_j(z)$ on $D \backslash S$ with multiplicity at least m_j $(j = 1, \dots, q)$, where $m_1, \dots, m_q$ are positive integers and may be ∞, with $\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q-(N+1)}{N}$. Then the holomorphic mapping f from $D \backslash S$ into $P^N(C)$ extends to a holomorphic mapping from D into $P^N(C)$.*

In the end, we will apply Theorem 2.1 to prove a quasi-normal criterion for families of meromorphic mappings of several complex variables into $P^N(C)$ with moving hyperplanes as follows.

Theorem 2.3. *Let F be a family of meromorphic mappings of a domain D in C^n into $P^N(C)$ and let $H_1(z), \dots, H_q(z)$ $(z \in D)$ be q $(\geqslant 2N+1)$ moving hyperplanes in $P^N(C)$ located in pointwise general position such that for any fixed compact subset K of D, the $2(n-1)$-dimensional Lebesgue areas of*

$$\left\{z \in \operatorname{supp} \nu(f, H_j);\ \nu(f, H_j)(z) < m_j\right\} \cap K \quad (j = 1, \dots, q)$$

regardless of multiplicities for all f in F are bounded above, where $\{m_j\}_{j=1}^{q}$ are fixed positive integers and may be ∞ with $\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q-(N+1)}{N}$. Then F is a quasi-normal family on D.

Now we shall present an outline of our proof of the main results. The main results in this paper are evolved by the Kiernan–Kobayashi–Kwack's extension theorem (see Lang [10, Theorem 5.2,

p. 58]). But the technique in the proof of Kiernan–Kobayashi–Kwack's extension theorem cannot be directly applied to prove our results. Thus our idea here is heavily based on the framework of the extension theorems of normal holomorphic mappings in several complex variables. Therefore, the key step in our proof is to prove that the $f: D\backslash S \to P^N(C)$ in Theorems 2.1 and 2.2 is a normal holomorphic mapping.

Our proof here is inspired by the proof of Theorem 3 (i.e., a big Picard theorem with fixed hyperplanes) in Tu [18]. By using a normality criterion with fixed hyperplanes in Tu [17], we can derive Theorem 3 in Tu [18] from the following result (see Aladro [1, Proposition 1.14]):

> Let f be a holomorphic mapping from a bounded domain Ω in C^n into $P^N(C)$ such that for every sequence of holomorphic mappings $\varphi_j(z)$ from the unit disc U in C into Ω, the sequence $\{f\circ\varphi_j(z)\}_{j=1}^{\infty}$ from U into $P^N(C)$ is a normal family on U. Then f is a normal holomorphic mapping from Ω into $P^N(C)$.

But in the case of moving hyperplane $H(z)$ in Theorems 2.1 and 2.2, the normality criterion with moving hyperplanes (e.g., Theorem 2.A) cannot be used to prove that $f: D\backslash S \to P^N(C)$ in Theorems 2.1 and 2.2 satisfies the assumption of the above result. This means that the above result cannot be applied to prove that $f: D\backslash S \to P^N(C)$ in Theorems 2.1 and 2.2 is a normal holomorphic mapping. Therefore, it seems that the idea in the proof of Theorem 3 (i.e., a big Picard theorem with fixed hyperplanes) in Tu [18] does not work for big Picard theorems with moving hyperplanes.

Hence, in order to overcome the difficulty, we use a different idea from that in Tu [18] to prove that the $f: D\backslash S \to P^N(C)$ in Theorems 2.1 and 2.2 is a normal holomorphic mapping by Theorem 2.A. Finally, we use the extension theorems of normal holomorphic mappings to finish our proof.

