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An improved upper bound for the argument of the Riemann zeta-function on the critical line II

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ABSTRACT

Text. This paper concerns the function $S(T)$, where $\pi S(T)$ is the argument of the Riemann zeta-function along the critical line. The main result is that

$$\left|S(T)\right| \leqslant 0.112 \log T + 0.278 \log\log T + 2.510,$$

which holds for all $T \geqslant e$.

Video. For a video summary of this paper, please click here or visit http://youtu.be/FldP0idE0aI.

1. Summary of results

This paper is the sequel to [15] and is related to [16]; reference will be made frequently to these papers. Write

$$\left|S(T)\right| \leqslant a \log T + b \log\log T + c, \quad \text{for } T \geqslant T_0, \tag{1.1}$$

whence Table 1 provides a brief historical summary.

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Table 1
Bounds on $S(T)$ in (1.1).

	a	b	c	T_0
Von Mangoldt [20] 1905	0.432	1.917	12.204	28.588
Grossmann [6] 1913	0.291	1.787	6.137	50
Backlund [1] 1914	0.275	0.979	7.446	200
Backlund [2] 1918	0.137	0.443	4.35	200
Rosser [11] 1939	1.12	0	9.5	1450
Rosser [12] 1941	0.137	0.443	1.588	1467
Trudgian [15] 2012	0.17	0	1.998	e
Trudgian (Theorem 1) 2012	0.112	0.278	2.510	e

Note that the result in [15] improves on that in [12] when $25 \leqslant T \leqslant 10^{15}$. The purpose of this article is to improve on the result in [12] for all T. This is achieved with the following theorem.

Theorem 1. *If $T \geqslant e$, then*

$$\left|S(T)\right| \leqslant 0.112 \log T + 0.278 \log\log T + 2.510.$$

This implies the following result concerning $N(T)$, the number of complex zeroes of $\zeta(s)$ with imaginary parts in $(0, T)$.

Corollary 1. *If $T \geqslant e$, then*

$$\left|N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8}\right| \leqslant 0.112 \log T + 0.278 \log\log T + 2.510 + \frac{0.2}{T}.$$

It is known (see, e.g., [3,13,4,18,19]) that[2]

$$\begin{aligned} \left|S(T)\right| &\leqslant 1, \quad \text{for } 0 \leqslant T \leqslant 280, \\ \left|S(T)\right| &\leqslant 2, \quad \text{for } 0 \leqslant T \leqslant 6.8 \times 10^6, \\ \left|S(T)\right| &\leqslant 2.31366, \quad \text{for } 0 \leqslant T \leqslant 5.45 \times 10^8. \end{aligned} \tag{1.2}$$

The approach taken in this paper is to prove results initially for $T > T_0 > 5.45 \times 10^8$, and then for all T using (1.2). Theorem 1 is sharper than Rosser's bound in [12] whenever $T \geqslant 25$; for smaller values of T one is better placed using (1.2), which is superior to both Theorem 1 and the bound in [12].

Explicit bounds on $S(T)$ are used in calculations concerning the zeros of the zeta-function — see, e.g., [11,12]. Hence there is some interest in obtaining, not necessary the smallest coefficient of $\log T$ in Theorem 1, but good bounds of the form $|S(T)| \leqslant \alpha \log T$ for all $T \geqslant T_0$. This is accomplished by choosing appropriate values of parameters η and r that appear in the bound for $S(T)$ in Section 6. The results are summarised in Table 2.

[2] Indeed, the first two lines in (1.2) are equivalent to Gram's Law holding for all $0 \leqslant T \leqslant 280$ and Rosser's Rule holding for all $0 \leqslant T \leqslant 6.8 \times 10^6$.

Table 2
Bounds on $|S(T)| \leqslant \alpha \log T$ for $T > T_0$.

T_0	α	η	r
10^6	0.258	0.31	2.23
10^7	0.245	0.26	2.31
10^8	0.234	0.23	2.36
10^9	0.226	0.21	2.40
10^{10}	0.218	0.19	2.44
10^{11}	0.212	0.17	2.48
10^{12}	0.207	0.15	2.53
10^{13}	0.203	0.14	2.54
10^{14}	0.198	0.13	2.53
10^{15}	0.195	0.13	2.49

Throughout the paper η denotes a parameter satisfying $0 < \eta \leqslant \frac{1}{2}$; θ denotes a complex number satisfying $|\theta| \leqslant 1$, and $s = \sigma + it$ where σ and t are real numbers.

