Methods of Mathematical Physics
Script of the Lecture

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Taking down:
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Contents

1 Two Classical Bodies

1.1 Kepler-Problem

o is a fixed center and $\mu > 0$ a positive constant. Let $q = x - o$ be the position of a particle (body) of mass $m > 0$ at $x \in \mathbb{R}^3$ relative to the fixed center o.

Definition 1.1: The *Newtonian gravitational field* determined by o and μ acts on a particle of mass $m > 0$ at $x \in \mathbb{R}^3$ by

$$-m\,\frac{\mu}{r^2}\,r^{-1}q$$

where $r = \|q\|_2$. Observe that $-r^{-1}q$ is a unit vector directed from x to o

$$\|-r^{-1}q\| = 1$$

→ *Universal law of gravitation.* □

Newton's Second Law: Newton's Second Law implies

$$m\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,q \;=\; -m\,\frac{\mu}{r^2}\,r^{-1}q \qquad q = (q_1, q_2, q_3) \in \mathbb{R}^3 \setminus \{o\}$$

For the moment, lets assume that there is a solution.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,q \;=\; -\frac{\mu}{r^2}\,r^{-1}q$$

We can write $\frac{\mathrm{d}}{\mathrm{d}t}\,q \;=\; v$

$$\Rightarrow \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\,p \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\,v \;=\; -\frac{\mu}{r^2}\,r^{-1}q \;=\; -\frac{\mu}{r^3}\,q \qquad (1.1)$$

Definition 1.2: $\forall a, b \in \mathbb{R}^3$ set

$$[a, b] \;:=\; \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

□

Proposition 1.1: The bracket $[\cdot,\cdot]$ defines a bilinear map from $\mathbb{R}^3 \oplus \mathbb{R}^3$ to $\mathbb{R}^3$ satisfying

1) $[a, b] = -[b, a]$
2) Jacobi identity

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] \;=\; 0$$

$\forall a, b, c \in \mathbb{R}^3$.

→ Look up the definition of Lie-Algebra. □

Definition 1.3: $\forall q, p \in \mathbb{R}^3$, the *linear momentum* is defined by

$$p \;:=\; mv$$

the *angular momentum* by

$$L \;:=\; [q, p]$$

the *Lenz vector* by

$$F \;:=\; \frac{1}{m}\,[p, L] - m\mu r^{-1}q$$

the *kinetic energy* by

$$T := \frac{1}{2m}\|p\|^2 = \frac{1}{2}m\|v\|^2$$

the *potential energy* by

$$U := -m\mu r^{-1} = -m\mu\|q\|^{-1}$$

and the *total energy* by

$$H := T + U = \frac{1}{2m}\|p\|^2 - m\mu r^{-1}$$

□

Proposition 1.2: Suppose $(q(t), p(t))$, $a \leq t \leq b$ is a solution of (1.1) (that means the Kepler problem). Then $\forall a \leq t \leq b$

$$\frac{\mathrm{d}}{\mathrm{d}t}L = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}F = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}H = 0$$

where

$$L(t) = [q(t), p(t)]$$

In other words, L, F, H are constant along the trajectories of the Kepler problem.

□

Proposition 1.3: $\forall a, b, c \in \mathbb{R}^3$

$$\begin{aligned}
[a, a] &= 0 \\
\langle [a, b], c\rangle &= \langle b, -[a, c]\rangle \\
\langle a, [a, b]\rangle &= 0 \\
\|[a, b]\|^2 &= \|a\|^2\|b\|^2 - \langle a, b\rangle^2 = \|a\|^2\|b\|^2\sin^2\theta \\
[[a, b], c] &= \langle a, c\rangle\, b - \langle b, c\rangle\, a
\end{aligned}$$

where θ is the angle between a and b. □

Proposition 1.4: $\forall p, q \in \mathbb{R}^3$, $L \neq 0$

$$\begin{aligned}
\langle q, L\rangle &= 0 \\
\langle p, L\rangle &= 0 \\
\langle F, L\rangle &= 0
\end{aligned}$$

q lies in the plane perpindicular to L. □

Proof:

$$\begin{aligned}
\langle q, L\rangle &= \langle q, [q, p]\rangle = 0 \\
\langle p, L\rangle &= \langle p, [q, p]\rangle = -\langle p, [p, q]\rangle = 0 \\
\langle F, L\rangle &= \frac{1}{m}\langle [p, L], L\rangle - \underbrace{m\mu\frac{1}{r}\langle q, L\rangle}_{0} = \frac{1}{m}\langle L, [p, L]\rangle = 0
\end{aligned}$$

(QED)

Proposition 1.5: Suppose $q(t)$, $a \leq t \leq b$, is any smooth curve in $\mathbb{R}^3$. If $q(t) \neq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^{-1}q\right) = -r^{-3}\frac{1}{m}\,[q, L]$$

where $r = r(t) = \|q(t)\|$. □

Proof:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^{-1}q\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\,r^{-1}\right)q + r^{-1}\,\frac{\mathrm{d}}{\mathrm{d}t}\,q \\ &= -r^{-2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,r\right)q + r^{-1}\,\frac{\mathrm{d}}{\mathrm{d}t}\,q \\ &= r^{-2}\left(rv - q\,\frac{\mathrm{d}}{\mathrm{d}t}\,r\right)\end{aligned}$$

where $v = \frac{\mathrm{d}}{\mathrm{d}t}\,q$.

$$\begin{aligned}2r\,\frac{\mathrm{d}}{\mathrm{d}t}\,r &= \frac{\mathrm{d}}{\mathrm{d}t}\,r^2 \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle q,q\right\rangle \\ &= \left\langle\frac{\mathrm{d}}{\mathrm{d}t}\,q,q\right\rangle + \left\langle q,\frac{\mathrm{d}}{\mathrm{d}t}\,q\right\rangle \\ &= \left\langle v,q\right\rangle + \left\langle q,v\right\rangle \\ &= 2\left\langle q,v\right\rangle\end{aligned}$$

From these equations follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^{-1}q\right) &= r^{-3}\left(\left\langle q,q\right\rangle v - \left\langle q,v\right\rangle q\right) \\ &= -r^{-3}\left(\left\langle q,v\right\rangle q - \left\langle q,q\right\rangle v\right) \\ &= -r^{-3}\left[q,\left[q,v\right]\right] \\ &= -\frac{1}{m}\,r^{-3}\left[q,\left[q,p\right]\right]\end{aligned}$$

where we use

$$\left[a,\left[a,b\right]\right] = \left\langle a,b\right\rangle a - \left\langle a,a\right\rangle b$$

(QED)

Remark: Let $a(t), b(t) \in \mathbb{R}^n$ be functions of t.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a(t),b(t)\right\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} a_i(t)b_i(t) \\ &= \sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\left(a_i(t)b_i(t)\right) \\ &= \sum_{i=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,a_i(t)\right)b_i(t) + a_i(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,b_i(t) \\ &= \sum_{i=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,a_i(t)\right)b_i(t) + \sum_{i=1}^{n} a_i(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,b_i(t) \\ &= \left\langle\frac{\mathrm{d}a}{\mathrm{d}t},b\right\rangle + \left\langle a,\frac{\mathrm{d}b}{\mathrm{d}t}\right\rangle\end{aligned}$$

□

Proof 1.2:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,L &= \frac{\mathrm{d}}{\mathrm{d}t}\left[q,p\right] \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\,q,p\right] + \left[q,\frac{\mathrm{d}}{\mathrm{d}t}\,p\right] \\ &= \left[v,p\right] + \left[q,-m\mu r^{-3}q\right] \\ &= m\left[v,v\right] + \left(-m\mu r^{-3}\right)\left[q,q\right] \\ &= 0\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} F &= \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}[q,L] - m\mu \frac{\mathrm{d}}{\mathrm{d}t}\left(r^{-1}q\right) \\
&= \frac{1}{m}\left[\frac{\mathrm{d}}{\mathrm{d}t}p, L\right] + \frac{1}{m}\left[p, \frac{\mathrm{d}}{\mathrm{d}t}L\right] - m\mu\left(-r^{-3}\,\frac{1}{m}\,[q,L]\right) \\
&= \frac{1}{m}\left[-m\mu r^{-3}q, L\right] - m\mu\left(-r^{-3}\,\frac{1}{m}\right)[q,L] \\
&= -\mu r^{-3}\,[q,L] + \mu r^{-3}\,[q,L] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2m}\,\|p\|^2 - m\mu r^{-1}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2m}\,(p,p) - m\mu r^{-1}\right) \\
&= \frac{1}{2m}\left(\frac{\mathrm{d}}{\mathrm{d}t}p, p\right) + \frac{1}{2m}\left(p, \frac{\mathrm{d}}{\mathrm{d}t}p\right) - \frac{\mathrm{d}}{\mathrm{d}t}\,m\mu r^{-1} \\
&= \frac{1}{m}\left(\frac{\mathrm{d}}{\mathrm{d}t}p, p\right) + m\mu r^{-2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,r \\
&= \frac{1}{m}\left(-m\mu r^{-3}q, p\right) + m\mu r^{-2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,r \\
&= -\mu r^{-3}\,(q, mv) + m\mu r^{-2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,r \\
&= -m\mu r^{-3}\left(q, \frac{\mathrm{d}}{\mathrm{d}t}q\right) + m\mu r^{-2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,r \\
&= -m\mu r^{-3}\,\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,(q,q) + m\mu r^{-2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,r \\
&= -m\mu r^{-3}\,\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,r^2 + m\mu r^{-2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,r \\
&= -m\mu r^{-2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,r + m\mu r^{-2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,r \\
&= 0
\end{aligned}$$

(QED)

Proposition 1.6: $\forall p, q \in \mathbb{R}^3$

$$\begin{aligned}
\|F\|^2 &= \frac{2}{m}\,\|L\|^2 H + m^2\mu^2 \\
\langle F, q\rangle &= \frac{1}{m}\,\|L\|^2 - m\mu r
\end{aligned}$$

□

Proof:

$$\begin{aligned}
\|F\|^2 &= \langle F, F\rangle \\
&= \left\langle \frac{1}{m}\,[p,L] - m\mu r^{-1}q, \frac{1}{m}\,[p,L] - m\mu r^{-1}q\right\rangle \\
&= \frac{1}{m^2}\,\|[p,L]\|^2 - 2\mu r^{-1}\,\langle [p,L]\,, q\rangle + m^2\mu^2 r^{-2}\|q\|^2 \\
&= \frac{1}{m^2}\left(\|p\|^2\|L\|^2 - \langle p, L\rangle^2\right) + 2\mu r^{-1}\,\langle L, [p,q]\rangle + m^2\mu^2 \\
&= \frac{1}{m^2}\,\|p\|^2\|L\|^2 - 2\mu r^{-1}\,\langle L, L\rangle + m^2\mu^2 \\
&= \frac{2}{m}\left(\frac{1}{2m}\,\|p\|^2 - m\mu r^{-1}\right)\|L\|^2 + m^2\mu^2
\end{aligned}$$

$$\begin{aligned}\langle F, q\rangle &= \left\langle \frac{1}{m}[p, L] - m\mu r^{-1}q, q\right\rangle \\ &= \frac{1}{m}\langle [p, L], q\rangle - m\mu r^{-1}\langle q, q\rangle \\ &= \frac{1}{m}\langle [q, p], L\rangle\end{aligned}$$

(QED)

Remark: The "TAO" of Schwarz's Inequality

$$\begin{aligned}\|[a, b]\|^2 &= \|a\|^2\|b\|^2 - \langle a, b\rangle^2 \\ \Rightarrow \langle a, b\rangle^2 &\le \|a\|^2\|b\|^2\end{aligned}$$

We prove this by simply calculating

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \sum_{i<j}(a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