3. Proofs of Theorems 2.1 and 2.2

Definition 3.1. Let $\Omega \subset C^n$ be a hyperbolic domain and let M be a complete complex Hermitian manifold with metric ds_M^2. A holomorphic mapping $f(z)$ from Ω into M is said to be a normal holomorphic mapping from Ω into M if and only if there exists a positive constant c such that for all $z \in \Omega$ and all $\xi \in T_z(\Omega)$,

$$\left|ds_M^2\big(f(z), df(z)(\xi)\big)\right| \leqslant cK_\Omega(z,\xi),$$

where $df(z)$ is the mapping from $T_z(\Omega)$ into $T_{f(z)}(M)$ induced by f and K_Ω denotes the infinitesimal Kobayashi metric on Ω.

For the detailed discussion of normal holomorphic mapping, see Aladro [1] and for the basic notation of hyperbolic space, see Lang [10] and Noguchi and Ochiai [12].

Lemma 3.2. *Let f be a holomorphic mapping of a hyperbolic domain D in C^n into $P^N(C)$. Then f is not normal on D if and only if there exist $\{p_i\} \subset D$, $\{r_i\}$ with $r_i > 0$ and $r_i \to 0^+$ and $\{u_i\} \subset C^n$ Euclidean unit vectors such that*

$$g_i(\xi) := f_i(p_i + r_i u_i \xi), \quad \xi \in C,$$

where $\lim_{i\to\infty} r_i/d(p_i, C^n\backslash D) = 0$ (where $d(p,q)$ is the Euclidean distance between p and q in C^n), converges uniformly on compact subsets of C to a non-constant holomorphic mapping g of C into $P^N(C)$.

Remark. The proof of Lemma 3.2 is only a slight modification of that of Theorem 3.1 in Aladro and Krantz [2]. Here we correct a mistake in Corollary 3.1 in Aladro and Krantz [2]. Cf. Theorem 6.4 in Hahn [8].

Lemma 3.3. *Let f be a holomorphic mapping from a bounded domain D in C^n into $P^N(C)$, and let $H_1(z), \dots, H_q(z)$ be q $(\geqslant 2N+1)$ moving hyperplanes in $P^N(C)$ located in pointwise general position on $\overline{D}$ such that $f(z)$ intersects $H_j(z)$ on D with multiplicity at least m_j $(j = 1, \dots, q)$, where $m_1, \dots, m_q$ are positive integers and may be ∞, with $\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q-(N+1)}{N}$. Then f is a normal holomorphic mapping from D into $P^N(C)$.*

Proof of Lemma 3.3. If f is not normal on D, then, by Lemma 3.2, there exist sequences $\{p_k\} \subset D$, $\{r_k\}$ with $r_k > 0$ and $r_k \to 0^+$ and $\{u_k\} \subset C^n$ Euclidean unit vectors, with $\lim_{i\to\infty} r_i/d(p_i, C^n \backslash D) = 0$, such that

$$g_k(\xi) := f(p_k + r_k u_k \xi),$$

where $\xi \in C$ satisfies $p_k + r_k u_k \xi \in D$, converges uniformly on compact subsets of C to a non-constant holomorphic mapping g of C into $P^N(C)$. We will prove that g must be a constant holomorphic mapping of C into $P^N(C)$, and get a contradiction.

Let the linear form

$$L_i(z)(Z) := a_{i1}(z)Z_1 + a_{i2}(z)Z_2 + \cdots + a_{iN+1}(z)Z_{N+1}$$

define the moving hyperplane $H_i(z)$ (z belongs to some neighborhood U of $\overline{D}$) in $P^N(C)$, where $Z = (Z_1, \dots, Z_{N+1}) \in C^{N+1}$, $a_{ij}(z)$ $(j = 1, \dots, N+1)$ are holomorphic functions on U without common zeroes $(i = 1, \dots, q)$. Since $\overline{D}$ is compact, without loss of generality, we assume that $\{p_k\}$ $(\subset D)$ converges to p_0 $(\in \overline{D})$.