2. Introduction

Write $h(s) = (s-1)\zeta(s)$, whence $h(s)$ is an entire function. The function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}s\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)h(s)$$

is entire and satisfies the functional equation

$$\xi(s) = \xi(1-s). \tag{2.1}$$

Let $N(T)$ denote the number of zeroes $\rho = \beta + i\gamma$ of $\zeta(s)$ for which $0 < \beta < 1$ and $0 < \gamma < T$. For any $\sigma_1 \in (1,2]$ form a rectangle with vertices at $\sigma_1 \pm iT$ and $1 - \sigma_1 \pm iT$. Let $\mathcal{C}$ denote the portion of the rectangle in the region $\Re(s) \geqslant \frac{1}{2}$ and $\Im(s) \geqslant 0$. Write $\mathcal{C}$ as the union of two straight lines, viz. let $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$, where $\mathcal{C}_1$ connects σ_1 to $\sigma_1 + iT$ and $\mathcal{C}_2$ connects $\sigma_1 + iT$ to $\frac{1}{2} + iT$. From Cauchy's theorem and (2.1) one deduces that $\pi N(T) = \Delta_{\mathcal{C}} \arg \xi(s)$. Thus

$$\pi N(T) = \Delta_{\mathcal{C}} \arg \pi^{-s/2} + \Delta_{\mathcal{C}} \arg s(s-1) + \Delta_{\mathcal{C}} \arg \Gamma\left(\frac{s}{2}\right) + \pi S(T), \tag{2.2}$$

where

$$\pi S(T) = \Delta_{\mathcal{C}_1} \arg \zeta(s) + \Delta_{\mathcal{C}_2} \arg h(s) - \Delta_{\mathcal{C}_2} \arg(s-1). \tag{2.3}$$

The only terms in (2.2) and (2.3) that require more than a passing mention are $\Delta_{\mathcal{C}_1} \arg \zeta(s), \Delta_{\mathcal{C}} \arg \Gamma(\frac{s}{2})$ and $\Delta_{\mathcal{C}_2} \arg h(s)$. For the first use

$$\left|\arg \zeta(\sigma_1 + it)\right| \leqslant \left|\log \zeta(\sigma_1 + it)\right| \leqslant \log \zeta(\sigma_1),$$

and for the second use

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \frac{\theta}{6|z|}$$

(see, e.g., [8, p. 294]), which is valid for $|\arg z| \leqslant \frac{\pi}{2}$. To estimate $\Delta_{\mathcal{C}_2} \arg h(s)$ write

$$f(s) = \frac{1}{2}\left\{h(s+iT)^N + h(s-iT)^N\right\}, \tag{2.4}$$

for some positive integer N, to be determined later. Thus $f(\sigma) = \Re h(\sigma + iT)^N$. Suppose that there are n distinct zeroes of $\Re h(\sigma + iT)^N$ for $\sigma \in \mathcal{C}_2$. These zeroes partition the segment into $n+1$ intervals. On each interval $\arg h(\sigma + iT)^N$ can increase by at most π, whence

$$\left|\Delta_{\mathcal{C}_2} \arg h(s)\right| = \frac{1}{N}\left|\Delta_{\mathcal{C}_2} \arg h(s)^N\right| \leqslant \frac{(n+1)\pi}{N}.$$

In conclusion, when $T \geqslant 1$

$$\begin{aligned}\left|N(T) - \frac{T}{2\pi}\log\frac{T}{2\pi e} - \frac{7}{8}\right| &\leqslant \frac{1}{4\pi}\tan^{-1}\frac{1}{2T} + \frac{T}{4\pi}\log\left(1 + \frac{1}{4T^2}\right) + \frac{1}{3\pi T} + \left|S(T)\right| \\ &\leqslant \frac{(n+1)}{N} + \frac{1}{\pi}\log\zeta(\sigma_1) + \frac{1}{T}.\end{aligned} \tag{2.5}$$

The inequality in (2.5) enables one to deduce Corollary 1 from Theorem 1.

3. Estimating n

One may estimate n with Jensen's Formula.

