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A moment inequality of the Marcinkiewicz–Zygmund type for some weakly dependent random fields

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ABSTRACT

The goal of this note is to give a new moment inequality for sums of some weakly dependent random fields. Our result extends moment bounds given by Wu (2007) or Liu and Lin (2009) for causal autoregressive processes and follows by using recursive applications of the Burkhölder inequality for martingales.

1. Introduction

In this paper, we consider random fields X indexed by $\mathbb{Z}^d$, $d \geq 1$, and which can be written as a Bernoulli shift, e.g.

$$X_t = H\left((\xi_{t-j})_{j\in\mathbb{Z}^d}\right), \quad t \in \mathbb{Z}^d, \tag{1}$$

where ξ is an independent and identically distributed E-valued random field and H is a real-valued measurable function. We consider here $E = \mathbb{R}^k$ with $k \in \mathbb{N}^* \cup \{\infty\}$ but a more general measurable space can be considered. A random field of the form (1) is strictly stationary. The most well-known examples of such random fields are linear random fields, i.e.

$$X_t = \sum_{j\in\mathbb{Z}^d} a_j \xi_{t-j}, \quad t \in \mathbb{Z}^d. \tag{2}$$

Spatial AR processes which can be written as linear random fields were extensively studied in spatial statistics (see Guyon, 1995, for a nice presentation).

Other (nonlinear) examples of such random fields are given in Doukhan and Truquet (2007) as solutions of autoregressive equations of the form

$$X_t = F\left((X_{t-j})_{j\in\mathbb{Z}^d}; \xi_t\right), \quad t \in \mathbb{Z}^d,$$

where F is a Lipschitzian function.

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In this note, we prove moment inequalities of the Marcinkiewicz–Zygmund type for the partial sums of a random field X of the form (1). More precisely we prove that for a real number $q > 1$,

$$\|S(B)\|_q \le C_q^d \sum_{k\in\mathbb{Z}^d} \|p_{d,k_d}^{(\xi)} \circ \cdots \circ p_{1,k_1}^{(\xi)}(X_0)\|_q |B|^{1/q'}, \tag{3}$$

where B is a finite subset of $\mathbb{Z}^d$, $S(B) = \sum_{j\in B} X_j$, $q' = \min(q, 2)$ and for $(s, \ell) \in \{1, \dots, d\} \times \mathbb{Z}$, $p_{s,\ell}^{(\xi)}$ denotes the projection operator defined for an integrable random variable Z by

$$p_{s,\ell}^{(\xi)}(Z) = Z - \mathbb{E}\left(Z/\sigma(\xi_j/j_s \neq \ell)\right). \tag{4}$$

The constant C_q is the universal constant of the Burkhölder inequality (see Hall and Heyde, 1980) and thus it does not depend on the distribution of the random field. $\|\cdot\|_q$ denotes the usual norm of $\mathbb{L}^q$. Defining

$$A_q = \sum_{k\in\mathbb{Z}^d} \|p_{d,k_d}^{(\xi)} \circ \cdots \circ p_{1,k_1}^{(\xi)}(X_0)\|_q, \tag{5}$$

A_q is a positive constant depending only on q and the distribution of the random field ξ. Obviously inequality (3) is interesting only when $A_q < \infty$. The latter condition basically indicates short-range dependence. Then we study in detail the finiteness of the constant A_q when the random field X can be written as a functional of linear fields.

An inequality of the form (3) is a useful tool for studying the behaviour of sums of dependent random variables. Strong laws of large numbers, the strong invariance principle and other variants of limit theorems are often based on this kind of inequality. Some existing methods for obtaining such inequalities use mixing coefficients. ϕ-mixing random fields are considered in Dedecker (2001) and Delyon (2009), and an inequality for α-mixing random fields is given in Doukhan (1994). Unfortunately, for random fields defined by (1), mixing conditions lead to restrictive assumptions on the random field distribution (see Guyon, 1995, for the case of linear random fields) or are impossible to check (see the counterexample of Andrews (1984) when $d = 1$). On the other hand, general weak dependence conditions formulated in terms of covariances of Lipschitzian functionals can be applied to random fields defined by (1) and corresponding moment inequalities are available. We refer the reader to Dedecker et al. (2007) and Doukhan et al. (2009) for weak dependence conditions that generalize strong mixing and to Bulinski and Shashkin (2006) for a generalization of associated random fields. Nevertheless, despite their generality, such approaches do not exploit the particular form of the random field when it can be written as a functional of independent random variables and lead to complicated and restrictive conditions for the representation (1). When $d = 1$, martingale decompositions and the Burkhölder inequality for martingales have been used to prove simple moment bounds for partial sums of causal processes of the form (2). We refer the reader to Wu (2007) and Liu and Lin (2009) for details. In this paper, we generalize this approach to random fields and we prove inequality (3) using a simple recursive application of the Burkhölder inequality. We next study in detail the application of such inequalities to partial sums of transforms of linear random fields.