Let g have a reduced representation

$$\tilde{g}(\xi) = \big(g_1(\xi), g_2(\xi), \dots, g_{N+1}(\xi)\big)$$

on C. We consider the entire function

$$G_i(\xi) := a_{i1}(p_0)g_1(\xi) + a_{i2}(p_0)g_2(\xi) + \cdots + a_{iN+1}(p_0)g_{N+1}(\xi)$$

on C for a fixed i $(i = 1, 2, \dots, q)$.

(I) If $G_i(\xi) \equiv 0$ on C or $G_i(\xi) \neq 0$ everywhere on C, then g intersects $H_i(p_0)$ on C with multiplicity ∞.

(II) Suppose that $G_i(\xi) \not\equiv 0$ on C and $G_i(\xi_0) = 0$ for some $\xi_0 \in C$. We will prove that ξ_0 is zero of $G_i(\xi)$ with multiplicity at least m_i.

Choose $r > 0$ such that ξ_0 is only zero point of $G_i(\xi)$ on $E := \{\xi \in C;\ |\xi - \xi_0| \leqslant r\}$. Since $\lim_{i\to\infty} r_i/d(p_i, C^n \backslash D) = 0$, take a positive integer k_0 such that for $k \geqslant k_0$, we have $p_k + r_k u_k \xi \in D$ for all $\xi \in E$. Then $g_k(\xi) := f(p_k + r_k u_k \xi)$ $(k \geqslant k_0)$ converges uniformly to g on E. Therefore, by the definition of convergence, ξ_0 $(\in E)$ has a relatively compact neighborhood in E (again denote the relatively compact neighborhood by E) such that each $g_k(\xi) := f(p_k + r_k u_k \xi)$ $(k \geqslant k_0)$ has a reduced representation

$$\tilde{g}_k(\xi) = \big(g_{1k}(\xi), g_{2k}(\xi), \dots, g_{N+1k}(\xi)\big)$$

on E and $\tilde{g}_k(\xi)$ converges to $\tilde{g}(\xi)$ uniformly on E as $k \to \infty$. Therefore,

$$G_{ik}(\xi) := a_{i1}(p_k + r_k u_k \xi) g_{1k}(\xi) + \cdots + a_{iN+1}(p_k + r_k u_k \xi) g_{N+1k}(\xi)$$

converges to $G_i(\xi)$ uniformly on E as $k \to \infty$. By Hurwitz theorem (see Conway [4]), there exists a positive integer M such that $G_{ik}(\xi)$ and $G_i(\xi)$ have the same number of zeros with counting multiplicities on E for $k \geqslant M$. By the definition of multiplicity we have the following fact: If $f(z)$ intersects $H_i(z)$ on D with multiplicity at least m_i, then $f(p_k + r_k u_k \xi)$ intersects $H_i(p_k + r_k u_k \xi)$ with multiplicity at least m_i, where $\xi \in C$ satisfies $p_k + r_k u_k \xi \in D$. Thus, for $k \geqslant k_0$, we have that $G_{ik}(\xi)$ has zero multiplicities $\geqslant m_i$ at all the zeros of $G_{ik}(\xi)$ on E because $g_k(\xi) = f(p_k + r_k u_k \xi)$ of E into $P^N(C)$ intersects $H_i(p_k + r_k u_k \xi)$ on E with multiplicity at least m_i. Therefore ξ_0 is zero of $G_i(\xi)$ with multiplicity at least m_i.

Combining (I) and (II), we have that g intersects $H_i(p_0)$ on C with multiplicity at least m_i $(i = 1, 2, \ldots, q)$. Since $H_1(z), \ldots, H_q(z)$ $(z \in \overline{D})$ are q $(\geqslant 2N + 1)$ moving hyperplanes in $P^N(C)$ located in pointwise general position, $H_1(p_0), \ldots, H_q(p_0)$ are q $(\geqslant 2N+1)$ hyperplanes in $P^N(C)$ located in general position. By Theorem 1.D, g must be a constant mapping of C into $P^N(C)$. We get a contradiction. The proof of Lemma 3.3 is completed. □

Lemma 3.4. *Let M be a complex manifold and let S be a complex analytic subset of M with* $\operatorname{codim} S \geqslant 2$. *Then $K_{M-S} = K_M$ on $M - S$ (i.e., the infinitesimal Kobayashi metric K_{M-S} is the restriction of K_M to $M - S$).*

For the proof of Lemma 3.4, see Proposition 1.2.22 in Noguchi and Ochiai [12].