□

Proposition 1.7: Suppose $L \neq 0$ and $F \neq 0$. Let f be the positively oriented angle between F and q in the plane perpendicular to L. Then

$$r = a\,\frac{1-e^2}{1+e\cos f}$$

where

$$r = \|q\| \qquad e := \frac{1}{m\mu}\|F\| \qquad a := \frac{\mu m}{2(-H)}$$

□

Proof:

$$\begin{aligned}\langle F, q\rangle &= \|F\|\,\|q\|\cos f \\ &= \|F\| r\cos f \\ &= \frac{1}{m}\|L\|^2 - m\mu r\end{aligned}$$

We now get

$$\begin{aligned}\|F\| r\cos f + m\mu r &= \frac{1}{m}\|L\|^2 \\ \Rightarrow (m\mu + \|F\|\cos f)r &= \frac{1}{m}\|L\|^2 \qquad (1.2)\end{aligned}$$

By using the equation

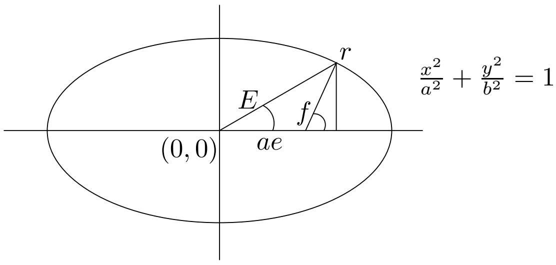
$$\begin{aligned}1 - e^2 &= 1 - (m\mu)^{-2}\|F\|^2 \\ &= 1 - (m\mu)^{-2}\left(\frac{2}{m}\|L\|^2 H + m^2\mu^2\right) \\ &= 1 - \left(\frac{2}{m^3\mu^2}\|L\|^2 H + 1\right) \\ &= \frac{2}{m^2\mu}(m\mu)^{-1}\|L\|^2(-H) \qquad (1.3)\end{aligned}$$

we can then derive

$$
\begin{aligned}
r &\overset{(1.2)}{=} \frac{\frac{1}{m}\|L\|^2}{m\mu + \|F\|\cos f} \\
&= \frac{1}{m}\frac{(m\mu)^{-1}\|L\|^2}{1+(m\mu)^{-1}\|F\|\cos f} \\
&= \frac{1}{m}\frac{(m\mu)^{-1}\|L\|^2}{1+e\cos f} \\
&= \frac{m\mu}{2(-H)}\frac{1-e^2}{1+e\cos f}
\end{aligned}
$$

(QED)

Definition 1.4: Looking at an ellipse



we call f *true anomaly* and E *eccentric anomaly.* □

Proposition 1.8: The two parametrisations

$$
\begin{aligned}
(x, y) &= (a\cos E, b\sin E) \\
(x, y) &= (r\cos f + ae, r\sin f)
\end{aligned}
$$

where

$$
r = a\frac{1-e^2}{1+e\cos f} \tag{1.4}
$$

both solve the ellipse equation

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
$$

□

Proof: This is obvious for the first parametrisation

$$
\frac{(a\cos E)^2}{a^2} + \frac{(b\sin E)^2}{b^2} = \cos^2 E + \sin^2 E = 1
$$

For the other one, we have to do some calculations: In the coordinate system with axes along F, $-[F, L]$ and L, we have $q = (q_1, q_2, 0)$ and

$$
r = \|q\| = \sqrt{q_1^2 + q_2^2 + 0^2} = \sqrt{q_1^2 + q_2^2}
$$

where

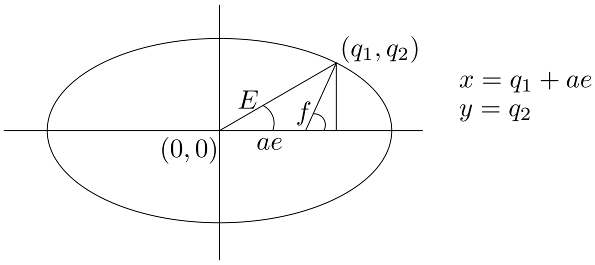
$$
q_1 = r\cos f \qquad q_2 = r\sin f
$$

Now we take equation (1.4) and play a little around

$$\begin{aligned}
r(1 + e\cos f) &= a(1 - e^2) \\
r + er\cos f &= a(1 - e^2) \\
\sqrt{q_1^2 + q_2^2} + eq_1 &= a(1 - e^2) \\
\sqrt{q_1^2 + q_2^2} &= a(1 - e^2) - eq_1 \\
q_1^2 + q_2^2 &= a^2(1 - e^2)^2 - 2eq_1a(1 - e^2) + e^2q_1^2 \\
(1 - e^2)q_1^2 + q_2^2 + 2a(1 - e^2)eq_1 &= a^2(1 - e^2)^2 \\
(1 - e^2)(q_1 + ae)^2 + q_2^2 &= a^2e^2\left(1 - e^2\right) + a^2(1 - e^2)^2 \\
&= (1 - e^2)\left[a^2e^2 + a^2(1 - e^2)\right] \\
&= a^2(1 - e^2)
\end{aligned}$$

which (by setting $b^2 := a^2(1 - e^2)$) finally gives us the equation

$$\Rightarrow \frac{(q_1 + ae)^2}{a^2} + \frac{q_2^2}{b^2} = 1$$



(QED)

Lets consider the two parametrisations

$$\begin{aligned}
(x, y) &= (a\cos E, b\sin E) \\
(x', y') &= (r\cos f, r\sin f)
\end{aligned}$$

where as usual

$$r(f) = a\,\frac{1 - e^2}{1 + e\cos f}$$

The first parametrisation is relative to the center of the ellipse $(0, 0)$ and the second one relative to its focus $(ae, 0)$. We now want to find a transformation from one to the other.

Proposition 1.9: The relation between the two angles f and E is given by the Fourier serie

$$f = E + 2\sum_{j=1}^{\infty} \frac{\beta^j}{j}\,\sin(jE)$$

where $\beta = \tan\frac{1}{2}\varphi$ and $\varphi = \arcsin e$. □

Proposition 1.10: Set $\xi := e^{if}$ and $\eta := e^{iE}$. Then

$$\xi = \eta\,\frac{1 - \beta\eta^{-1}}{1 - \beta\eta}$$

is equivalent to

$$\tan\frac{1}{2}f = \frac{1 + \beta}{1 - \beta}\tan\frac{1}{2}E$$

□

Exercise 1.1: Prove Proposition 1.10. □

Proof 1.9: By looking at the picture above, we get

$$\begin{aligned} r(f)\cos f &= a\cos E - ae \\ r(f)\sin f &= b\sin E \end{aligned}$$

Claim 1:

$$r^2(f) = a^2(1-e\cos E)^2$$

Now we write

$$\begin{aligned} \cos f &= \cos^2\left(\frac{1}{2}f\right) - \sin^2\left(\frac{1}{2}f\right) \\ &= 2\cos^2\left(\frac{1}{2}f\right) - \left(\cos^2\left(\frac{1}{2}f\right) + \sin^2\left(\frac{1}{2}f\right)\right) \\ &= 2\cos^2\left(\frac{1}{2}f\right) - 1 \end{aligned}$$

It follows

$$\begin{aligned} r\left(2\cos^2\left(\frac{1}{2}f\right) - 1\right) &= r\cos f = a\left(\cos E - e\right) \\ \Rightarrow 2r\cos^2\left(\frac{1}{2}f\right) &= a(\cos E - e) + r \\ &= a(\cos E - e) + a(1 - e\cos E) \\ &= a(\cos E - e + 1 - e\cos E) \\ &= a(1-e)(1+\cos E) \end{aligned}$$

Putting this together, we get

$$r\cos^2\left(\frac{1}{2}f\right) = a(1-e)\cos^2\left(\frac{1}{2}E\right)$$

Claim 2:

$$r\sin^2\left(\frac{1}{2}f\right) = a(1+e)\sin^2\left(\frac{1}{2}E\right)$$

From this follows

$$\frac{\sin^2\frac{1}{2}f}{\cos^2\frac{1}{2}f} = \frac{(1+e)}{(1-e)}\frac{\sin^2\frac{1}{2}E}{\cos^2\frac{1}{2}E}$$

which we bring to the form

$$\tan\left(\frac{1}{2}f\right) = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\left(\frac{1}{2}E\right)$$

We know, $0 < e < 1$, so there is a unique φ satisfying

$$e = \sin\varphi \qquad 0 < \varphi < \frac{1}{2}\pi$$

Claim 3: By setting $\beta := \tan\frac{1}{2}\varphi$ follows

(i) $0 < \beta < 1$

(ii) $e = \frac{2\beta}{1+\beta^2}$, $\beta = \frac{1-(1-e^2)^{1/2}}{e}$

(iii) $\left(\frac{1+e}{1-e}\right)^{1/2} = \frac{1+\beta}{1-\beta}$

(iv) $e = 2\beta\left(1 + \mathcal{O}(\beta^2)\right)$

So we get rid of the square root

$$\tan\frac{1}{2}f \quad = \quad \frac{1+\beta}{1-\beta}\tan\frac{1}{2}E$$

Proposition 1.10 says, that this is equivalent to

$$\xi = \eta\,\frac{1-\beta\eta^{-1}}{1-\beta\eta}$$

with $\xi = e^{if}$ and $\eta = e^{iE}$. It follows

$$\begin{aligned} if &= \log\xi = \log\left(\eta\,\frac{1-\beta\eta^{-1}}{1-\beta\eta}\right) = \log\eta + \log\left(\frac{1-\beta\eta^{-1}}{1-\beta\eta}\right) \\ &= iE + \log\left(1-\beta\eta^{-1}\right) - \log\left(1-\beta\eta\right) \end{aligned}$$

We know

$$\log(1-x) \quad = \quad -\sum_{n=1}^{\infty}\frac{x^n}{n}$$

$|\beta| < 1$, $1 = |\eta| = |\eta^{-1}|$ and $|\beta\eta| = |\beta\eta^{-1}| < 1$

$$\begin{aligned} \Rightarrow if &= iE + (-1)\sum_{j=1}^{\infty}\frac{(\beta\eta^{-1})^j}{j} - (-1)\sum_{j=1}^{\infty}\frac{(\beta\eta)^j}{j} \\ \Rightarrow f &= E + i\sum_{j=1}^{\infty}\frac{(\beta\eta^{-1})^j}{j} - i\sum_{j=1}^{\infty}\frac{(\beta\eta)^j}{j} \\ &= E - i\sum_{j=1}^{\infty}\frac{\beta^j}{j}\left(\eta^j - \eta^{-j}\right) \end{aligned}$$

We have $\eta - \eta^{-j} = e^{ijE} - e^{-ijE} = 2i\sin jE$, which gives us the desired equation

$$f \quad = \quad E + 2\sum_{j=1}^{\infty}\frac{\beta^j}{j}\sin jE$$

The derivation of this is simply

$$\Rightarrow \frac{\mathrm{d}f}{\mathrm{d}E} \quad = \quad 1 + 2\sum_{j=1}^{\infty}\beta^j\cos jE$$

(QED)

Does this mean, that we could write every periodic function as sum of sines and cosines?

Exercise 1.2: Prove Claim 1,2 and 3 of Proof 1.10. □

1.2 The complex logarithm

Suppose $z \neq 0$, then write

$$z \quad = \quad re^{i\theta} \quad = \quad r(\cos\theta + i\sin\theta) \quad = \quad \sum_{j=0}^{\infty}\frac{(i\theta)^j}{j!}$$

How do I choose θ? The problem is, that $\forall k \in \mathbb{Z}$

$$e^{i\theta} \quad = \quad e^{i(\theta+2\pi k)}$$

so there are infinitely many possibilities.

Definition 1.5: The *principal branche* of $\log z$ is defined on the slit plane

$$\{z = re^{i\theta} \mid 0 < r < \infty, -\pi < \theta < \pi\} \quad := \quad \mathbb{C} \setminus (-\infty, 0]$$

by

$$\begin{aligned} \log z &:= \log(re^{i\theta}) = \log r + \log e^{i\theta} \\ &= \log r + i\theta \end{aligned}$$

□

From this follows

$$\log 1 = 0 \qquad \log x = \log_{(\text{“old”})} x \quad \forall x \in \mathbb{R}$$

so $\log z$ extends $\log x$.