2. Moment bounds

We first state basic properties of specific conditional expectations and operators $p_{s,\ell}^{(\xi)}$ defined in (4).

Lemma 1. *Let Z be an integrable random variable, ξ an i.i.d. random field and q a real number with $q \ge 1$. Then:*

1. *If $A, B \subset \mathbb{Z}^d$,*

$$\mathbb{E}\left(\mathbb{E}\left(Z/\xi_A\right)/\xi_B\right) = \mathbb{E}\left(Z/\xi_{A\cap B}\right).$$

2. *If $A \subset \mathbb{Z}^d$ and $B \subset \mathbb{Z}^d \setminus A$,*

$$\|\mathbb{E}\left(Z/\xi_{A\cup B}\right) - \mathbb{E}\left(Z/\xi_A\right)\|_q \le \left\|Z - \mathbb{E}\left(Z/\xi_{\mathbb{Z}^d\setminus B}\right)\right\|_q.$$

3. *If $(s, s', \ell, \ell') \in \{1, \dots, d\}^2 \times \mathbb{Z} \times \mathbb{Z}$,*

$$\|p_{s,\ell}^{(\xi)} \circ p_{s',\ell'}^{(\xi)}(Z)\|_q \le 2\min\left(\|p_{s,\ell}^{(\xi)}(Z)\|_q, \|p_{s',\ell'}^{(\xi)}(Z)\|_q\right).$$

Proof of Lemma 1. 1. The result follows from the independence properties of the random field ξ.

2. From the point 1, we have

$$\mathbb{E}\left(Z/\xi_{A\cup B}\right) - \mathbb{E}\left(Z/\xi_A\right) = \mathbb{E}\left(Z - \mathbb{E}\left(Z/\xi_{\mathbb{Z}^d/B}\right)/\xi_{A\cup B}\right),$$

and the result is a consequence of the Jensen inequality.

3. We have from the point 1

$$p_{s,\ell}^{(\xi)} \circ p_{s',\ell'}^{(\xi)}(Z) = Z - \mathbb{E}\left(Z/\xi_A\right) + \mathbb{E}\left(Z - \mathbb{E}\left(Z/\xi_A\right)/\xi_B\right),$$

for $A = \{j \in \mathbb{Z}^d/j_s \neq \ell\}$ and $B = \{j \in \mathbb{Z}^d/j_{s'} \neq \ell'\}$ and from the triangular inequality and Jensen inequality, we obtain

$$\|p_{s,\ell}^{(\xi)} \circ p_{s',\ell'}^{(\xi)}(Z)\|_q \le 2\|p_{s,\ell}^{(\xi)}(Z)\|_q.$$

Exchanging A and B, the result follows. □

The main result of the paper can now be stated.

Theorem 1. *Let X be a real centered random field indexed by $\mathbb{Z}^d$, $d \geq 1$, and such that $\mathbb{E}|X_t| < \infty$ for all $t \in \mathbb{Z}^d$. Assume furthermore that there exists an i.i.d. random field ξ such that $\sigma(X) \subset \sigma(\xi)$. Then for a real number $q > 1$, we have with $q' = \min(q, 2)$*

$$\|S(B)\|_q \leq C_q^d \sum_{k \in \mathbb{Z}^d} \left(\sum_{t \in B} \|p^{(\xi)}_{d, t_d + k_d} \circ \cdots \circ p^{(\xi)}_{1, t_1 + k_1}(X_t)\|_q^{q'} \right)^{1/q'}, \tag{6}$$

where C_q denotes the universal constant of the Burkhölder inequality for martingales.