Lemma 1 *(Jensen's Formula). Let $f(z)$ be holomorphic for $|z-a| \leqslant R$ and non-vanishing at $z = a$. Suppose there are n zeroes of $f(z)$ in the circle. Let these be z_k, where $k = 1, 2, \ldots, n$, and let $|z_k - a| = r_k$. Then*

$$\log\frac{R^n}{r_1 r_2 \cdots r_n} = \frac{1}{2\pi}\int_0^{2\pi} \log f\left(a + Re^{i\phi}\right) d\phi - \log\left|f(a)\right|. \tag{3.1}$$

For a complex-valued function $F(s)$, and for $\delta > 0$ define $\Delta_+ \arg F(s)$ to be the change in argument of $F(s)$ as σ varies from $\frac{1}{2}$ to $\frac{1}{2} + \delta$, and define $\Delta_- \arg F(s)$ to be the change in argument of $F(s)$ as σ varies from $\frac{1}{2}$ to $\frac{1}{2} - \delta$.

The following lemma is now used to invoke Backlund's trick. Backlund's trick is to show that if there are zeroes of $\Re F(\sigma + iT)^N$ on the line $\sigma \in [\frac{1}{2}, \sigma_1]$, then there are zeroes on the line $\sigma \in [1 - \sigma_1, \frac{1}{2}]$.

Lemma 2. *(See [16].)*

(i) *Let N be a positive integer and let $T \geqslant T_0 \geqslant 1$. Suppose that there is an upper bound E that satisfies*

$$\left|\Delta_+ \arg F(s) + \Delta_- \arg F(s)\right| \leqslant E,$$

where $E = E(\delta, T_0)$. Suppose further that there exists an $n \geqslant 3 + \lfloor NE/\pi \rfloor$ for which

$$n\pi \leqslant \left|\Delta_{\mathcal{C}_3} \arg F(s)^N\right| < (n+1)\pi. \tag{3.2}$$

Then there are at least n zeroes of $\Re F(\sigma + iT)^N$ for $\sigma \in [\frac{1}{2}, \sigma_1]$, and at least $n - 2 - \lfloor NE/\pi \rfloor$ zeroes in $\sigma \in [1 - \sigma_1, \frac{1}{2}]$.

(ii) *Denote the zeroes in $[\frac{1}{2}, \sigma_1]$ by $\rho_\nu = a_\nu + iT$ where $\frac{1}{2} \leqslant a_n \leqslant a_{n-1} \leqslant \cdots \leqslant \sigma_1$, and the zeroes in $[1 - \sigma_1, \frac{1}{2}]$ as $\rho'_\nu = a'_\nu + iT$ where $1 - \sigma_1 \leqslant a'_1 \leqslant a'_2 \leqslant \cdots \leqslant \frac{1}{2}$. Then*

$$a_\nu \geqslant 1 - a'_\nu, \quad \text{for } \nu = 1, 2, \ldots, n - 2 - \lfloor NE/\pi \rfloor, \tag{3.3}$$

and, if η is defined by $\sigma_1 = \frac{1}{2} + \sqrt{2}(\eta + \frac{1}{2})$, then

$$\prod_{\nu=1}^{n} |1 + \eta - a_\nu| \prod_{\nu=1}^{n-2-\lfloor NE/\pi \rfloor} \left|1 + \eta - a'_\nu\right| \leqslant \left(\frac{1}{2} + \eta\right)^{2n-2-\lfloor NE/\pi \rfloor}. \tag{3.4}$$

Proof. See [16, Lemma 2]. □

3.1. Calculation of E

When $F(s) = h(s) = (s-1)\zeta(s)$, one may use [16, (5.4)] with $n_K = 1$ to estimate E in Lemma 2.

$$E \leqslant G(\delta, T) = \left(-\frac{5}{4} + \frac{\delta}{2}\right) \tan^{-1} \frac{\frac{1}{2} + \delta}{T} - \left(\frac{5}{4} + \frac{\delta}{2}\right) \tan^{-1} \frac{\frac{1}{2} - \delta}{T} + \frac{5}{2} \tan^{-1} \frac{1}{2T}$$
$$- \frac{T}{4} \log\left[1 + \frac{2\delta^2(T^2 - \frac{1}{4}) + \delta^4}{(T^2 + \frac{1}{4})^2}\right] + \frac{4\theta}{3T}.$$

One can show that $G(\delta, T)$ is decreasing in T and increasing in δ. Therefore, since, in Lemma 2 (i) one takes $\sigma_1 = \frac{1}{2} + \sqrt{2}(\frac{1}{2} + \eta)$, it follows that $\delta \leqslant \sqrt{2}(\frac{1}{2} + \eta) \leqslant \sqrt{2}$, whence

$$E \leqslant G(\sqrt{2}, T_0) \leqslant \frac{4.4}{T_0},$$

for $T \geqslant T_0$.