Exercise 1.3: Suppose $z, z' \in \mathbb{C} \setminus (-\infty, 0]$ and $-\pi < \theta + \theta' < \pi$, then

$$\log(z + z') = \log z + \log z'$$

where $\log z$ is principal branche on $\mathbb{C} \setminus (-\infty, 0]$. □

Exercise 1.4: $0 < \beta < 1$, $\eta = e^{iE}$. Show that

$$1 - \beta\eta^{-1},\ 1 - \beta\eta,\ \frac{1 - \beta\eta^{-1}}{1 - \beta\eta},\ \eta\,\frac{1 - \beta\eta^{-1}}{1 - \beta\eta} \in \mathbb{C} \setminus (-\infty, 0]$$

□

Remark: If you can do both exercices, we get

$$\log\left[\eta\left(\frac{1 - \beta\eta^{-1}}{1 - \beta\eta}\right)\right] = \log\eta + \log\left(1 - \beta\eta^{-1}\right) - \log\left(1 - \beta\eta\right)$$

□

1.3 Where are we?

Theorem 1.1: Suppose $(q(t), p(t))$, $-\infty < t < \infty$ is an orbit (a solution) to the Kepler problem with $L, F \neq 0$ and total energy $H < 0$. If $q(0)$ is on the upper (lower) half of the ellipse

$$r = a\,\frac{1 - e^2}{1 + e\cos f}$$

in the plane perpendicular to L, where

$$e = (m\mu)^{-1}\,\|F\| \qquad a = \frac{\mu m}{2(-H)}$$

Let τ be the first time before (after) $t = 0$ that $r = a(1 - e)$

Set

$$u = u(t) := \left(\frac{2|H|}{m}\right)^{1/2} \int_{\tau}^{t} \frac{1}{r(\sigma)}\,\mathrm{d}\sigma$$

Then

$$\left(\frac{\mathrm{d}}{\mathrm{d}u}(r - a)\right)^2 + (r - a)^2 = a^2 e^2 \quad \Rightarrow \quad r(u) = a(1 - e\cos u)$$

Moreover, the eccentric anomaly is related to the *mean (avarage) anomaly* $M = n(t - \tau)$, where $n = \mu^{1/2} a^{-3/2}$, by *Kepler's equation*

$$n(t - \tau) = E - e\sin E$$

□

Corollary 1.1: Define the function φ by

$$\varphi(E,t) \;:=\; E - e\sin E - n(t-\tau) \;=\; 0$$

It follows

$$\Rightarrow \frac{\mathrm{d}\varphi}{\mathrm{d}E} \;=\; 1 - e\cos E \;\neq\; 0$$

and we can use the implicit function theorem. □

Remark: u is strictly monotone increasing and $u(\tau) = 0$, so there is an inverse function $t = t(u)$.
u is the eccentric anomaly E. □

Lemma 1.1:

$$H \;=\; \frac{m}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{1}{2m}\,r^{-2}\|L\|^2 - m\mu r^{-1}$$

where $r = \|q\|$. □

Proof:

$$\begin{aligned}
\|p\|^2 &= r^{-2}\|q\|^2\|p\|^2 \\
&= r^{-2}\left(\langle q,p\rangle^2 + \|q\|^2\|p\|^2 - \langle q,p\rangle^2\right) \\
&= r^{-2}\left(\langle q,p\rangle^2 + \|L\|^2\right) \\
&= r^{-2}\left(r^2 m^2\left(\frac{\mathrm{d}}{\mathrm{d}t}\,r\right)^2 + \|L\|^2\right)
\end{aligned}$$

using

$$\begin{aligned}
2r\,\frac{\mathrm{d}}{\mathrm{d}t}\,r &= \frac{\mathrm{d}}{\mathrm{d}t}\,r^2 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\,\|q\|^2 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\,\langle q,q\rangle \\
&= 2\left\langle q,\,\frac{\mathrm{d}}{\mathrm{d}t}\,q\right\rangle \;=\; \frac{2}{m}\,\langle q,p\rangle
\end{aligned}$$

Now, we substitute

$$\begin{aligned}
H &= \frac{1}{2m}\,\|p\|^2 - m\mu r^{-1} \\
&= \frac{m}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{1}{2m}\,r^{-2}\|L\|^2 - m\mu r^{-1}
\end{aligned}$$

(QED)

Proof of Theorem 1.1: If $H < 0$, then we know from Lemma 1.1

$$\begin{aligned}
\frac{m}{2}\left(r\,\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{1}{2m}\,\|L\|^2 &= -|H|r^2 + m\mu r \\
&= -|H|\left(r^2 - 2\,\frac{\mu m}{2|H|}\,r\right) \\
&= -|H|\left(r^2 - 2\,\frac{\mu m}{2|H|}\,r + \frac{\mu^2 m^2}{4|H|^2} - \frac{\mu^2 m^2}{4|H|^2}\right) \\
&= -|H|\left(\left(r - \frac{\mu m}{2|H|}\right)^2 - \frac{\mu^2 m^2}{4|H|^2}\right) \\
&= -|H|\left((r-a)^2 - a^2\right) \\
&= -|H|\,(r-a)^2 + |H|a^2
\end{aligned}$$

So we have

$$\frac{m}{2|H|}\left(r\,\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{\mu m}{2|H|m^2\mu}\,\|L\|^2 \;=\; -(r-a)^2 + a^2 \tag{1.5}$$

Now set

$$u(t) \;:=\; \left(\frac{2|H|}{m}\right)^{1/2} \int_\tau^t \frac{1}{r(\sigma)}\,\mathrm{d}\sigma$$

It follows

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= \left(\frac{2|H|}{m}\right)^{1/2}\frac{1}{r(t)}\\ \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\mathrm{d}r}{\mathrm{d}u}\,\frac{\mathrm{d}u}{\mathrm{d}t} \;=\; \frac{\mathrm{d}r}{\mathrm{d}u}\left(\frac{2|H|}{m}\right)^{1/2}\frac{1}{r(t)}\end{aligned}$$

which is equivalent to

$$r\,\frac{\mathrm{d}r}{\mathrm{d}t} \;=\; \left(\frac{2|H|}{m}\right)^{1/2}\frac{\mathrm{d}r}{\mathrm{d}u} \tag{1.6}$$

With (1.5) we get

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}u}\,r\right)^2 + (r-a)^2 &= a^2 - a\,\frac{1}{m^2\mu}\,\|L\|^2\\ &\overset{(1.3)}{=} a^2 - a^2(1-e^2)\\ &= a^2e^2\\ \Rightarrow \left(\frac{\mathrm{d}}{\mathrm{d}u}(r-a)\right)^2 + (r-a)^2 &= a^2e^2\end{aligned}$$

We solve this equation and get the solution

$$r(u) \;=\; a(1-e\cos u)$$

From (1.6) we get

$$r\,\mathrm{d}u \;=\; \left(\frac{2|H|}{m}\right)^{1/2}\mathrm{d}t \;=\; \left(\frac{\mu}{a}\right)^{1/2}\mathrm{d}t$$

We now integrate in two different ways

$$\begin{aligned}\left(\frac{\mu}{a}\right)^{1/2}(t-\tau) &= \int_\tau^t \left(\frac{\mu}{a}\right)^{1/2}\mathrm{d}t\\ &= \left(\frac{\mu}{a}\right)^{1/2}\int_0^u \frac{\mathrm{d}t}{\mathrm{d}u}\,\mathrm{d}u\\ &= \int_0^u r(u)\,\mathrm{d}u\\ &= \int_0^u a(1-e\cos u)\,\mathrm{d}u\\ &= a(u-e\sin u)\end{aligned}$$

and finally get Kepler's equation

$$\mu^{1/2}a^{-3/2}\,(t-\tau) \;=\; E - e\sin E$$

(QED)

Definition 1.6: We define the *Kepler function* K by

$$K(E) \; := \; E - e \sin E$$

where $0 < e < 1$ is fixed. □

Exercise 1.5: For each $0 < e < 1$ the Kepler function $K(E)$ defines a smooth odd monotone map $K : \mathbb{R} \to K(\mathbb{R}) = \mathbb{R}$ (a bijection). Moreover $\forall n \in \mathbb{Z}$

$$K(n\pi) \; = \; n\pi$$

and

$$K(E + 2\pi) \; = \; K(E) + 2\pi$$

The inverse map has the same proporties. □

Remark: We take the derivative of the above function and get that

$$\frac{\mathrm{d}K}{\mathrm{d}E}(E + 2\pi) \; = \; \frac{\mathrm{d}K}{\mathrm{d}E}(E)$$

which means that $\frac{\mathrm{d}K}{\mathrm{d}E}$ is a periodic function of E of period 2π, as well as

$$\frac{\mathrm{d}}{\mathrm{d}M} E(M) \; = \; \frac{\mathrm{d}}{\mathrm{d}M}(K^{-1})(M)$$

□

1.4 Fourer Series (Preview)

Definition 1.7: Let $f(x)$ be a complex valued function of x on $\mathbb{R}$. By definition $f(x)$ is a *periodic function* of x with period 2π, when $f(x + 2\pi) = f(x)$ $\forall x \in \mathbb{R}$.

We also define

$$C^0_{\text{per}}[0, 2\pi] \; := \; \{\text{continuous complex valued } 2\pi\text{-periodic functions on } \mathbb{R}\}$$

For each $k \geq 0$

$$C^k_{\text{per}}[0, 2\pi] \; := \; \left\{ f(x) \in C^0_{\text{per}}[0, 2\pi] \;\middle|\; \frac{\mathrm{d}^j}{\mathrm{d}x} f(x) \in C^0_{\text{per}}[0, 2\pi] \text{ for } j = 0, ..., k \right\}$$

□

Exercise 1.6: Let be $f \in C^0_{\text{per}}[0, 2\pi]$ and $-\infty < a < b < \infty$, $b - a = 2\pi$.

$$\Rightarrow \int_a^b f(x) \, \mathrm{d}x \; = \; \int_0^{2\pi} f(x) \, \mathrm{d}x$$

□

Theorem 1.2: Let be $f(x) \in C^k_{\text{per}}[0, 2\pi]$ for some $k \geq 1$. Set $\forall j \in \mathbb{Z}$

$$\widehat{f}(j) \; := \; \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ijx} \, \mathrm{d}x$$

Then

$$f(x) \; = \; \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ijx}$$

which converges absolutely and uniformly on $[0, 2\pi]$ and

$$\max_{0\le x\le 2\pi}\left|f(x) - \sum_{j=-N}^{N}\widehat{f}(j)e^{ijx}\right| \;\le\; \left\|\frac{\mathrm{d}^k f}{\mathrm{d}x^k}\right\|_\infty \frac{1}{N^{k-\frac{1}{2}}}$$

where $\|g\|_\infty = \max_{0\le x\le 2\pi}|g(x)|$. Moreover,

$$\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2\,\mathrm{d}x \;=\; \sum_{j=-\infty}^{\infty}|\widehat{f}(j)|^2$$

□

Exercise 1.7: Let $f \in C^1_{\mathrm{per}}[0, 2\pi]$ be a real valued function. Then

$$f(x) \;=\; \frac{1}{2}\,a_0 + \sum_{j=1}^{\infty} a_j\cos jx + b_j\sin jx$$

where

$$a_j \;=\; \frac{1}{\pi}\int_0^{2\pi} f(x)\cos jx\,\mathrm{d}x \qquad j\ge 0$$

$$b_j \;=\; \frac{1}{\pi}\int_0^{2\pi} f(x)\sin jx\,\mathrm{d}x \qquad j\ge 1$$

If $f(x)$ is even (i.e. $f(-x) = f(x)$)

$$f(x) \;=\; \frac{1}{2}\,a_0 + \sum_{j=1}^{\infty} a_j\cos jx$$

where

$$a_j \;=\; \frac{2}{\pi}\int_0^{\pi} f(x)\cos jx\,\mathrm{d}x \qquad j\ge 0$$

and similarly for odd functions (i.e. $f(-x) = -f(x)$). □

Remark: We know, that

$$\frac{\mathrm{d}}{\mathrm{d}M}\,E \;=\; \frac{\mathrm{d}}{\mathrm{d}M}\,K^{-1}(M)$$

is a real, smooth, even and periodic function of period 2π, so

$$\frac{\mathrm{d}E}{\mathrm{d}M} \;=\; \frac{1}{2}\,a_0 + \sum_{j=1}^{\infty} a_j\cos jx$$