Remarks. 1. The proof of inequality (6) does not use the representation (1) and the stationarity assumption. Note that for a stationary random field, inequality (6) coincides with (3).

2. Let X be a random field of the form (1) and such that $A_q < \infty$ for a real number $q > 1$. Then the Móricz theorem (Móricz, 1983) can be used to deduce from the bounds (3) a moment inequality for partial maxima when B is a block of $\mathbb{Z}^d$, i.e.

$$B = ((a_1, b_1] \times \cdots \times (a_d, b_d]) \cap \mathbb{Z}^d.$$

More precisely, if $q > 2$, we have

$$\| \max_{W \lhd B} |S(W)| \|_q \leq (5/2)^{d/q} \left(1 - 2^{\frac{2-q}{2q}}\right)^{-d} C_q^d A_q |B|^{1/2},$$

where the notation $W \lhd B$ means $W \subset B$ and W is a block of $\mathbb{Z}^d$ with the same minimal vertex as B, i.e.

$$W = ((a_1, c_1] \times \cdots \times (a_d, c_d]) \cap \mathbb{Z}^d.$$

In fact the result of Móricz is analytic, in the sense that it does not involve any dependence properties. When $q \in (1, 2]$, it gives the bound

$$\| \max_{W \lhd B} |S(W)| \|_q \leq 5^{d/q} 2^{d(q-1)/q} C_q^d A_q |B|^{1/q} \prod_{i=1}^{d} \log(b_i - a_i).$$

Proof of Theorem 1. 1. We first prove the result for $d = 1$. For $j \in \mathbb{Z}$ and an integrable random variable Z, we define

$$\mathcal{P}_i(Z) = \mathbb{E}\left(Z/\mathcal{F}_i\right) - \mathbb{E}\left(Z/\mathcal{F}_{i-1}\right),$$

where $\mathcal{F}_i = \sigma(\xi_j / j \leq i)$. We use the following decomposition:

$$\begin{aligned} X_i &= X_i - \mathbb{E}\left(X_i/\mathcal{F}_i\right) + \mathbb{E}\left(X_i/\mathcal{F}_i\right) \\ &= \sum_{j \geq 1} \mathcal{P}_{i+j}(X_i) + \sum_{j \geq 0} \mathcal{P}_{i-j}(X_i) \\ &= \sum_{j \in \mathbb{Z}} \mathcal{P}_{i+j}(X_i). \end{aligned}$$

Then we have $S(B) = \sum_{j \in \mathbb{Z}} \sum_{i \in B} \mathcal{P}_{i+j}(X_i)$, and $\left(\mathcal{P}_{i+j}(X_i)\right)_{i \in \mathbb{Z}}$ is a martingale difference for the filtration $\left(\mathcal{F}_{i+j}\right)_{i \in \mathbb{Z}}$. The Burkhölder inequality leads to

$$\|S(B)\|_q \leq \sum_{j \in \mathbb{Z}} \left\| \sum_{i \in B} \mathcal{P}_{i+j}(X_i) \right\|_q \leq C_q \sum_{j \in \mathbb{Z}} \left\{ \mathbb{E} \left| \sum_{i \in B} \mathcal{P}_{i+j}(X_i)^2 \right|^{q/2} \right\}^{1/q}.$$

If $q \geq 2$, the triangular inequality for the $\mathbb{L}^{q/2}$ norm leads to

$$\|S(B)\|_q \leq C_q \sum_{j \in \mathbb{Z}} \left\{ \sum_{i \in B} \|\mathcal{P}_{i+j}(X_i)\|_q^2 \right\}^{1/2}.$$

Setting $A = \{\ell \in \mathbb{Z} / \ell \leq i + j - 1\}$ and $B = \{i + j\}$, an application of point 2 in Lemma 1 yields

$$\|\mathcal{P}_{i+j}(X_i)\|_q \leq \|p^{(\xi)}_{1, i+j}(X_i)\|_q.$$

Inequality (6) follows.