3.2. Applying Jensen's Formula

In Lemma 1, take $a = 1+\eta$, $f(z)$ as in (2.4), and $R = r(\frac{1}{2}+\eta)$, where $r > 1$. Suppose that there are n zeroes of $\Re h(\sigma + iT)^N$ for $\sigma \in [\frac{1}{2}, \sigma_1]$, where $\sigma_1 = \frac{1}{2} + \sqrt{2}(\eta + \frac{1}{2})$. Initially, to take advantage of Backlund's trick, one needs $1 + \eta - r(\frac{1}{2} + \eta) \leqslant 1 - \sigma_1$, so that all of the zeroes are included in the contour. The argument in [16, §4.1] shows that one can use any $r > 1$. The following results are simplified greatly if one imposes an upper bound on r. Indeed, to use (4.10) requires $r(\frac{1}{2} + \eta) \leqslant \frac{3}{2} + \eta \leqslant 2$.

To apply Jensen's Formula it is necessary to show that $f(1 + \eta)$ is non-zero: this is easy to do upon invoking an observation due to Rosser [12]. Write $h(1+\eta+iT) = Ke^{i\psi}$, where $K > 0$. Choose a sequence of N's tending to infinity for which $N\psi$ tends to zero modulo 2π. Thus

$$\frac{f(1+\eta)}{|h(1+\eta+iT)|^N} \to 1. \tag{3.5}$$

It follows from (3.1) and (3.4) that

$$n \leqslant \frac{1}{4\pi \log r} J - \frac{1}{2\log r} \log\bigl|f(1+\eta)\bigr| + \frac{1}{2} + \frac{NE}{2\pi},$$

where

$$J = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \log\left|f\left(1+\eta+r\left(\frac{1}{2}+\eta\right)e^{i\phi}\right)\right| d\phi.$$

First one may bound $\log|f(1+\eta)|$ using (3.5) and the trivial bound $|\zeta(s)| \geqslant \frac{\zeta(2\sigma)}{\zeta(\sigma)}$. Thus

$$\log\bigl|f(1+\eta)\bigr| = N\log\bigl|h(1+\eta+iT)\bigr| + o(1) \geqslant N\left(\log T + \log\frac{\zeta(2+2\eta)}{\zeta(1+\eta)}\right) + o(1).$$

4. Estimating the integrals

To estimate J one seeks good bounds for $\zeta(s)$ over the range

$$1+\eta - r\left(\frac{1}{2}+\eta\right) \leqslant \sigma \leqslant 1+\eta+r\left(\frac{1}{2}+\eta\right).$$

One may estimate $\zeta(s)$ trivially when $\sigma > 1$; applying the functional equation gives an estimate when $\sigma < 0$. One may also estimate $\zeta(s)$ when $\sigma = \frac{1}{2}$. Therefore by the Phragmén–Lindelöf principle one may obtain bounds for $\zeta(s)$ when $\sigma \in [1 + \eta - r(\frac{1}{2} + \eta), \frac{1}{2}]$, $\sigma \in [\frac{1}{2}, 1+\eta]$ and $\sigma \geqslant 1+\eta$. This was used in [15].

The key idea in this paper is to divide J further by considering the bound on $\zeta(1+it)$ as given in [17]. An application of the functional equation gives a bound on $\zeta(it)$. Therefore one may obtain bounds for $\zeta(s)$ when $\sigma \in [1+\eta - r(\frac{1}{2}+\eta), 0]$, $\sigma \in [0, \frac{1}{2}]$, $\sigma \in [\frac{1}{2}, 1]$, $\sigma \in [1, 1+\eta]$ and $\sigma \geqslant 1+\eta$. To this end, write

$$R_0 = \left\{s\colon\ 1+\eta \leqslant \sigma \leqslant 1+\eta + r\left(\frac{1}{2}+\eta\right)\right\} = \left\{\phi\colon\ -\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2}\right\},$$
$$R_1 = \{s\colon\ 1 \leqslant \sigma \leqslant 1+\eta \text{ and } t \geqslant T\} = \left\{\phi\colon\ \frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2} + \phi_1\right\},$$
$$R_2 = \left\{s\colon\ \frac{1}{2} \leqslant \sigma \leqslant 1 \text{ and } t \geqslant T\right\} = \left\{\phi\colon\ \frac{\pi}{2} + \phi_1 \leqslant \phi \leqslant \frac{\pi}{2} + \phi_2\right\},$$
$$R_3 = \left\{s\colon\ 0 \leqslant \sigma \leqslant \frac{1}{2} \text{ and } t \geqslant T\right\} = \left\{\phi\colon\ \frac{\pi}{2} + \phi_2 \leqslant \phi \leqslant \frac{\pi}{2} + \phi_3\right\},$$
$$R_4 = \left\{s\colon\ 1+\eta - r\left(\frac{1}{2}+\eta\right) \leqslant \sigma \leqslant 0 \text{ and } t \geqslant T\right\} = \left\{\phi\colon\ \frac{\pi}{2} + \phi_3 \leqslant \phi \leqslant \pi\right\},$$