□

Definition 1.8: The function $e^{ix\sin\theta}$ is a 2π-periodic function, so we know (by using the Fourier expansion theorem), that

$$e^{ix\sin\theta} \;=\; \sum_{n=-\infty}^{\infty}\widehat{e^{ix\sin\theta}}(n)e^{in\theta}$$

where we call

$$J_n(x) \;:=\; \widehat{e^{ix\sin\theta}}(n) \;=\; \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}e^{-in\theta}\,\mathrm{d}\theta$$

the *Bessel function of Index* $n\in\mathbb{Z}$. □

Proposition 1.11: $\forall j \geq 0$

$$\begin{aligned} a_j &= \frac{1}{2\pi} \int_0^\pi \frac{\mathrm{d}}{\mathrm{d}M} K^{-1}(M) \cos jM \,\mathrm{d}M \\ &= \frac{2}{\pi} \int_0^\pi \cos(jE - je\sin E)\,\mathrm{d}E \quad = \quad 2J_j(je) \end{aligned}$$

In particular, for $j = 0$

$$\frac{2}{\pi} \int_0^\pi \frac{\mathrm{d}}{\mathrm{d}M} K^{-1}(M)\,\mathrm{d}M \quad = \quad 2$$

In other words

$$\frac{\mathrm{d}}{\mathrm{d}M} E(M) \quad = \quad 1 + 2\sum_{j=1}^{\infty} J_j(je) \cos jM \tag{1.7}$$

□

Proof: We first write

$$\begin{aligned} 1 \quad = \quad \frac{\mathrm{d}}{\mathrm{d}E} E \quad &= \quad \frac{\mathrm{d}}{\mathrm{d}E} K^{-1}(K(E)) \\ &= \quad \left(\frac{\mathrm{d}}{\mathrm{d}M} K^{-1}\left(K(E)\right)\right) \frac{\mathrm{d}}{\mathrm{d}E} K(E) \end{aligned} \tag{1.8}$$

which we use in

$$\begin{aligned} &\frac{2}{\pi} \int_0^\pi \frac{\mathrm{d}}{\mathrm{d}M} K^{-1}(M) \cos jM \,\mathrm{d}M \\ &= \quad \frac{2}{\pi} \int_{K^{-1}(0)}^{K^{-1}(\pi)} \left(\frac{\mathrm{d}}{\mathrm{d}M} K^{-1}(K(E))\right) \cos\left(jK(E)\right) \frac{\mathrm{d}}{\mathrm{d}E} K(E)\,\mathrm{d}E \\ &\overset{(1.8)}{=} \quad \frac{2}{\pi} \int_0^\pi \cos\left(jK(E)\right)\,\mathrm{d}E \\ &= \quad \frac{2}{\pi} \int_0^\pi \cos\left(j(E - e\sin E)\right)\,\mathrm{d}E \\ &= \quad \frac{2}{\pi} \int_0^\pi \frac{1}{2} \left(e^{-ij(E-e\sin E)} + e^{ij(E-e\sin E)}\right)\,\mathrm{d}E \\ &= \quad 2\,\frac{1}{2\pi} \int_0^\pi e^{-ij(E-e\sin E)}\,\mathrm{d}E + 2\,\frac{1}{2\pi} \int_0^\pi e^{ij(E-e\sin E)}\,\mathrm{d}E \\ &= \quad 2\,\frac{1}{2\pi} \int_0^\pi e^{-ij(E-e\sin E)}\,\mathrm{d}E + 2\,\frac{1}{2\pi} \int_{-0}^{-\pi} e^{ij((-E)-e\sin(-E))}(-1)\,\mathrm{d}E \\ &= \quad 2\,\frac{1}{2\pi} \int_{-\pi}^\pi e^{-ij(E-e\sin E)}\,\mathrm{d}E \\ &= \quad 2\,\frac{1}{2\pi} \int_{-\pi}^\pi e^{i(je)\sin E} e^{-ijE}\,\mathrm{d}E \\ &= \quad 2J_j(je) \end{aligned}$$

(QED)

Corollary 1.2: By integration of (1.7), we get

$$E(M) = M + 2\sum_{n\geq 1} \frac{1}{n} J_n(ne)\sin nM + E_0$$

where

$$M = \frac{\mu^{1/2}}{a^{3/2}}(t-\tau)$$

□

Theorem 1.3: Suppose $L \neq 0$, $H < 0$ and $(x(t), y(t))$ are coordinates perpendicular to L. Then

$$\begin{aligned} x(t) &= -\frac{3}{2}ae + 2a\sum_{n\geq 1}\frac{1}{n^2}\frac{\mathrm{d}}{\mathrm{de}} J_n(ne)\cos\left(n\mu^{1/2}a^{-3/2}(t-\tau)\right) \\ y(t) &= a\,\frac{2}{e}(1-e^2)^{1/2}\sum_{n\geq 1}\frac{1}{n} J_n(ne)\sin\left(n\mu^{1/2}a^{-3/2}(t-\tau)\right) \end{aligned}$$

□

Theorem 1.4: "Betty Bossi" Recipe for elliptical orbits
Let $q_0, v_0 \in \mathbb{R}$ be initial data for the Kepler Problem with $L_0 = L(q_0, p_0) \neq 0$ ($p_0 = mv_0$), $F_0 = F(q_0, p_0) \neq 0$ and $H_0 = H(q_0, p_0) < 0$. Let x, y be Cartesian coordinates in the plane perpendicular to L_0. Precisely, x is the coordinate along the F_0-axis and y along the $-[F_0, L_0]$-axis. i.e. we are using

$$\frac{F_0}{\|F_0\|}, \qquad \frac{-[F_0, L_0]}{\|[F_0, L_0]\|}, \qquad \frac{L_0}{\|L_0\|}$$

as orthonormal basis for $\mathbb{R}^3$. Set

$$e := (m\mu)^{-1}\|F_0\|, \qquad a := \frac{\mu m}{2(-H_0)}, \qquad \tau := \mu^{-1/2}a^{3/2}(e\sin E_0 - E_0)$$

where $-\pi < E_0 < \pi$ is the unique solution to

$$\langle q_0, v_0\rangle = (\mu a)^{1/2}\sin E_0$$

and $\|q_0\| = a(1 - e\cos E_0)$.

Then, the unique solution to the Kepler Problem with initial data q_0, v_0 is given by

$$\begin{aligned} x(t) &= -\frac{3}{2}ae + 2a\sum_{n\geq 1}\frac{1}{n^2}\frac{\mathrm{d}}{\mathrm{de}} J_n(ne)\cos\left(\mu^{1/2}a^{-3/2}n(t-\tau)\right) \\ y(t) &= a\,\frac{2}{e}\left(1-e^2\right)^{1/2}\sum_{n\geq 1}\frac{1}{n} J_n(ne)\sin\left(\mu^{1/2}a^{-3/2}n(t-\tau)\right) \end{aligned}$$

of course, the point $(x(t), y(t))$ moves on the ellipse

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1$$

□

Proof: We know

$$\begin{aligned} x &= a(\cos E - e) \\ y &= a(1-e^2)^{1/2}\sin E = b\sin E \end{aligned}$$

and

$$\cos E(M) = \cos K^{-1}(M)$$

is a smooth, real, even and periodic function of M of period 2π because

$$K^{-1}(M+2\pi) = K^{-1}(M)+2\pi$$

By Fourier Expansion we get

$$\cos K^{-1}(M) = \frac{1}{2}\,a_0 + \sum_{n\geq 1} a_n \cos nM$$

where

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi \cos K^{-1}(M)\cos nM\,\mathrm{d}M \\
&= \frac{2}{\pi}\cos\left(K^{-1}(M)\right)\frac{1}{n}\sin nM\Big|_0^\pi \\
&\quad -\frac{2}{\pi}\int_0^\pi \left(\frac{\mathrm{d}}{\mathrm{d}M}\cos\left(K^{-1}(M)\right)\right)\frac{1}{n}\sin nM\,\mathrm{d}M \\
&= -\frac{2}{\pi}\int_0^\pi \left(-\sin\left(K^{-1}(M)\right)\right)\left(\frac{\mathrm{d}}{\mathrm{d}M}K^{-1}(M)\right)\frac{1}{n}\sin nM\,\mathrm{d}M \\
&= \frac{2}{\pi}\int_0^\pi \sin E\left(\frac{1}{n}\sin nK(E)\right)\frac{\mathrm{d}}{\mathrm{d}M}K^{-1}(M)\Big|_{M=K(E)}\frac{\mathrm{d}M}{\mathrm{d}E}\,\mathrm{d}E \\
&= \frac{1}{n}\frac{2}{\pi}\int_0^\pi \sin E\sin nK(E)\,\mathrm{d}E \\
&= \frac{1}{n}\frac{1}{\pi}\int_0^\pi \cos\left(nK(E)-E\right)-\cos\left(nK(E)+E\right)\,\mathrm{d}E \\
&= \frac{1}{n\pi}\int_0^\pi \cos\left((n-1)E-ne\sin E\right)\,\mathrm{d}E \\
&\quad -\frac{1}{n\pi}\int_0^\pi \cos\left((n+1)E-ne\sin E\right)\,\mathrm{d}E
\end{aligned}$$

Claim 1:

$$\frac{2}{\pi}\int_0^\pi \cos K^{-1}(M)\cos nM\,\mathrm{d}M = \frac{1}{2}\left(J_{n-1}(ne)-J_n(ne)\right)$$

Claim 2: With all that, we can finish the "Betty Bossi" Recipe. (QED)

Exercise 1.8: Prove Claim 1 and 2 of the proof above. □

Back to the Kepler Problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\,p = -m\mu r^{-3}q \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,q = \frac{1}{m}\,p$$

where $r=\|q\|$. We expand to

$$\begin{aligned}
U(q) &= -m\mu r^{-1} \\
\nabla_q U &= \begin{pmatrix}\frac{\partial}{\partial q_1}U\\ \frac{\partial}{\partial q_2}U\\ \frac{\partial}{\partial q_3}U\end{pmatrix} = m\mu r^{-3}\begin{pmatrix}q_1\\ q_2\\ q_3\end{pmatrix} = m\mu r^{-3}q
\end{aligned}$$

which brings the Kepler problem to the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,q \;=\; \frac{1}{m}\,p \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,p \;=\; -\nabla_q U$$

We do the same with

$$\begin{aligned} T(p) &= \frac{1}{2m}\|p\|^2 \;=\; \frac{1}{2m}\left(p_1^2+p_2^2+p_3^2\right) \\ \nabla_p T &= \frac{p}{m} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\,q \end{aligned}$$

With the total energy $H(p,q) = T(p) + U(q)$ we then get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,q \;=\; \nabla_p H \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,p \;=\; -\nabla_q H$$

Now we build the 6×6 matrix

$$J \;:=\; \begin{pmatrix} 0 & \mathbf{1}_3 \\ -\mathbf{1}_3 & 0 \end{pmatrix} \qquad \mathbf{1}_3 \;=\; \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

Defining $\nabla := \begin{pmatrix} \nabla_q & \nabla_p \end{pmatrix}^T$ we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} \;=\; \begin{pmatrix} 0 & \mathbf{1}_3 \\ -\mathbf{1}_3 & 0 \end{pmatrix}\nabla H \;=\; J\nabla H$$

We know that

$$J^T \;=\; -J, \qquad J^2 \;=\; -\mathbf{1}_6$$

Now, the Kepler problem is an equation with a Hamiltonian vectorfield.