If now $q \in (1, 2]$, inequality (6) is a consequence of the previous remark and the bound

$$\left(\sum_{i \in B} \mathcal{P}_{i+j}(X_i)^2 \right)^{q/2} \leq \sum_{i \in B} |\mathcal{P}_{i+j}(X_i)|^q.$$

2. We show the result for $d \geq 2$ using induction on d. Suppose that the bound (6) holds for any random field indexed by $\mathbb{Z}^s$, $s \leq d-1$, and satisfying the assumptions of Theorem 1. We have $X = (X_{i,j})_{i\in\mathbb{Z}, j\in\mathbb{Z}^{d-1}}$ and

$$S(B) = \sum_{i\in A} Y_i, \qquad Y_i = \sum_{j\in B_i} X_{i,j},$$

where $B = \cup_{i\in A}\{i\}\times B_i$. It is obvious that $(Y_i)_{i\in\mathbb{Z}}$ satisfies the assumptions of Theorem 1 with the random field $\xi'_i = \xi_{i,\mathbb{Z}^{d-1}}$, $i \in \mathbb{Z}$. Then, using inequality (6) for $d = 1$, we obtain

$$\|S(B)\|_q \leq C_q \sum_{k_1\in\mathbb{Z}} \left(\sum_{i\in A} \|p^{(\xi')}_{1,i+k_1}(Y_i)\|_q^{q'}\right)^{1/q'}. \tag{7}$$

Note also that for $\ell \in \mathbb{Z}$, we have $p^{\xi'}_{1,\ell} = p^{\xi}_{1,\ell}$. Then for $k_1 \in \mathbb{Z}$ and $i \in A$, we have $p^{(\xi')}_{1,i+k_1}(Y_i) = \sum_{j\in B_i} p^{(\xi)}_{1,k_1}(X_{i,j})$. Since the random field $\left(p^{(\xi)}_{1,k_1}(X_{i,j})\right)_{j\in B_i}$ is indexed by $\mathbb{Z}^{d-1}$ and satisfies the assumptions of Theorem 1 with the i.i.d. random field $\xi'' = \left(\xi_{\mathbb{Z},j}\right)_{j\in\mathbb{Z}^{d-1}}$, we can apply the induction hypothesis. We observe that for $s \in \{1,\ldots,d-1\}$, $p^{(\xi'')}_{s,k} = p^{(\xi)}_{s+1,k}$. Then we get the following bound:

$$\begin{aligned}\|p^{(\xi)}_{1,i+k_1}(Y_i)\|_q &\leq C_q^{d-1} \sum_{k_2,\ldots,k_d\in\mathbb{Z}} \left(\sum_{j\in B_i} \left\| p^{(\xi)}_{d,j_{d-1}+k_d} \circ \cdots \circ p^{(\xi)}_{2,j_1+k_2}\left(p^{(\xi)}_{1,i+k_1}(X_{i,j})\right)\right\|_q^{q'}\right)^{1/q'} \\ &\leq C_q^{d-1} \sum_{k_2,\ldots,k_d\in\mathbb{Z}} \left(\sum_{j\in B_i} \|p^{(\xi)}_{d,j_{d-1}+k_d} \circ \cdots \circ p^{(\xi)}_{2,j_1+k_2} \circ p^{(\xi)}_{1,i+k_1}(X_{i,j})\|_q^{q'}\right)^{1/q'}.\end{aligned}$$

Then, using the triangular inequality for the norm $\|x\| = \left(\sum_i |x_i|^{q'}\right)^{1/q'}$, we obtain

$$\left(\sum_{i\in A} \|p^{(\xi)}_{1,i+k_1}(Y_i)\|_q^{q'}\right)^{1/q'} \leq C_q^{d-1} \sum_{k_2,\ldots,k_d\in\mathbb{Z}} \left(\sum_{i\in A}\sum_{j\in B_i} \|p^{(\xi)}_{d,j_{d-1}+k_d} \circ \cdots \circ p^{(\xi)}_{1,i+k_1}(X_{i,j})\|_q^{q'}\right)^{1/q'},$$

and the bound (6) follows from (7) using the equality $\sum_{i\in A}\sum_{j\in B_i} = \sum_{(i,j)\in B}$. □

The iterations involving the operators $p^{(\xi)}_{s,\ell}$ are not always easy to evaluate. The following corollary will be useful for verifying the condition $A_q < \infty$ for most of the examples.