where

$$\phi_1 = \sin^{-1}\frac{\eta}{r(\frac{1}{2}+\eta)}, \qquad \phi_2 = \sin^{-1}\frac{1}{r}, \qquad \phi_3 = \sin^{-1}\frac{1+\eta}{r(\frac{1}{2}+\eta)}.$$

By the reflection principle, one may then write

$$J = \int\limits_{R_0} + 2\int\limits_{R_1} + 2\int\limits_{R_2} + 2\int\limits_{R_3} + 2\int\limits_{R_4}.$$

In estimating each integral some small error terms labelled $\epsilon_0, \dots, \epsilon_4$ are encountered. Since these are all $O(T_0^{-1})$, they have been estimated with a great deal of alacrity. It is neither essential nor insightful to strive for the smallest bounds on these terms.

4.1. Convexity bounds

To estimate $\zeta(s)$ on R_0 one may use the trivial estimate $|\zeta(s)| \leqslant \zeta(\sigma)$. On $R_1, \dots, R_4$ one can use the following version of the Phragmén–Lindelöf principle.

Lemma 3. *Let a, b, Q and k be real numbers, such that $Q+a > 1$, and let $f(s)$ be regular analytic in the strip $a \leqslant \sigma \leqslant b$ and satisfy the growth condition*

$$\left|f(s)\right| < C\exp\left\{e^{k|t|}\right\},$$

for a certain $C > 0$ and for $0 < k < \pi/(b-a)$. Also assume that

$$\left|f(s)\right| \leqslant \begin{cases} A|Q+s|^{\alpha_1}(\log|Q+s|)^{\alpha_2} & \text{for } \Re(s) = a, \\ B|Q+s|^{\beta_1}(\log|Q+s|)^{\beta_2} & \text{for } \Re(s) = b, \end{cases} \tag{4.1}$$

where $\alpha_1 \geqslant \beta_1$ and where $\alpha_1, \alpha_2, \beta_1, \beta_2 \geqslant 0$. Then throughout the strip $a \leqslant \sigma \leqslant b$ the following holds

$$\left|f(s)\right| \leqslant \left\{A|Q+s|^{\alpha_1}\left|\log(Q+s)\right|^{\alpha_2}\right\}^{\frac{b-\sigma}{b-a}}\left\{B|Q+s|^{\beta_1}\left|\log(Q+s)\right|^{\beta_2}\right\}^{\frac{\sigma-a}{b-a}}. \tag{4.2}$$

Proof. This extends a result due to Rademacher [10, pp. 66–67] where $\alpha_2 = \beta_2 = 0$. Form the function

$$F(s) = f(s)\phi(s;Q)E^{-1}e^{-vs}\left\{\log(Q+s)\right\}^{\frac{\alpha_2(\sigma-b)+\beta_2(a-\sigma)}{b-a}},$$

where $\phi(s;Q)$ is the function of [10, Theorem 1], and E and ν are determined by $A = Ee^{\nu a}$ and[3] $B = Ee^{\nu b}$. Since $Q + a > 1$, the function $F(s)$ is holomorphic in the strip $a \leqslant \sigma \leqslant b$. The proof now proceeds as in [10]. □

Lemma 3 will be applied to $R_1, \ldots, R_4$ where it will be convenient to write $|Q+s|$, which appears in (4.1) and (4.2), in terms of T. If $Q = Q_0 \leqslant 1000$ and $T \geqslant T_0 \geqslant 10^6$ one may write

$$\left|\log|Q_0+s| - \log T\right| \leqslant \frac{6}{T_0} \leqslant 10^{-5}. \tag{4.3}$$

If, in addition, $Q_0 \geqslant 1$, then $|\arg(Q_0+s)| \leqslant \frac{\pi}{2}$ on $R_1, \ldots, R_4$. Using this, (4.3), and the identity

$$|\log z| = \left|\log|z|\right|\left\{1 + \left(\frac{\arg z}{\log|z|}\right)^2\right\}^{1/2},$$

one deduces that

$$\left|\log(Q_0+s)\right| \leqslant 1.007\log T, \qquad \log\left|\log(Q_0+s)\right| \leqslant \log\log T + 0.007. \tag{4.4}$$