Definition 1.9: Let be $(q,p) \in \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{R}^{2n}$, $f, g \in C^\infty(\Omega)$, where f and g are functions of q and p and $\Omega \subset \mathbb{R}^n$. The *Poisson bracket* is defined by

$$\{f,g\} \;=\; \sum_{j=1}^{n}\left(\frac{\partial}{\partial q_j}\,f\,\frac{\partial}{\partial p_j}\,g - \frac{\partial}{\partial p_j}\,f\,\frac{\partial}{\partial q_j}\,g\right)$$

□

We have

$$\nabla f \;=\; \begin{pmatrix} \nabla_q f \\ \nabla_p f \end{pmatrix} \qquad \{f,g\}(q,p) \;=\; \langle \nabla f, J\nabla g\rangle_{\mathbb{R}^{2n}}\,(q,p)$$

and further

$$\begin{aligned} \{q_i,p_j\} &= \sum_{k=1}^{n}\frac{\partial q_i}{\partial q_k}\,\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\,\frac{\partial p_j}{\partial q_k} \;=\; \sum_{k=1}^{n}\frac{\partial q_i}{\partial q_k}\,\frac{\partial p_j}{\partial p_k} \;=\; \delta_{ij} \\ \{q_i,q_j\} &= 0 \\ \{p_i,p_j\} &= 0 \\ \{q_j,f\} &= \sum_{k=1}^{n}\frac{\partial q_j}{\partial q_k}\,\frac{\partial f}{\partial p_k} - \frac{\partial f}{\partial q_k}\,\frac{\partial q_j}{\partial p_k} \;=\; \frac{\mathrm{d}f}{\mathrm{d}p_j} \\ \{p_j,f\} &= -\frac{\partial f}{\partial q_j} \end{aligned}$$

Now we can write

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} \;=\; \{q_j,H\} \qquad \frac{\mathrm{d}p_j}{\mathrm{d}t} \;=\; -\{p_j,H\}$$

Proposition 1.12: Let $\Omega \subset \mathbb{R}^{2n}$ be a domain (open connected set).

Then, the Poisson bracket $\{,\}$ defines a bilinear map from $C^\infty(\Omega) \times C^\infty(\Omega)$ to $C^\infty(\Omega)$ satisfying $\forall f, g, h \in C^\infty(\Omega)$

i) Antisymmetry
$$\{f, g\} = -\{g, f\}$$

ii) Jacobi Identity
$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$$

iii) $\{f, \{g, h\}\} = \{f, g\}h + g\{f, h\}$

This means, $(C^\infty(\Omega), \{,\})$ is a Lie Algebra (with infinite dimension). □

Definition 1.10: The *Hamiltonian trajectory* $\phi_h^t(q, p)$ in Ω is the unique integral curve of V_h passing through $(q, p) \in \Omega$ at time $t = 0$ i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\phi_h^t(q,p) = V_h\left(\phi_h^t(q,p)\right) \qquad \phi_h^t(q,p)\big|_{t=0} = (q_0, p_0)$$

□

Proposition 1.13: The function $f \in C^\infty(\Omega, \mathbb{R})$ is constant along all the trajectories $\phi_h^t(q, p)$ of V_h if and only if $\{f, h\} = 0$ on Ω. □

Proof: The Hamiltonian vector field $V_h(q, p)$ corresponding to the Hamiltonian function h is
$$V_h := J\nabla h = \begin{pmatrix} \nabla_p h \\ -\nabla_q h \end{pmatrix}$$
where
$$J := \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \qquad \nabla := \begin{pmatrix} \nabla_q \\ \nabla_p \end{pmatrix}$$
Now we get
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\, f\left(\phi_h^t(q,p)\right) &= \left\langle \nabla f\left(\phi_h^t(q,p)\right), \frac{\mathrm{d}}{\mathrm{d}t}\,\phi_h^t(q,p)\right\rangle_{\mathbb{R}^{2n}} \\
&= \left\langle \begin{pmatrix} \nabla_q \\ \nabla_p \end{pmatrix} f\left(\phi_h^t(q,p)\right), \begin{pmatrix} \nabla_p h\left(\phi_h^t(q,p)\right) \\ -\nabla_q h\left(\phi_h^t(q,p)\right) \end{pmatrix} \right\rangle_{\mathbb{R}^{2n}} \\
&= \langle \nabla_q f, \nabla_p h\rangle_{\mathbb{R}^n} + \langle \nabla_p f, -\nabla_q h\rangle_{\mathbb{R}^n} \\
&= \left\langle \begin{pmatrix} \nabla_q \\ \nabla_p \end{pmatrix} f, J \begin{pmatrix} \nabla_q \\ \nabla_p \end{pmatrix} h \right\rangle_{\mathbb{R}^n} \\
&= \sum_{j=1}^{n} \frac{\partial f}{\partial q_j}\frac{\partial h}{\partial p_j} - \frac{\partial h}{\partial q_j}\frac{\partial f}{\partial p_j} \\
&= \{f, h\}\left(\phi_h^t(q,p)\right) \qquad (1.9)
\end{aligned}$$

$\Rightarrow f$ is conserved along the orbits of V_h if $\{f, h\} = 0$. (QED)

We go back to the Kepler problem
$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \frac{1}{m}\,p(t) \qquad \frac{\mathrm{d}p(t)}{\mathrm{d}t} = -m\mu r^{-3} q(t)$$
$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}H}{\mathrm{d}t} = 0$$
Now we can say, that the total energy is conserved $\Leftrightarrow \{H, H\} = 0$.

Proposition 1.14: $\{L_i, H\} = 0$, $i = 1, 2, 3$, $\{F_j, H\} = 0$, $j = 1, 2, 3$ □

Exercise 1.9: Prove Proposition 1.14 and for $L = [q, p] = (L_1, L_2, L_3)$ prove

$$\{L_1, L_3\} = L_3 \qquad \{L_2, L_3\} = L_1 \qquad \{L_3, L_1\} = L_2$$

□

Exercise 1.10: Set $L_a := a_1L_1 + a_2L_2 + a_3L_3$ and show

$$\{L_a, L_b\} = L_{[a,b]}$$

i.e. $a \in \mathbb{R}^3 \to L_a \in C^\infty(\mathbb{R}^6, \mathbb{R})$ is an iso of Lie algebra. □

Proposition 1.15: Let $(L, [,])$ be an arbitrary Lie Algebra (like $(\mathbb{R}^3, [,])$, $(C^\infty(\Omega, \mathbb{R}), \{,\})$). Fix $x \in L$. The set $\{y \in L \mid [y, x] = 0\}$ is a Lie subalgebra of L. □

Application: Let be $h \in C^\infty(\Omega, \mathbb{R})$.

$$\Rightarrow \{f \mid \{f, h\} = 0\}$$

is a Lie algebra under $\{,\}$.

Question: What is $\{f \mid \{f, H\} = 0\}$, where H is the Kepler Hamiltonian.

Definition 1.11: Define

$$\begin{aligned} \mathscr{P}(\mathbb{R}^6) &:= \left\{ 0 \neq \sum_{i_1,\ldots,i_6 \in \mathbb{N}} c_{i_1\ldots i_6} q_1^{i_1} q_2^{i_2} q_3^{i_3} p_1^{i_4} p_2^{i_5} p_3^{i_6} \mid \begin{array}{l} c_{i_1\ldots i_6} \in \mathbb{R}, \\ \#\text{Coefficients} < \infty \end{array} \right\} \\ &= \text{Vector space of all polynomials in } q_1, q_2, q_3, p_1, p_2, p_3 \end{aligned}$$

So $q_1^2 + q_2^2 + q_3^2$, $p_1^2 + p_2^2 + p_3^3 \in \mathscr{P}(\mathbb{R}^6)$, $L_1 = q_2p_3 - q_3p_2$, $L_2 = q_3p_1 - q_1p_3$, $L_3 = q_1p_2 - q_2p_1 \in \mathscr{P}(\mathbb{R}^6)$.

$f, g \in \mathscr{P}(\mathbb{R}^6)$, $\{f, g\} \in \mathscr{P}(\mathbb{R}^6)$, $(L_1, L_2, L_3) = [q, p]$ □

Proposition 1.16: $(\mathscr{P}(\mathbb{R}^6), \{,\})$ is a Lie algebra, actually a sub algebra of $(C^\infty(\mathbb{R}^6), \{,\})$. □

Lemma 1.2:

$$\{p_ip_j, q_kp_\ell\} = p_jq_k\delta_{i\ell} - q_ip_\ell\delta_{jk}$$

□

Exercise 1.11: Prove Lemma 1.2. □

Proposition 1.17: $\{L_1, L_2\} = L_3$, $\{L_3, L_1\} = L_2$, $\{L_2, L_3\} = L_1$ □

Exercise 1.12: Prove Proposition 1.17 using Lemma 1.2. □

In other words, the 3-dimensional subspace of $\mathscr{P}(\mathbb{R}^6)$ spanned by L_1, L_2 and L_3 is a Lie subalgebra isomorphic to $o(3)$.

$$\begin{aligned} M_n(\mathbb{C}) &:= \{A = (a_{ij}) \mid a_{ij} \in \mathbb{C},\, i, j = 1, ..., n\} \\ M_n(\mathbb{R}) &:= \{A = (a_{ij}) \mid a_{ij} \in \mathbb{R},\, i, j = 1, ..., n\} \end{aligned}$$

Definition 1.12: $\forall A, B \in M_n(\mathbb{C})$, the set $[A, B] := AB - BA$ is called the comutator of A and B. □

Proposition 1.18: $\forall A, B, C \in \mathbb{M}_n(\mathbb{C})$

$$\begin{aligned} [A, B] &= -[B, A] \\ [A, [B, C]] + [C, [A, B]] + [B, [C, A]] &= 0 \end{aligned}$$

and $[,] : \mathbb{M}_n(\mathbb{C}) \times \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is bilinear. □

Definition 1.13: Define

$$\begin{aligned} \mathrm{bl}_n(\mathbb{C}) &:= (\mathbb{M}_n(\mathbb{C}), [,]) \\ \mathrm{bl}_n(\mathbb{R}) &:= (\mathbb{M}_n(\mathbb{R}), [,]) \end{aligned}$$

Obviously, these are Lie algebras, called the general linear complex/real algebra. □

Definition 1.14: $\forall n \in \mathbb{N},\, n \geq 2$

$$o(n) := \{A \in \mathrm{gl}_n(\mathbb{R}) \mid A^T = -A\}$$

$(o(n), [,])$ is a Lie algebra called the *orthogonal algebra*. □

Proposition 1.19: $(\mathbb{R}^3, [,])$, $(\{L_a = a_1 L_1 + a_2 L_2 + a_3 L_3 \mid (a_1, a_2, a_3) = a \in \mathbb{R}^3\}, \{,\})$ are all isomorphic. □

$\mathbb{R}^3 \ni a \to L_a \in \mathrm{span}(L_1, L_2, L_3)$ is an isomorphism

$$\{L_a, L_b\} = L_{[a,b]}$$

$$\delta_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \delta_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \delta_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$a \in \mathbb{R}^3 \to a_1\delta_1 + a_2\delta_2 + a_3\delta_3 \in o(3)$ is a Lie algebra isomorphism.

Remember

$$\begin{aligned} F &= \frac{1}{m}\,[p, L] - m\mu r^{-1} q \\ F_1 &= \frac{1}{m}\,(p_2 L_3 - p_3 L_2) - m\mu r^{-1} q_1 \\ &= \frac{1}{m}\left(\|p\|^2 q_1 - \langle q, p\rangle\, p_1\right) - m\mu r^{-1} q_1 \\ F_2 &= \frac{1}{m}\left(\|p\|^2 q_2 - \langle q, p\rangle\, p_2\right) - m\mu r^{-1} q_2 \\ F_3 &= \frac{1}{m}\left(\|p\|^2 q_3 - \langle q, p\rangle\, p_3\right) - m\mu r^{-1} q_3 \end{aligned}$$

Proposition 1.20: Set $L_a = a_1 L_1 + a_2 L_2 + a_3 L_3,\; F_b = b_1 F_1 + b_2 F_2 + b_3 F_3$, $a, b \in \mathbb{R}^3$.