Corollary 1. *Let X be a real random field satisfying the assumptions of Theorem 1. Then*

$$A_q \leq 2^{d-1} \sum_{s=1}^{d} \sum_{n\in\mathbb{N}} (2n+1)^{d-1} \sum_{|k_s|=n} c_{s,k_s}, \tag{8}$$

where for $s \in \{1,\ldots,d\}$,

$$c_{s,k_s} = \left(\sum_{t\in B} \|p^{(\xi)}_{s,t_s+k_s}(X_t)\|_q^{q'}\right)^{1/q'}.$$

Remarks. 1. When the random field X can be written as in (1) we have the bound

$$\|p^{(\xi)}_{s,t_s+k_s}(X_t)\|_q \leq \|X_t - \widetilde{X}_{t,s,k_s}\|_q, \tag{9}$$

where $\widetilde{X}_{t,s,k_s} = H\left(\left(\widetilde{\xi}_{t-j}\right)_{j\in\mathbb{Z}^d}\right)$, $\widetilde{\xi}_{t-j} = \xi_{t-j}$ if $j_s \neq -k_s$ and ξ'_{t-j} otherwise, ξ' being a copy of ξ. When $d = 1$, the right hand side of (9) is the coefficient denoted by $\theta_{q,k}$ used by Wu (2005) to measure the dependence of a stationary process X. In this case, condition $A_q < \infty$ holds when

$$\sum_{k\in\mathbb{Z}} \|p^{(\xi)}_{1,k}(X_0)\|_q < \infty,$$

which is similar to the condition $\sum_{k\leq 0} \theta_{q,k} < \infty$ used by Liu and Lin (2009) (see Lemma A1 and Lemma A2 of their paper) for a causal stationary process

$$X_t = H\left(\left(\xi_{t-j}\right)_{j\geq 0}\right), \quad t \in \mathbb{Z}.$$

2. Note that when $d = 1$, A_q is finite if and only if the right hand side of inequality (8) is finite. For $d \geq 2$ this is no longer true. For example, for the simple example of linear random fields (2), we have

$$A_q = \sum_{k\in\mathbb{Z}^d} |a_k| \times \|\xi_0\|_q.$$

In contrast, when $q = 2$ and ξ_0 is square integrable, the right hand side of (8) is finite provided for $s = 1, \ldots, d$,

$$\sum_{n\in\mathbb{N}} n^{d-1} \sqrt{\sum_{|j_s|=n} a_j^2} < \infty,$$

which is more restrictive than the summability of coefficients (a_j).

Proof of Corollary 1. From point 3 in Lemma 1, a straightforward induction leads to the following inequality:

$$\|p^{(\xi)}_{d,t_d+k_d} \circ \cdots \circ p^{(\xi)}_{1,t_1+k_1}(X_t)\|_q \leq 2^{d-1} \wedge_{s=1}^d \|p^{(\xi)}_{\ell,k_s+t_s}(X_t)\|_q. \tag{10}$$

Assume that the random field X satisfies the assumptions of Theorem 1. We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}\left(\sum_{t\in B} \wedge_{s=1}^d \|p^{(\xi)}_{s,t_s+k_s}(X_t)\|_q\right)^{1/q'} &\leq \sum_{s=1}^d \sum_{n\in\mathbb{N}} \sum_{k/\|k\|_\infty=|k_s|=n} c_{s,k_s}\\ &\leq \sum_{s=1}^d \sum_{n\in\mathbb{N}} (2n+1)^{d-1} \sum_{|k_s|=n} c_{s,k_s}.\end{aligned}$$

Then the conclusion of Corollary 1 follows. □

3. Examples

In the sequel ξ denotes an i.i.d. random field satisfying $\mathbb{E}\xi_0 = 0$. For a random field X of the form (1), inequality (3) is interesting provided the constant A_q defined in (5) is finite. For the simple case of a linear random field (2), we have

$$A_q = \sum_{k\in\mathbb{Z}^d} |a_j| \|\xi_0\|_q.$$

Then provided that $\xi_0 \in \mathbb{L}^q$, we have $A_q < \infty$ if and only if $\sum_{j\in\mathbb{Z}^d} |a_j| < \infty$. The latter condition is a weak dependence condition since in this case the autocovariances of the field are summable.