4.2. Bounds on R_0

On R_0

$$\left|h(s)\right| \leqslant \left|\eta + r\left(\frac{1}{2}+\eta\right)e^{i\phi} \pm iT\right|\zeta\left(1+\eta+r\left(\frac{1}{2}+\eta\right)\cos\phi\right),$$

whence

$$\frac{R_0}{N} \leqslant \pi(\log T + \epsilon_0) + \int_{-\pi/2}^{\pi/2} \log\zeta\left(1+\eta+r\left(\frac{1}{2}+\eta\right)\cos\phi\right)d\phi,$$

[3] Note that in [10, (3.7)] there is a typo: $B = e^{\nu b}$ should read $B = Ee^{\nu b}$.

where

$$\epsilon_0 = \frac{r(\frac{1}{2}+\eta)}{T_0} + \frac{\{\eta + r(\frac{1}{2}+\eta)\}^2}{2T_0^2} \leqslant \frac{2}{T_0} + \frac{25}{8T_0^2} \leqslant \frac{3}{T_0}.$$

4.3. Bounds on R_1

On $\sigma = 1+\eta$ bound $\zeta(\sigma)$ trivially, whence, for any $Q_0 \geqslant 0$

$$\left|h(1+\eta+it)\right| \leqslant \left|Q_0 + (1+\eta+it)\right|\zeta(1+\eta). \tag{4.5}$$

Since $\zeta(1+it) \ll \log t$, on $\sigma = 1$ one may use an estimate of the form

$$\left|\zeta(1+it)\right| \leqslant \theta_1 \log t, \quad \text{for } t \geqslant t_0. \tag{4.6}$$

Providing that Q_0 is large enough it follows that

$$\left|h(1+it)\right| \leqslant \left|Q_0 + (1+it)\right| \log\left|Q_0 + (1+it)\right|, \tag{4.7}$$

for all t. It follows from Lemma 3, (4.3), (4.4), (4.5) and (4.7) that

$$\frac{R_1}{N} \leqslant \left(\log T + 10^{-5}\right)\phi_1 + \log\zeta(1+\eta)\left[\phi_1 - \frac{r(\frac{1}{2}+\eta)(1-\cos\phi_1)}{\eta}\right] + (\log\log T + 0.007 + \log\theta_1)\left[\frac{r(\frac{1}{2}+\eta)(1-\cos\phi_1)}{\eta}\right].$$

4.4. Bounds on R_2

Suppose that one is equipped with a bound

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leqslant k_1 t^{k_2} (\log t)^{k_3}, \quad \text{for } t \geqslant t_0,$$

in which $0 \leqslant k_3 \leqslant 10$, say. This upper bound on k_3 is imposed merely to simplify the resulting error term. The approximate functional equation for $\zeta(s)$, see, e.g. [14, (5.1.8)] shows that $k_2 \leqslant \frac{1}{4}$.

It follows that

$$\left|h\left(\frac{1}{2}+it\right)\right| \leqslant k_1 \left|Q_0 + \left(\frac{1}{2}+it\right)\right|^{k_2+1} \left(\log\left|Q_0 + \left(\frac{1}{2}+it\right)\right|\right)^{k_3}, \quad \text{for } t \geqslant t_0, \tag{4.8}$$

for any $Q_0 \geqslant 0$. It is always possible to choose Q_0 large enough so that (4.8) holds for all t. It follows from Lemma 3, (4.3), (4.4), (4.7) and (4.8) that

$$\begin{aligned}\frac{R_2}{N} \leqslant{}& \left(\log T + 10^{-5}\right)\left[2k_2 r\left(\frac{1}{2}+\eta\right)(\cos\phi_1 - \cos\phi_2) + (\phi_2-\phi_1)(1-2k_2\eta)\right] \\ &+ 2\log k_1\left[r\left(\frac{1}{2}+\eta\right)(\cos\phi_1-\cos\phi_2) - \eta(\phi_2-\phi_1)\right] \\ &+ 2\left(\frac{1}{2}+\eta\right)\log\theta_1\left\{\phi_2-\phi_1+r(\cos\phi_2-\cos\phi_1)\right\} \\ &+ 2(\log\log T + 0.007)\left[r\left(\frac{1}{2}+\eta\right)(1-k_3)(\cos\phi_2-\cos\phi_1)\right. \\ &\left.+ (\phi_2-\phi_1)\left(\frac{1}{2}+\eta-k_3\eta\right)\right].\end{aligned}$$