$\Rightarrow \forall a, b \in \mathbb{R}^3$

$$\begin{aligned} \{L_a, L_b\} &= L_{[a,b]} \\ \{L_a, F_b\} &= F_{[a,b]} \\ \{F_a, F_b\} &= \frac{2}{m}\,(-H)\, L_{[a,b]} \end{aligned}$$

□

Exercise 1.13: Prove Proposition 1.20. □

Definition 1.15: Let be $\Omega_- = \{(q,p) \in \mathbb{R}^6 \mid H(q,p) < 0\}$.

$$E_a := \left(\frac{2}{m}(-H)\right)^{-1/2} F_a \in C^\infty(\Omega_-)$$

□

Proposition 1.21: For all $(a,b) \in \Omega_-$

$$\begin{aligned} \{L_a, L_b\} &= L_{[a,b]} \\ \{L_a, E_b\} &= E_{[a,b]} \\ [E_a, E_b] &= L_{[a,b]} \end{aligned}$$

Moreover

$$\begin{aligned} \{L_a, H\} &= 0 \\ \{E_a, H\} &= 0 \end{aligned}$$

□

Conclude:

$$(\mathrm{span}(L_1, L_2, L_3, E_1, E_2, E_3), \{,\})$$

is a Lie subalgebra of $(C^\infty(\Omega_-), \{,\})$. In fact this is $o(4)$.

The set

$$\mathrm{GL}_n(\mathbb{R}) = \{A \in \mathbb{M}_n(\mathbb{R}) \mid \det A \neq 0\} \subset \mathbb{R}^{n\times n}$$

is an open set. We have

$$\mathbf{1}_n \in \mathrm{GL}_n \qquad A(t) \in \mathrm{GL}_n(\mathbb{R}),\ -\varepsilon < t < \varepsilon \qquad A(0) = \mathbf{1}_n$$

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t)\bigg|_{t=0} = B$$

Now let be $A(t) \in \mathrm{SO}(3)$, $A(0) = \mathbf{1}_3$. We know

$$A(t)A^T(t) = \mathbf{1}_3 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(A(t)A^T(t)\right) = 0$$

$$\Rightarrow \left(\frac{\mathrm{d}}{\mathrm{d}t} A(t)\right) A^T(t) + A(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(A^T(t)\right) = 0$$

Using

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A^T\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} A\right)^T$$

we get

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} A(t)\right) A^T(t) + A(t)\left(\frac{\mathrm{d}}{\mathrm{d}t} A(t)\right)^T = 0$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} A(0) + \left(\frac{\mathrm{d}}{\mathrm{d}t} A(0)\right)^T = 0$$

So $\frac{\mathrm{d}}{\mathrm{d}t} A(0) \in o(3) = \{B \in \mathbb{M}_3(\mathbb{R}) \mid B^T = -B\}$.

Definition 1.16: If $A \in \mathbb{M}_n(\mathbb{C})$, we define

$$e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

□

Definition 1.17: For $1 \leq \ell < k \leq 4$ set $I^{k\ell} = (\delta^k_m \delta^\ell_n - \delta^k_n \delta^\ell_m)$ like for example

$$I^{21} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have $I^{kk} = 0$, $I^{k\ell} = -I^{\ell k}$ and $[I^i, J^j] = 0 \; \forall i, j$. □

Exercise 1.14:

$$[I^{ij}, I^{k\ell}] = \delta_{jk} I^{i\ell}$$

□

Set

$$\begin{aligned} I^1 &= \frac{1}{2}\left(I^{23} - I^{14}\right) & J^1 &= \frac{1}{2}\left(I^{23} + I^{14}\right) \\ I^2 &= \frac{1}{2}\left(I^{13} - I^{24}\right) & J^2 &= \frac{1}{2}\left(I^{13} + I^{24}\right) \\ I^3 &= \frac{1}{2}\left(I^{12} - I^{34}\right) & J^3 &= \frac{1}{2}\left(I^{12} + I^{24}\right) \end{aligned}$$

Exercise 1.15: $I^1, I^2, I^3, J^1, J^2, J^3$ are a basis for $o(4)$ and

$$\{I^i, I^j\} = \varepsilon^{ijk} I^k \qquad \{I^i, J^j\} = \varepsilon^{ijk} J^k$$

where $1 \leq i, j, k \leq 3$, $\varepsilon^{123} = 1$, $\varepsilon^{ijk} = 1$, $\varepsilon^{\pi(1)\pi(2)\pi(3)} = \mathrm{Sg}(\pi)$, $\varepsilon^{iij} = 0$. □

Set

$$A_a = \frac{1}{2}\left(L_a + E_a\right) \qquad B_b = \frac{1}{2}\left(L_b - E_b\right)$$

$a, b \in \mathbb{R}^3$.

$$\begin{aligned} A_1 &= \frac{1}{2}\left(L_1 + E_1\right) & B_1 &= \frac{1}{2}\left(L_1 - E_1\right) \\ A_2 &= \frac{1}{2}\left(L_2 + E_2\right) & B_2 &= \frac{1}{2}\left(L_2 - E_2\right) \\ A_3 &= \frac{1}{2}\left(L_3 + E_3\right) & B_3 &= \frac{1}{2}\left(L_3 - E_3\right) \end{aligned}$$

Now we know

$$\{\mathrm{span}(L_1, L_2, L_3, E_1, E_2, E_3), \{,\}\} \approx o(4) = o(3) \oplus o(3)$$

2 Fourier Series

2.1 Finite Fourier Series

Definition 2.1: Define the set

$$C^0_{\text{per}}[0,2\pi) \quad := \quad \{f(x) \in C^0(\mathbb{C}) \mid f(x+2\pi) = f(x)\, \forall x \in \mathbb{R}\}$$

□

The points $2\pi/n$, $2 \cdot 2\pi/n$, ..., $k \cdot 2\pi/n$, ..., $n \cdot 2\pi/n = 2\pi$ divide $[0, 2\pi)$ into subintervals of equal length $2\pi/n$ and they look like

$$\left[\frac{2\pi}{n}(k-1), \frac{2\pi}{n}k\right)$$

We sample f at the points $\frac{2\pi}{n}k$, $k = 1, ..., n$

$$f^{(n)} \quad := \quad \left(f\left(\frac{2\pi}{n}\right), f\left(\frac{2\pi}{n}2\right), ..., f\left(\frac{2\pi}{n}n\right)\right) \in \mathbb{C}^n$$

Definition 2.2: Let be $f, g \in C^0_{\text{per}}[0, 2\pi)$

$$\left\langle f^{(n)}, g^{(n)} \right\rangle \quad := \quad \sum_{k=1}^{n} f\left(\frac{2\pi}{n}k\right) g^\star\left(\frac{2\pi}{n}k\right)$$

we call the *average inner product.* □

Now write

$$\begin{aligned}\frac{1}{n}\left\langle f^{(n)}, g^{(n)} \right\rangle &= \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{2\pi}{n}k\right) g^\star\left(\frac{2\pi}{n}k\right) \\ &= \frac{1}{2\pi}\frac{2\pi}{n}\sum_{k=1}^{n} f\left(\frac{2\pi}{n}k\right) g^\star\left(\frac{2\pi}{n}k\right)\end{aligned}$$

and the limit

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\left\langle f^{(n)}, g^{(n)} \right\rangle &= \lim_{n\to\infty}\frac{1}{2\pi}\frac{2\pi}{n}\sum_{k=1}^{n} f\left(\frac{2\pi}{n}k\right) g^\star\left(\frac{2\pi}{n}k\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x) g^\star(x)\,\mathrm{d}x\end{aligned}$$

Definition 2.3: For $f, g \in C^0_{\text{per}}[0, 2\pi)$ define the *inner product* by

$$\langle f, g \rangle \quad := \quad \frac{1}{2\pi}\int_0^{2\pi} f(x) g^\star(x)\,\mathrm{d}x$$

□

We can see, that

$$\frac{1}{2\pi}\int_0^{2\pi} 1\,\mathrm{d}x \quad = \quad 1$$

Look at the number $e^{\frac{2\pi i}{n}}$. It's absolute value is one and

$$\left(e^{\frac{2\pi i}{n}}\right)^n \quad = \quad 1$$

For $n \geq 2$ we get

$$\sum_{k=0}^{n-1} e^{\frac{2\pi i}{n}\,k} \;=\; \sum_{k=0}^{n-1} \left(e^{\frac{2\pi i}{n}}\right)^k \;=\; \frac{1-e^{2\pi i}}{1-e^{\frac{2\pi i}{n}}} \;=\; 0$$

For $k = 1, ..., n$ set

$$e_k \;:=\; \left(e^{\frac{2\pi i}{n}\,k}, e^{\frac{2\pi i}{n}\,2k}, ..., e^{\frac{2\pi i}{n}\,nk}\right) \;=\; (e_{k1}, ..., e_{kn})$$

where

$$e_{kj} \;:=\; e^{2\pi i j\,\frac{k}{n}}$$

Now we get

$$\begin{aligned}
\frac{1}{n}\,\langle e_k, e_\ell\rangle_{\mathbb{C}^n} \;&=\; \frac{1}{n}\sum_{j=1}^{n} e_{kj} e_{\ell j}^{\star} \;=\; \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i j\,\frac{k}{n}} \left(e^{2\pi i j\,\frac{\ell}{n}}\right)^{\star} \\
&=\; \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i j\,\frac{k}{n}} e^{-2\pi i j\,\frac{\ell}{n}} \\
&=\; \frac{1}{n}\sum_{j=1}^{n} e^{\frac{2\pi i}{n}\,j(k-\ell)} \;=\; \frac{1}{n}\sum_{j=0}^{n-1} e^{\frac{2\pi i}{n}\,j(k-\ell)} \\
&=\; \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell \end{cases}
\end{aligned}$$

and we even have a sample

$$e_k \;=\; \left(e^{ik\,\frac{2\pi}{n}}, ..., e^{ik\,\frac{2\pi}{n}\,j}, ..., e^{ik\,\frac{2\pi}{n}\,n}\right) \;=\; \left(e^{ikx}\right)^{(n)}$$

Proposition 2.1: Fix $n \geq 1$. For each $j = 1, ..., n$ set

$$e_j \;:=\; \frac{1}{\sqrt{n}}\left(e^{2\pi i j\,\frac{1}{n}}, e^{2\pi i j\,\frac{2}{n}}, ..., e^{2\pi i j\,\frac{n-1}{n}}, e^{2\pi i j\,\frac{n}{n}}\right) \in \mathbb{C}^n$$

Then $\langle e_i, e_j\rangle = \delta_{ij}$. That is, $e_1, ..., e_n$ is an orthonormal basis for $\mathbb{C}^n$ which we call the *finite fourier transform basis*. Notice that $e_j^{\star} = e_{n-j}$ for $j = 1, ..., n-1$. □

Definition 2.4: For each $f \in \mathbb{C}^n$ set

$$\begin{aligned}
D_+ f \;&:=\; (f_2 - f_1, f_3 - f_2, ..., f_1 - f_n) \\
D_- f \;&:=\; (f_1 - f_n, f_2 - f_1, ..., f_n - f_{n-1})
\end{aligned}$$

D_+, D_- are linear transformations on $\mathbb{C}^n$, i.e.

$$D_{\pm}\left(\lambda f + \mu g\right) \;=\; \lambda D_{\pm} f + \mu D_{\pm} g$$

□

Example 2.1:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\, f(x) \;&=\; \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon) - f(x)}{\varepsilon} \\
f^{(n)} \;&=\; \left(f\left(\frac{2\pi}{n}\right), ..., f\left(\frac{2\pi\, n}{n}\right)\right) \\
D_+ f^{(n)} \;&=\; \left(..., f\left(\frac{2\pi}{n}\,(j+1)\right) - f\left(\frac{2\pi}{n}\,j\right), ...\right)
\end{aligned}$$

so we've got a differential operator.