A precise evaluation of the constant A_q may also be possible when the random field X has a polynomial expression with respect to the coordinates of ξ. We investigate, as an example, a moment bound for the covariances of linear random fields.

Corollary 2. *Let $Y_t = \sum_{j\in\mathbb{Z}^d} a_j \xi_{t-j}$, $t \in \mathbb{Z}^d$, with $\sum_{j\in\mathbb{Z}^d} |a_j| < \infty$, and for a given $h \in \mathbb{Z}^d$,*

$$X_t = Y_t Y_{t+h} - \mathbb{E}\left(Y_t Y_{t+h}\right).$$

Assume that $\xi_0 \in \mathbb{L}^{2m}$ with $m > 1$. Then for $1 < q \leq m$, $A_q < \infty$.

Proof of Corollary 2. Let $k \in \mathbb{Z}^d$ and $B = p^{(\xi)}_{d,t_d+k_d} \circ \cdots \circ p^{(\xi)}_{1,t_1+k_1}(X_t)$. Then we are going to prove

$$B = \sum_{(\alpha,\beta)\in I} \left(\sum_{j\in A_\alpha} a_j \xi_{t-j}\right) \times \left(\sum_{j\in A_\beta} a_{j+h} \xi_{t-j}\right) + a_{-k} a_{-k+h}\left(\xi_{t+k}^2 - \mathbb{E}\xi_0^2\right), \tag{11}$$

where

$$\begin{aligned}&I = \cap_{s=1}^d \{(\alpha,\beta) \in \{0,1\}^d \times \{0,1\}^d / (\alpha,\beta) \neq (0,0) \text{ and } \alpha_s\beta_s = 0\},\\ &A_\alpha = \cap_{s=1}^d \{j \in \mathbb{Z}^d / j_s = -k_s \text{ if } \alpha_s = 0 \text{ and } \alpha_s = 1 \text{ otherwise }\}.\end{aligned}$$

One can easily see that if $(\alpha,\beta) \in I$, then $\alpha \neq \beta$ and since there exists $s \in \{1,\ldots,d\}$ such that $\alpha_s = 0$ and $\beta_s = 1$ (or the contrary) then the sums $\sum_{j\in A_\alpha} a_j \xi_{t-j}$ and $\sum_{j\in A_\beta} a_j \xi_{t-j}$ are independent. Before giving a proof of (11), we show why the conclusion of Corollary 2 holds. From (11) we obtain the following bound:

$$\|B\|_q \leq \sum_{(\alpha,\beta)\in I} \sum_{j\in A_\alpha} |a_j| \cdot \sum_{j\in A_\beta} |a_{j+h}| \|\xi_0\|_q^2 + \left|a_{-k} a_{-k+h}\left(\xi_0^2 - \sigma^2\right)\right|,$$

where $\sigma^2 = \mathbb{E}\xi_0^2$. Let $q \in (1, m]$. Using assumptions on ξ and coefficients (a_j), $A_q < \infty$ if for $(\alpha, \beta) \in I$,

$$G = \sum_{k\in\mathbb{Z}^d}\left(\sum_{j\in A_\alpha}|a_j| \cdot \sum_{j\in A_\beta}|a_{j+h}|\right) < \infty. \tag{12}$$

If $\alpha = 0$, then $A_\alpha = -k$ and $G \le \sum_j |a_j| \times \sum_j |a_{j+h}| < \infty$. The same holds if $\beta = 0$. Now suppose that $\ell = \sharp\{s \in \{1,\dots,d\}/\alpha_s = 0\}$ satisfies $1 \le \ell \le d-1$. Without loss of generality, we assume that $\alpha_s = 0$, $s \le \ell$. Then by definition of I, we have $\beta_s = 0$ if $s \ge \ell + 1$. In this case, we obtain

$$\begin{aligned} G &\le \sum_{k\in\mathbb{Z}^d}\left(\sum_{j_{\ell+1},\dots,j_d}|a_{-k_1,\dots,-k_\ell,j_{\ell+1},\dots,j_d}|\right)\cdot\left(\sum_{j_1,\dots,j_\ell}|a_{j_1+h_1,\dots,j_\ell+h_\ell,-k_{\ell+1}+h_{\ell+1},\dots,-k_d+h_d}|\right) \\ &\le \sum_{j\in\mathbb{Z}^d}|a_j|\cdot\sum_{j\in\mathbb{Z}^d}|a_{j+h}|. \end{aligned}$$

Then $G < \infty$ and the conclusion of Corollary 2 follows.