4.5. Bounds on R_3

The functional equation

$$\zeta(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\zeta(1-s) \tag{4.9}$$

(see, e.g., [14, Ch. II]), the estimate

$$\left|\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\right| \leqslant \left(\frac{|1+s|}{2}\right)^{\frac{1}{2}-\sigma}, \tag{4.10}$$

for $-\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}$ (see [9, p. 197]), and (4.6) show that

$$\left|h(it)\right| \leqslant (2\pi)^{-\frac{1}{2}}|Q_0+it|^{\frac{3}{2}}\log|Q_0+it|, \tag{4.11}$$

for all t, if Q_0 is sufficiently large. It follows from Lemma 3, (4.3), (4.4), (4.8) and (4.11) that

$$\begin{aligned}\frac{R_3}{N} \leqslant{}& \left(\log T + 10^{-5}\right)\left[r\left(\frac{1}{2}+\eta\right)(1-2k_2)(\cos\phi_2-\cos\phi_3)\right. \\ &\left.+ (\phi_3-\phi_2)\left\{2k_2(1+\eta)+\frac{1}{2}-\eta\right\}\right] \\ &+ 2(\log\log T + 0.007)\left[(k_3-1)r\left(\frac{1}{2}+\eta\right)(\cos\phi_3-\cos\phi_2)\right. \\ &\left.+ (\phi_3-\phi_2)\left\{k_3(1+\eta)-\left(\frac{1}{2}+\eta\right)\right\}\right] \\ &+ (1+2\eta)\log\left\{\theta_1(2\pi)^{-1/2}\right\}\left[(\phi_2-\phi_3)+r(\cos\phi_2-\cos\phi_3)\right] \\ &+ 2\log k_1\left[(1+\eta)(\phi_3-\phi_2) - r\left(\frac{1}{2}+\eta\right)(\cos\phi_2-\cos\phi_3)\right].\end{aligned}$$

4.6. Bounds on R_4

Eqs. (4.9), (4.10), and the trivial estimate for $\zeta(s)$ show that

$$\left|h(q+it)\right| \leqslant (2\pi)^{q-\frac{1}{2}}\left|Q_0+(q+it)\right|^{\frac{3}{2}-q}\zeta(1-q), \tag{4.12}$$

for all t and for all $Q_0 \geqslant 2$, where $q = 1+\eta - r(\frac{1}{2}+\eta)$.

Lemma 3, (4.3), (4.4), (4.11) and (4.12) show that

$$\begin{aligned}\frac{R_4}{N} \leqslant{}& \left(\log T + 10^{-5}\right)\left[\left(\frac{1}{2}-\eta\right)\left(\frac{\pi}{2}-\phi_3\right) + r\left(\frac{1}{2}+\eta\right)\cos\phi_3\right] \\ &+ \left(\log\log T + \log\theta_1 + 0.007\right)\left[\frac{r(\frac{1}{2}+\eta)(\frac{\pi}{2}-\phi_3-\cos\phi_3)}{r(\frac{1}{2}+\eta)-(1+\eta)}\right] \\ &+ \log 2\pi\left[\left(\frac{1}{2}+\eta\right)\left(\frac{\pi}{2}-\phi_3\right) - r\left(\frac{1}{2}+\eta\right)\cos\phi_3\right] \\ &+ \log\zeta\left(r\left(\frac{1}{2}+\eta\right)-\eta\right)\left[\frac{r(\frac{1}{2}+\eta)-(\frac{\pi}{2}-\phi_3)(1+\eta)}{r(\frac{1}{2}+\eta)-(1+\eta)}\right].\end{aligned}$$

One therefore has all of the results needed to bound $|S(t)|$. This produces an expression the inelegance of which prohibits its being inserted in this paper. The next section will provide some specific information.