Let be $f, g \in C^\infty_{\text{per}}[0, 2\pi)$

$$\begin{aligned}
\left\langle f, \frac{\mathrm{d}}{\mathrm{d}x}\, g \right\rangle &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \left(\frac{\mathrm{d}}{\mathrm{d}x}\, g(x) \right)^\star \mathrm{d}x \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(x)\, \frac{\mathrm{d}}{\mathrm{d}x}\, g^\star(x)\, \mathrm{d}x \\
&= \frac{1}{2\pi}\, f(x) g^\star(x) \Big|_0^{2\pi} - \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\mathrm{d}}{\mathrm{d}x}\, f(x) \right) g^\star(x)\, \mathrm{d}x \\
&= -\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\mathrm{d}}{\mathrm{d}x}\, f(x) \right) g^\star(x)\, \mathrm{d}x \\
&= \left\langle -\frac{\mathrm{d}}{\mathrm{d}x}\, f, g \right\rangle
\end{aligned}$$

Lets interpret this. We know $\frac{\mathrm{d}}{\mathrm{d}x} : C^\infty_{\text{per}}[0, 2\pi) \to C^\infty_{\text{per}}[0, 2\pi)$ is a linear transformation of the infinite dimensional complex vector space $C^\infty_{\text{per}}[0, 2\pi)$. We have

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x}$$

where we used the boundary conditions. We remember the self adjoint matrices

$$A^\dagger = A$$

□

Proposition 2.2: $\frac{1}{i} \frac{\mathrm{d}}{\mathrm{d}x}$ is a self adjoint operator $C^\infty_{\text{per}}[0, 2\pi)$, i.e.

$$\left\langle f, \frac{1}{i}\, \frac{\mathrm{d}}{\mathrm{d}x}\, g \right\rangle = \left\langle \frac{1}{i}\, \frac{\mathrm{d}}{\mathrm{d}x}\, f, g \right\rangle$$

for all $f, g \in C^\infty_{\text{per}}[0, 2\pi)$. □

Proof: Integrate by parts and use the periodic boundary conditions and

$$\left(\frac{1}{i} \right)^\star = -\frac{1}{i}$$

(QED)

Example 2.2: $f \in C^\infty_{\text{per}}[0, 2\pi)$

$$\frac{1}{i}\, \frac{\mathrm{d}}{\mathrm{d}x}\, f = \lambda f \qquad \frac{1}{i}\, \frac{\mathrm{d}}{\mathrm{d}x}\, e^{ijx} = j e^{ijx}$$

for all $j \in \mathbb{Z}$, so

$$\operatorname{spec} \left(\frac{1}{i}\, \frac{\mathrm{d}}{\mathrm{d}x} \right) \supset \mathbb{Z}$$

□

Proposition 2.3: For all $j \in \mathbb{Z}$, e^{ijx} is an eigenfunction in $C^\infty_{\text{per}}[0, 2\pi)$ of $\frac{1}{i} \frac{\mathrm{d}}{\mathrm{d}x}$ with eigenvalue j. i.e.

$$\frac{1}{i}\, \frac{\mathrm{d}}{\mathrm{d}x}\, e^{ijx} = j e^{ijx}$$

We have

$$\langle e^{ijx}, e^{ikx} \rangle = \delta_{jk}$$

□

Proof: For $j \neq k$ we have

$$
\begin{aligned}
j\left\langle e^{ijx}, e^{ikx}\right\rangle &= \left\langle j e^{ijx}, e^{ikx}\right\rangle \\
&= \left\langle \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x} e^{ijx}, e^{ikx}\right\rangle \\
&= \left\langle e^{ijx}, \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x} e^{ikx}\right\rangle \\
&= \left\langle e^{ijx}, k e^{ikx}\right\rangle \\
&= k\left\langle e^{ijx}, e^{ikx}\right\rangle
\end{aligned}
$$

This can be true if and only if $\left\langle e^{ijx}, e^{ikx}\right\rangle = 0$. And

$$
\begin{aligned}
\left\langle e^{ijx}, e^{ijx}\right\rangle &= \frac{1}{2\pi}\int_0^{2\pi} e^{ijx}(e^{ijx})^\star\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{ijx}e^{-ijx}\,\mathrm{d}x \\
&= 1
\end{aligned}
$$

(QED)

Theorem 2.1: Suppose $\left\langle f, e^{ijx}\right\rangle = 0$ for all $j \in \mathbb{Z}$. Then $f = 0$, that is e^{ijx}, $j \in \mathbb{Z}$ is a basis for $C^\infty_{\text{per}}[0, 2\pi)$. □

Remark: If we take a function and write it as

$$
f = \sum_{j=-\infty}^{\infty} \widehat{f}(j)e^{ijx} \quad \Rightarrow \quad \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x} f = \sum_{j=-\infty}^{\infty} j\widehat{f}(j)e^{ijx}
$$

We can write

$$
\langle v, D_+ w\rangle_{\mathbb{C}^n} = \left\langle D_+^\dagger v, w\right\rangle_{\mathbb{C}^n}
$$

□

Exercise 2.1:

$$
D_+^\dagger = D_-
$$

□

Remark:

$$
\langle v, D_+ w\rangle = \langle D_- v, w\rangle
$$

where $D_+, D_- : \mathbb{C}^n \to \mathbb{C}^n$. Can we diagonalize D_+ and D_-?

$$
\begin{aligned}
(D_+ e_j)_k &= \frac{1}{\sqrt{n}} e^{2\pi ij\,\frac{k+1}{n}} - \frac{1}{\sqrt{n}} e^{2\pi ij\,\frac{k}{n}} \\
&= \frac{1}{\sqrt{n}} e^{2\pi ij\,\frac{k}{n}} e^{2\pi ij\,\frac{1}{n}} - \frac{1}{\sqrt{n}} e^{2\pi ij\,\frac{k}{n}} \\
&= \left(e^{2\pi ij\,\frac{1}{n}} - 1\right)\frac{1}{\sqrt{n}} e^{2\pi ij\,\frac{k}{n}} \\
&= \left(e^{2\pi ij\,\frac{1}{n}} - 1\right)(e_j)_k
\end{aligned}
$$

and

$$
\begin{aligned}
(D_- e_j)_k &= \frac{1}{\sqrt{n}} e^{2\pi ij\,\frac{k}{n}} - \frac{1}{\sqrt{n}} e^{2\pi ij\,\frac{k-1}{n}} \\
&= \left(1 - e^{-2\pi ij\,\frac{1}{n}}\right)\frac{1}{\sqrt{n}} e^{2\pi ij\,\frac{k}{n}}
\end{aligned}
$$

So we get

$$D_+ e_j = \left(e^{2\pi ij\,\frac{1}{n}} - 1\right) e_j$$
$$D_- e_j = \left(1 - e^{-2\pi ij\,\frac{1}{n}}\right) e_j$$

□

Theorem 2.2: D_+ and D_- are diagonal in the basis

$$e_j = \left(e^{2\pi ij\,\frac{1}{n}}, ..., e^{2\pi ij\,\frac{k}{n}}, ..., e^{2\pi ij\,\frac{n}{n}}\right) \qquad j = 1, ..., n$$

The eigenvalues of D_+ are $e^{2\pi ij\,\frac{1}{n}} - 1$, $j = 1, ..., n$ and the eigenvalues of D_- are $1 - e^{-2\pi ij\,\frac{1}{n}}$, $j = 1, ..., n$. So this basis diagonalizes the differentiation. □

Theorem 2.3: Finite Fourier Theorem
Let be $f = (f_1, ..., f_n) \in \mathbb{C}^n$. Set $\widehat{f_j} := \langle f, e_j \rangle_{\mathbb{C}^n}$ for $j = 1, ..., n$.

$$\Rightarrow f = \sum_{j=1}^{n} \widehat{f}_j e_j$$

In particular, for $k = 1, ..., n$

$$f_k = \sum_{j=1}^{n} \frac{1}{\sqrt{n}}\, \widehat{f}_j e^{2\pi ij\,\frac{k}{n}}$$

□

Definition 2.5: Define the matrix

$$F := \begin{pmatrix} F_{11} & \dots & F_{1n} \\ \vdots & & \vdots \\ F_{n1} & \dots & F_{nn} \end{pmatrix}$$

with entries

$$F_{jk} = \frac{1}{\sqrt{n}}\, e^{-2\pi ij\,\frac{k}{n}}$$

So

$$Ff = \widehat{f} \qquad \widehat{f} = (\widehat{f}_1, ..., \widehat{f}_n)$$

where F is called the *Finite Fourier Transform.* □

Proposition 2.4:

$$F^\dagger = F^{-1}$$

i.e. F is unitary. □

Exercise 2.2:

$$F^4 = \mathbf{1}$$

□

Remark:

$$\begin{aligned} Fv &= \lambda v \\ F^2 v &= \lambda F v = \lambda^2 v \\ F^3 v &= \lambda^2 F v = \lambda^3 v \\ F^4 v &= \lambda^4 v = v \end{aligned}$$

This implies

$$\lambda^4 = 1 \quad \Rightarrow \quad \lambda \in \{\pm 1, \pm i\}$$

□

Example 2.3: Wave propagation in a chain of identical ions

⊠

□

Proposition 2.5: Suppose n is even. Then

$$\begin{aligned} f_k &= \sum_{j=-\frac{n}{2}+1}^{\frac{n}{2}-1} \widehat{f}_j \frac{1}{\sqrt{n}} e^{2\pi i j \frac{k}{n}} + \frac{1}{\sqrt{n}} \left\langle f, e_{n/2} \right\rangle e^{2\pi i \frac{n}{2} \frac{k}{n}} \\ &= \sum_{j=-\frac{n}{2}+1}^{\frac{n}{2}-1} \widehat{f}_j \frac{1}{\sqrt{n}} e^{2\pi i j \frac{k}{n}} + (-1)^k \frac{1}{\sqrt{n}} \left\langle f, e_{n/2} \right\rangle \end{aligned}$$

where $\widehat{f}_j = \langle f, e_j \rangle$. □

Proof: We use $e_{n+j} = e_j$

$$\begin{aligned} f_k &= \sum_{j=1}^{n} \widehat{f}_j \frac{1}{\sqrt{n}} e^{2\pi i j \frac{k}{n}} \\ &= \sum_{j=1}^{\frac{n}{2}} \widehat{f}_j \frac{1}{\sqrt{n}} e^{2\pi i j \frac{k}{n}} + \sum_{j=\frac{n}{2}+1}^{n} \widehat{f}_j \frac{1}{\sqrt{n}} e^{2\pi i j \frac{k}{n}} \\ &= \sum_{j=1}^{\frac{n}{2}} \widehat{f}_j \frac{1}{\sqrt{n}} e^{2\pi i j \frac{k}{n}} + \sum_{j=-\frac{n}{2}+1}^{0} \widehat{f}_{n+j} \frac{1}{\sqrt{n}} e^{2\pi i (n+j) \frac{k}{n}} \\ &= \sum_{j=1}^{\frac{n}{2}-1} \widehat{f}_j \frac{1}{\sqrt{n}} e^{2\pi i j \frac{k}{n}} + \widehat{f}_{\frac{n}{2}} \frac{1}{\sqrt{n}} e^{2\pi i \frac{n}{2} \frac{k}{n}} \end{aligned}$$

and then we can write

$$e^{2\pi i \frac{n}{2} \frac{k}{n}} = e^{k\pi i} = (-1)^k$$

(QED)

Definition 2.6: The 1 dimensional discrete *Laplace operator* Δ is defined by

$$\Delta := D_- D_+ = \begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 1 \\ 1 & -2 & 1 & & & \\ 0 & 1 & -2 & & & \\ \vdots & & & \ddots & & 1 \\ 1 & & \dots & & 1 & -2 \end{pmatrix}$$

□

Exercise 2.3: Compute this matrix by yourself, i.e. find matrices for D_- and D_+. □

Remark: The Laplace operator on $\mathbb{R}^n$ is

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_1^2}$$

□

Proposition 2.6:

$$F\Delta F^{-1} \;=\; \begin{pmatrix} -4\sin^2\frac{\pi}{n} & & & & 0 \\ & -4\sin^2\frac{2\pi}{n} & & & \\ & & -4\sin^2\frac{3\pi}{n} & & \\ & & & \ddots & \\ 0 & & & & -4\sin^2\frac{n\pi}{n} \end{pmatrix}$$

and therefore for $j = 1, ..., n$

$$\Delta e_j \;=\; \left(-4\sin^2\frac{j\pi}{n}\right) e_j$$

□

Exercise 2.4: Prove this Proposition. □

Example 2.4: Waves in a chain of identical ions

⊠

Let $1, ..., n$ be the equilibrium positions of n identical ions of mass M. Now consider small oscillations u_k, so $k + u_k$ is the position on the line of the ion that oscillates around k. The springs have constant K and $u_{n+1} = u_1$.