Proof of Theorem 2.1. Fix a point $p_0 \in S$, take a bounded neighborhood U of p_0 in C^n (i.e., U is hyperbolic) with $\overline{U} \subset D$. By Lemma 3.3 f is a normal holomorphic mapping from $U - S$ into $P^N(C)$.

Thus by Definition 3.1 and the definition of the integrated distance there exists a positive constant c such that

$$d_{P^N}\big(f(z), f(w)\big) \leqslant c d^K_{U-S}(z, w)$$

for all $z, w \in U - S$, where d^K_{U-S} and d_{P^N} denote the Kobayashi distance on $U - S$ and the Fubini–Study distance on $P^N(C)$, respectively. For any $z_0 \in U \cap S$, let $\{z_i\}_{i=1}^{\infty}$ be a sequence of points of $U - S$ so as to converge to z_0. By Lemma 3.4, we have

$$d_{P^N}\big(f(z_i), f(z_j)\big) \leqslant c d^K_{U-S}(z_i, z_j) = c d^K_U(z_i, z_j).$$

Then $\{f(z_i)\}_{i=1}^{\infty}$ is a Cauchy sequence of $P^N(C)$ and hence $\{f(z_i)\}_{i=1}^{\infty}$ converges to a point $a_0 \in P^N(C)$. It is easy to check that a_0 is independent of the choice of $\{z_i\}$ as far as it converges to z_0. Then $f(z)$ has an extension $\tilde{f}(z)$ on U so as to be holomorphic on $U - S$ and continuous on U and hence $\tilde{f}(z)$ is holomorphic on U by the Riemann extension theorem. The proof of Theorem 2.1 is completed. □

Remark. The latter half of proof of Theorem 2.1 can be given by (3) of Corollary 2.5 in Joseph and Kwack [9]. We include the proof here for the completeness.

Proof of Theorem 2.2. Fix a point $p_0 \in S$, take a bounded neighborhood U of p_0 in C^n (i.e., U is hyperbolic) with $\overline{U} \subset D$. By Lemma 3.3 f is a normal holomorphic mapping from $U - S$ into $P^N(C)$. By assumption of Theorem 2.2, S is an analytic subset of a domain U in C^n with codimension one, whose singularities are normal crossings. Hence $f(z)$ extends to a holomorphic mapping from U into $P^N(C)$ by Theorem 2.3 in Joseph and Kwack [9]. The proof of Theorem 2.2 is completed. □

4. Proof of Theorem 2.3

To prove our results, we need some preparations.

We define the limit of a sequence $\{F_k\}_{k=1}^{\infty}$ of closed subsets of a locally compact Hausdorff space M as follows:

Definition 4.1. A point x of M is called a limit point of $\{F_k\}$ if there exist an integer k_0 and points $a_k \in F_k$ $(k > k_0)$ such that $x = \lim a_k$. A point of M is called a cluster point of $\{F_k\}$ if it is a limit point of some subsequence of $\{F_k\}$. If the set of limit points coincides with the set of cluster points, $\{F_k\}$ is said to converge to this set F, and write $\lim F_k = F$. (For the detailed discussion of this convergent concept, see Stoll [16, pp. 196–201])

Lemma 4.2. *Let $\{N_i\}$ be a sequence of pure $(n-1)$-dimensional analytic subsets of a domain D in C^n. Suppose that the $2(n-1)$-dimensional Lebesgue areas of $N_i \cap K$ regardless of multiplicities $(i = 1, 2, \ldots)$ are bounded above for any fixed compact subset K of D and $\{N_i\}$ converges to N as a sequence of closed subsets of D. Then N is either empty or a pure $(n-1)$-dimensional analytic subset of D. (See Stoll [16, Proposition 4.11] or Bishop [3, Theorem 1] for more general analytic subsets).*