Now we prove (11). We first introduce some notation. For $s \in \{1,\dots,d\}$, let

$$\begin{aligned} I^{(s)} &= \cap_{i=1}^s\{(\alpha_1,\dots,\alpha_s,\beta_1,\dots,\beta_s) \in \{0,1\}^{2s}\ \ \alpha_i\beta_i = 0\}, \\ A_\alpha^{(s)} &= \cap_{i=1}^s\{j \in \mathbb{Z}^d\ /\ j_i = -k_i \text{ if } \alpha_i = 0, \text{ and } j_i \ne -k_i \text{ otherwise }\}, \\ B_s &= p^{(\xi)}_{s,t_s+k_s}\circ\cdots\circ p^{(\xi)}_{1,t_1+k_1}(X_t). \end{aligned}$$

Using finite induction on the set $\{1,\dots,d\}$, we are going to prove that for $s \in \{1,\dots,d\}$,

$$B_s = \sum_{(\alpha,\beta)\in I^{(s)}}\left(\sum_{j\in A_\alpha^{(s)}}a_j\xi_{t-j}\right)\times\left(\sum_{j\in A_\beta^{(s)}}a_{j+h}\xi_{t-j}\right) - \sum_{j_i=-k_i,i=1,\dots,s}a_ja_{j+h}\sigma^2. \tag{13}$$

Observe that $B_d = B$. The proof of this induction uses the following remark: for two subsets C and D of $\mathbb{Z}^d$, we have for $s \in \{1,\dots,d\}$ and $Z_t = \sum_{j\in C}a_j\xi_{t-j}\cdot\sum_{j\in D}a_{j+h}\xi_{t-j}$,

$$\begin{aligned} p_{s,t_s+k_s}(Z_t) = &\sum_{j\in C,j_s=-k_s}a_j\xi_{t-j}\cdot\sum_{j\in D,j_s\ne-k_s}a_{j+h}\xi_{t-j} + \sum_{j\in C,j_s\ne-k_s}a_j\xi_{t-j}\cdot\sum_{j\in D,j_s=-k_s}a_{j+h}\xi_{t-j} \\ &-\sigma^2\sum_{j\in C\cap D,j_s=-k_s}a_ja_{j+h}. \end{aligned}$$

- The case $s = 1$ is an easy consequence of the previous remark.
- Suppose that equality (13) holds for B_s. Then applying again the previous equality to $B_{s+1} = p_{s+1,k_{s+1}}(B_s)$, equality (13) easily follows for B_{s+1}. $\square$

The next corollary gives a sufficient condition for having $A_q < \infty$ for some locally Hölderian functionals of linear random fields. Here we apply Corollary 1.

Corollary 3. *Let $h : \mathbb{R} \mapsto \mathbb{R}$ be a function such that there exist $a \ge 0$, $b \in (0,1]$ and a positive constant K satisfying*

$$\left|h(x) - h(y)\right| \le K\left(1 + |x|^a + |y|^a\right)\left|x - y\right|^b, \quad x, y \in E.$$

For $t \in \mathbb{Z}^d$, let $Y_t = \sum_{j\in\mathbb{Z}^d}a_j\xi_{t-j}$, $t \in \mathbb{Z}^d$ and $X_t = h(Y_t) - \mathbb{E}h(Y_t)$. Assume that $\mathbb{E}|\xi_0|^m < \infty$ with $m > 1$, $m \ge (a+b)q$ and $q > 1$. If for $s \in \{1,\dots,d\}$

$$\sum_{n\ge0}(n+1)^{d-1}\left(\sum_{j/|j_s|=n}a_j^2\right)^{b/2} < \infty, \tag{14}$$

then $A_q < \infty$.

In particular, if $a_j = O\left(\|j\|^{-\alpha}\right)$ with $\alpha > \frac{(2+b)d-b}{2b}$, condition (14) is satisfied.