5. Specific values of k_1, k_2, k_3 and θ_1

Cheng and Graham [5, Thm. 3] proved that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leqslant 1.457t^{1/6}\log t + 40.995t^{1/6} + 1.863\log t + 123.125, \tag{5.1}$$

and that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leqslant 6t^{1/4} + 41.129, \tag{5.2}$$

where both (5.1) and (5.2) are valid for $t \geqslant e$. They combine these results with a computational check to show that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leqslant 3t^{1/6}\log t,$$

for $t \geqslant e$. This can be improved by combining (5.1) not with (5.2) but with the estimate

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leqslant \frac{4}{(2\pi)^{\frac{1}{4}}}t^{1/4} \tag{5.3}$$

(see, e.g., [7, Lemma 2]), which is valid for $t \geqslant 1$. This shows that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leqslant 2.38t^{1/6}\log t, \tag{5.4}$$

for $t \geqslant e$. Since (5.4) improves on (5.3) only when $t \geqslant 4\times 10^{19}$, one does not expect that (5.4) will be useful for any moderate value of t. Indeed, this is borne out in calculations. Henceforth the bound in (5.3) is used, whence $k_1 = 4/(2\pi)^{1/4}$, $k_2 = \frac{1}{4}$ and $k_3 = 0$.

As for θ_1, Backlund [2, (53)] proved that $|\zeta(1+it)| \leqslant \log t$ for $t \geqslant 50$. In [17] the author proved that $|\zeta(1+it)| \leqslant \frac{3}{4}\log t$ for $t \geqslant 3$. Henceforth one may take $\theta_1 = \frac{3}{4}$.

One can now take $Q_0 = 2$, whence all of (4.5), (4.7), (4.8), (4.11), (4.12) are satisfied.

6. Conclusion

Combining the results of Sections 3–5 shows that, when $T \geqslant T_0 \geqslant 1$,

$$\left|S(T)\right| \leqslant a\log T + b\log\log T + c, \tag{6.1}$$

where

$$a = \frac{\phi_1\eta + \phi_3(1+\eta) - \pi(\frac{1}{2}+\eta) + r(\frac{1}{2}+\eta)(\cos\phi_1 + \cos\phi_3)}{4\pi\log r},$$

$$b = \frac{\frac{1}{2}+\eta}{2\pi\log r}\Biggl\{-2\phi_1 - 2\phi_3 + 4\phi_2 + r\biggl(\frac{1-\cos\phi_1}{\eta} + \frac{\frac{\pi}{2}-\phi_3-\cos\phi_3}{r(\frac{1}{2}+\eta)-(1+\eta)}$$
$$- 2\cos\phi_1 - 2\cos\phi_3 + 4\cos\phi_2\biggr)\Biggr\},$$

$$c = \frac{\log\frac{\zeta(1+\eta)}{\zeta(2+2\eta)}}{2\log r} + \frac{1}{\pi}\log\zeta\left(\frac{1}{2}+\sqrt{2}\left(\eta+\frac{1}{2}\right)\right) + \frac{\int_{-\pi/2}^{\pi/2}\log\zeta(1+\eta+r(\frac{1}{2}+\eta)\cos\phi)\,d\phi}{4\pi\log r}$$
$$+ \Biggl\{\log\zeta(1+\eta)\left[\phi_1 + \frac{r(\frac{1}{2}+\eta)(\cos\phi_1 - 1)}{\eta}\right] + \left(\frac{1}{2}+\eta\right)\left(\frac{\pi}{2}-\phi_2-r\cos\phi_2\right)\log 2\pi$$
$$- 2\log k_1\left[r\left(\frac{1}{2}+\eta\right)(2\cos\phi_2 - \cos\phi_1 - \cos\phi_3) + \phi_2 - \phi_3 + \eta(2\phi_2 - \phi_1 - \phi_3)\right]$$
$$+ \left[\frac{(1+\eta)(\frac{\pi}{2}-\phi_3) - r(\frac{1}{2}+\eta)\cos\phi_3}{1+\eta-r(\frac{1}{2}+\eta)}\right]\log\zeta\left[r\left(\frac{1}{2}+\eta\right)-\eta\right]$$
$$+ \log(3/4)\left(\frac{1}{2}+\eta\right)\left[\frac{r(1-\cos\phi_1)}{\eta} + \frac{r(\frac{\pi}{2}-\phi_3-\cos\phi_3)}{r(\frac{1}{2}+\eta)-(1+\eta)}\right.$$
$$\left.+ 4(\phi_2 + r\cos\phi_2) - 2(\phi_1 + r\cos\phi_1) - 2(\phi_3 + r\cos\phi_3)\right]\Biggr\}/(2\pi\log r) + 0.003.$$

Taking $\eta = 0.077$ and $r = 2.051$ proves Theorem 1 for $T \geqslant 5.45\times 10^8$. When $T < 5.45\times 10^8$, Theorem 1 follows from (1.2).

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Appendix A. Supplementary material

The online version of this article contains additional supplementary material.
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