Lemma 4.3. *Let $\{N_i\}$ be a sequence of pure $(n-1)$-dimensional analytic subsets of a domain D in C^n. If the $2(n-1)$-dimensional Lebesgue areas of $N_i \cap K$ regardless of multiplicities $(i = 1, 2, \ldots)$ are bounded above for any fixed compact subset K of D, then $\{N_i\}$ is normal as a family of closed subsets of D. (See Stoll [16, Proposition 4.12]).*

Stoll [16] introduced the concept of convergence of a net of divisors. In the special case which we use, his definition reduces to the following:

Definition 4.4. Let $\{\nu_i\}_{i=1}^{\infty}$ be a sequence of non-negative divisors on a domain D in C^n. It is said to converge to a non-negative divisor ν on D if and only if any $a \in D$ has a neighborhood U such that there exist holomorphic functions $h_i(z)$ $(\not\equiv 0)$ and $h(z)$ $(\not\equiv 0)$ on U with $\nu_i(z) = \nu_{h_i}(z)$ and $\nu(z) = \nu_h(z)$ on U such that $h_i(z)$ converges to $h(z)$ uniformly on compact subsets of U.

Lemma 4.5. *A sequence $\{\nu_i\}$ of non-negative divisors on a domain D in C^n is normal in the sense of the convergence of divisors on D if and only if the $2(n-1)$-dimensional Lebesgue areas of $\nu_i \cap E$ $(i = 1, 2, \ldots)$ with counting multiplicities are bounded above for any fixed compact set E of D. (See Stoll [16, Theorem 2.24].)*

Proof of Theorem 2.3. Take any sequence $\{f_i\} \subset F$. By the assumption of Theorem 2.3 and Lemma 4.3 we can find a subsequence (again denoted by $\{f_i\}$) such that

$$\lim_{i\to\infty} \overline{\{z \in \operatorname{supp} \nu(f_i, H_k(f_i));\ \nu(f_i, H_k(f_i))(z) < m_k\}} = S_k$$

$(k = 1, \dots, q)$ as a sequence of closed subsets of D, where S_k are either empty or pure $(n-1)$-dimensional analytic sets of D by Lemma 4.2. Let $E := \bigcup_{k=1}^{q} S_k$. Then E is either empty or a pure $(n-1)$-dimensional analytic set of D and hence E is a nowhere dense analytic set of D.

Now we shall prove that $\{f_n(z)\}_{n=1}^{\infty}$ has a compactly convergent subsequence on $D - E$. For any fixed point z_0 in $D - E$, there exist an integer i_0 and a neighborhood $U(z_0)$ in $D - E$ such that

$$\bigl\{z \in \operatorname{supp} \nu\bigl(f_i, H_k(f_i)\bigr);\ \nu\bigl(f_i, H_k(f_i)\bigr)(z) < m_k\bigr\} \cap U(z_0) = \emptyset$$

for $i \geqslant i_0$ and $k = 1, \dots, q$. Hence by Theorem 2.1, $\{f_i(z)\}_{i=i_0}^{\infty}$ is a sequence of holomorphic mappings of $U(z_0)$ into $P^N(C)$ and by Theorem 2.A $\{f_i(z)\}_{i=i_0}^{\infty}$ has a subsequence which converges uniformly on compact subsets of $U(z_0)$ to a holomorphic mapping of $U(z_0)$ into $P^N(C)$. Therefore, by the usual diagonal argument, we can find a subsequence $\{f_{i_j}(z)\}$ so as to converge uniformly on compact subsets of $D - E$ to a holomorphic mapping of $D - E$ into $P^N(C)$ and hence $\{f_{i_j}(z)\}$ is quasi-regular on D. The proof of Theorem 2.3 is completed. □

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