Remarks. 1. Corollary 3 shows that inequality $A_q < \infty$ holds with a more restrictive condition than $\sum_{j\in\mathbb{Z}^d}|a_j| < \infty$. For example, when $a_j = O\left(\|j\|^{-\alpha}\right)$ the summability of coefficients $(a_j)_j$ holds when $\alpha > d$, but $\frac{(2+b)d-b}{2b} > d$ when $d \ge 2$.

2. The result of Corollary 1 can also be applied if the functional h of Corollary 3 has some discontinuities. However, additional assumptions could be required in order to bound explicitly the quantity $\|p_{s,\ell}^{(\xi)}(X_0)\|_q$. For example if $h(x) = \mathbb{1}_{x>t}$, one can use an approximation of h by the Lipschitz function $h_\epsilon(x) = \mathbb{1}_{x>t} + \left(\frac{1}{\epsilon}x + 1 - \frac{t}{\epsilon}\right)\mathbb{1}_{t-\epsilon<x\leq t}$. We obtain
$$\|p_{s,\ell}^{(\xi)}(X_0)\|_q \leq 2\|X_0 - h_\epsilon(Y_0)\|_q + \|p_{s,\ell}^{(\xi)}(h_\epsilon(Y_0))\|_q.$$
Then we can bound the second term as in the proof of Corollary 3 and we get
$$\|p_{s,\ell}^{(\xi)}(X_0)\|_q \leq 2\mathbb{P}(t-\epsilon < Y_0 \leq t) + \frac{2}{\epsilon}\|\xi_0\|_q\sqrt{\sum_{j_s=-\ell} a_j^2}.$$
Some regularity assumptions have to be made for optimizing the previous bound in ϵ (e.g. $\|p_{s,\ell}^{(\xi)}(X_0)\|_q \sim \left(\sum_{j_s=-\ell} a_j^2\right)^{1/4}$ if Y_0 has a locally bounded density).

Proof of Corollary 3. We first derive a bound for the quantities $\|p_{s,t_s+k_s}(X_t)\|_q$, $s = 1, \ldots, d$, and $q \leq \frac{m}{a+b}$. For a copy ξ' of ξ, we set
$$Z = K\left(1 + \left|\sum_{j\in\mathbb{Z}^d} a_j\xi_{t-j}\right|^a + \left|\sum_{j/j_s\neq -k_s} a_j\xi_{t-j} + \sum_{j/j_s=-k_s} a_j\xi'_{t-j}\right|^a\right)\left|\sum_{j/j_s=-k_s} a_j(\xi_{t-j} - \xi'_{t-j})\right|^b.$$
Then using the assumption on h, it is not difficult to prove that $\|p_{s,t_s+k_s}(X_t)\|_q \leq \|Z\|_q$. If $a > 0$, we consider two positive numbers p and r satisfying $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, $ar \leq m$ and $m \geq bp > 1$. Such a choice is possible from the assumptions on m. In the sequel C denotes a generic positive constant not depending on the index k or j. Then using the previous bound and Hölder inequality, we obtain
$$\|p_{s,t_s+k_s}(X_t)\|_q \leq C\mathbb{E}^{1/p}\left|\sum_{j/j_s=-k_s} a_j(\xi_{t-j} - \xi'_{t-j})\right|^{bp}.$$
Note that the previous bound holds also when $a = 0$ if we set $p = m/b$. From the Burkhölder inequality, we obtain
$$\|p_{s,t_s+k_s}(X_t)\|_q \leq C\left(\sum_{j/j_s=-k_s} a_j^2\right)^{b/2}. \tag{15}$$
The conclusion of the corollary follows easily from (15) and Corollary 1.

Suppose now $a_j = O\left(\|j\|^{-\alpha}\right)$ with $\alpha > \frac{(2+b)d-b}{2b}$. Then we have for $s \in \{1, \ldots, d\}$
$$\sum_{j/j_s=n} a_j^2 \leq C \sum_{j/\|j\|\geq n, j_s=n} \|j\|^{-2\alpha} \leq Cn^{-2\alpha+d-1}.$$
Using the same bound for $\sum_{j/j_s=-n} a_j^2$, we deduce that condition (14) is satisfied and the result follows. □

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