The potential energy is

$$U(u_1, ..., u_n) \;=\; \frac{K}{2}\sum_{k=1}^{n}(u_k - u_{k+1})^2$$

The equations of motion are

$$\begin{aligned} M\ddot{u}_\ell &= -\frac{\partial}{\partial u_\ell} U \\ &= -\frac{\partial}{\partial u_\ell}\frac{K}{2}\left((u_\ell - u_{\ell+1})^2 + (u_{\ell-1} + u_\ell)^2\right) \\ &= K\left(u_{\ell+1} + u_{\ell-1} - 2u_\ell\right) \end{aligned}$$

Set the boundary values $u_k(0) = u_{0k}$, $\dot{u}_k(0) = v_k$, where

$$u_0 \;=\; \begin{pmatrix} u_{01} \\ \vdots \\ u_{0n} \end{pmatrix} \qquad v \;=\; \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad u \;=\; \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

are given. So we have the boundary problem

$$M\ddot{u} \;=\; K\Delta u \qquad u(0) \;=\; u_0 \qquad \dot{u}(0) \;=\; v \tag{2.1}$$

where Δ is the 1 dimensional discrete Laplace operator.

Try to solve this with the finite fourier transform with the convention

$$\widehat{f}_j \;=\; (Ff)_j \;=\; \sum_{k=1}^{n} f_k \frac{1}{\sqrt{n}} e^{-2\pi i\,\frac{jk}{n}}$$

so

$$(F\Delta f)_j \;=\; (F\Delta F^{-1}Ff)_j \;=\; -4\sin^2\left(\frac{\pi j}{n}\right)(Ff)_j$$

(2.1) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Fu)_j \;=\; -\frac{4K}{M}\sin^2\left(\frac{\pi j}{n}\right)(Fu)_j$$

and the general solution we get over

$$(Fu)_j = a_j \cos(\omega_j t) + b_j \sin(\omega_j t)$$

with general a_j, b_j and

$$\omega_j = 2\sqrt{\frac{K}{M}}\left|\sin\left(\frac{\pi j}{n}\right)\right|$$

So we get

$$u_k(t) = (F^{-1}Fu(t))_k = \sum_{j=1}^{n}(a_j \cos\omega_j t + b_j \sin\omega_j t)\,\frac{1}{\sqrt{n}}\,e^{2\pi i\,\frac{jk}{n}}$$

or vectorial

$$u(t) = \sum_{j=1}^{n}(a_j \cos\omega_j t + b_j \sin\omega_j t)\,e_j$$

which finally is the general solution of (2.1).

Now we look at the boundaries

$$\begin{aligned} u_0 &= u(0) = \sum_{j=1}^{n} a_j e_j = F^{-1}a \\ v &= \dot{u}(0) = \sum_{j=1}^{n} b_j\omega_j e_j = F^{-1}\begin{pmatrix} b_1\omega_1 \\ \vdots \\ b_n\omega_n \end{pmatrix} \end{aligned}$$

and we can write

$$a_j = (Fu_0)_j = \langle u_0, e_{-j}\rangle \qquad b_j\omega_j = (Fv)_j = \langle v, e_{-j}\rangle$$

So, the boundary problem has the solution

$$\begin{aligned} u(t) &= \sum_{j=1}^{n}\left(\langle u_0, e_{-j}\rangle \cos\omega_j t + \langle v, e_{-j}\rangle\,\frac{1}{\omega_j}\,\sin\omega_j t\right) e_j \\ u_k(t) &= \sum_{j=1}^{n}\left(\langle u_0, e_{-j}\rangle \cos\omega_j t + \langle v, e_{-j}\rangle\,\frac{1}{\omega_j}\,\sin\omega_j t\right)\frac{1}{\sqrt{n}}\,e^{2\pi i\,\frac{jk}{n}} \end{aligned}$$

Note that $\omega_{n-j} = \omega_j$, because

$$\begin{aligned} \omega_{n-j} &= 2\sqrt{\frac{K}{M}}\left|\sin\left(\frac{\pi(n-j)}{n}\right)\right| = 2\sqrt{\frac{K}{M}}\left|\sin\left(\pi - \frac{\pi j}{n}\right)\right| \\ &= 2\sqrt{\frac{K}{M}}\left|\sin\left(\frac{\pi j}{n}\right)\right| \\ &= \omega_j \end{aligned}$$

Consider a practical boundary problem

$$u_{0k} = \varepsilon\delta_{k\ell} \qquad v_k = 0 \qquad k = 0, ..., n$$

We need

$$\begin{aligned} \langle u_0, e_{-j}\rangle &= \sum_{k=1}^{n}\varepsilon\delta_{k\ell}\,\frac{1}{\sqrt{n}}\,e^{-2\pi i\,\frac{jk}{n}} \\ &= \frac{\varepsilon}{\sqrt{n}}\,e^{-2\pi i\,\frac{j\ell}{n}} \\ \langle v, e_{-j}\rangle &= 0 \\ \Rightarrow u_k(t) &= \sum_{j=1}^{n}\frac{\varepsilon}{\sqrt{n}}\,e^{-2\pi i\,\frac{j\ell}{n}}\cos(\omega_j t)\,\frac{1}{\sqrt{n}}\,e^{2\pi i\,\frac{jk}{n}} \\ &= \frac{\varepsilon}{n}\sum_{j=1}^{n}\cos(\omega_j t)\,e^{2\pi i j\,\frac{k-\ell}{n}} \end{aligned}$$

Let be n even, so $n = 2m$, $m \in \mathbb{N}$

$$\begin{aligned} u_k(t) &= \frac{\varepsilon}{n} \sum_{j=1}^{m} \cos(\omega_j t)\, e^{2\pi i j \,\frac{k-\ell}{n}} + \frac{\varepsilon}{n} \sum_{j=0}^{m-1} \cos(\omega_{n-j} t)\, e^{2\pi i (n-j) \frac{k-\ell}{n}} \\ &= \frac{\varepsilon}{n} \left(\cos(\omega_m t)\, e^{2\pi i m \,\frac{k-\ell}{n}} + \cos(\omega_n t)\, e^{-2\pi i 0 \,\frac{k-\ell}{n}} \right) \\ &\quad + \frac{\varepsilon}{n} \sum_{j=1}^{m-1} \cos(\omega_j t) \left(e^{2\pi i j \,\frac{k-\ell}{n}} + e^{-2\pi i j \,\frac{k-\ell}{n}} \right) \\ &= \frac{\varepsilon}{n} \left((-1)^{k-\ell} \cos\left(2\sqrt{\frac{K}{M}}\, t \right) + 1 \right) \\ &\quad + 2\,\frac{\varepsilon}{n} \sum_{j=1}^{m-1} \cos(\omega_j t) \cos\left(2\pi j\, \frac{k-\ell}{n} \right) \end{aligned}$$

An other method would be to separate the variables. For this, make the assumption

$$u(t) = \phi(t) f \qquad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in \mathbb{R}^n$$

After putting this into (2.1), we get

$$M \ddot{\phi}(t) f = K \phi(t) \Delta f$$

If $\phi(t) \neq 0$, this means

$$\Delta f = \frac{M}{K} \frac{\ddot{\phi}(s)}{\phi(s)}\, f$$

This implies, that f is an eigenvector and the term before it is an eigenvalue

$$\lambda = \frac{M}{K} \frac{\ddot{\phi}(s)}{\phi(s)}$$

This produces an ordinary differential equation

$$\ddot{\phi}(t) = \frac{\lambda K}{M}\, \phi(t)$$

But we already know, what the eigenvalues and eigenvectors of Δ are. The general solution of the ordinary differential equation is

$$\phi_j(t) = a_j \cos \omega_j t + b_j \sin \omega_j t \qquad \omega_j = 2\sqrt{\frac{K}{M}} \left| \sin\left(\frac{\pi j}{n} \right) \right|$$

The solutions, coming from the assumption, are

$$\phi_j(t) e_j$$

and the space of the solutions is a vector space, so

$$\sum_{j=1}^{n} \phi_j(t) e_j$$

as well is a solution.

To generalize this, take a matrix $A \in \mathbb{R}^{n \times n}$, which is self adjoint with respect to an inner product $(\cdot, \cdot)$. Let be $\mu_1, ..., \mu_n$ the eigenvalues of A and $f_1, ..., f_n$ their $(\cdot, \cdot)$-Orthonormalbasis of eigenvectors. The solution of the boundary problem

$$\ddot{u} = Au$$

with $u(0) = u_0$, $\dot{u}(0) = v$ is

$$u(t) = \sum_{j=1}^{n} \left((u_0, f_j) \frac{e^{\sqrt{\mu_j}\, t} + e^{-\sqrt{\mu_j}\, t}}{2} + (v, f_j) \frac{e^{\sqrt{\mu_j}\, t} - e^{-\sqrt{\mu_j}\, t}}{2\sqrt{\mu_j}\, i} \right) f_j$$

□

Example 2.5: Wave propagation in a diatomic chain of ions
Let be $n = 2\nu$, where ν is an odd number. Now consider ν identical ions of mass M at the odd points $1, 3, ..., n-1$ and ν identical ions of mass m at the even points $2, 4, ..., n$. As before, let be u_k the displacement from the equilibrium, so

$$\begin{aligned}
U &= \frac{K}{2}\sum_{k=1}^{n}(u_k - u_{k+1})^2 + \frac{K}{2}\,n \\
M\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,u_1 &= -K\,(2u_1 - u_n - u_2) \\
m\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,u_2 &= -K\,(2u_2 - u_1 - u_3) \\
M\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,u_3 &= -K\,(2u_3 - u_2 - u_4) \\
&\vdots \\
m\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,u_n &= -K\,(2u_n - u_{n-1} - u_1)
\end{aligned}$$

We write this with the matrix

$$\rho = \begin{pmatrix} M & & & & & 0 \\ & m & & & & \\ & & M & & & \\ & & & m & & \\ & & & & \ddots & \\ 0 & & & & & m \end{pmatrix}$$

and the Laplace-operator Δ to get the boundary problem

$$\begin{aligned}
\rho\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,u &= K\Delta u \qquad u(0) = u_0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,u(0) = v \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,u &= K\rho^{-1}\Delta u =: KAu
\end{aligned}$$

with $u = (u_1, ..., u_n)$. Define the inner product

$$\langle v, w\rangle_\rho := \langle \rho v, w\rangle = Mv_1w_1 + mv_2w_2 + ... + mv_nw_n$$

$A := \rho^{-1}\Delta$ is self adjoint with respect to $\langle\cdot,\cdot\rangle_\rho$, that is

$$\langle Av, w\rangle_\rho = \langle v, Aw\rangle_\rho$$

We make the ansatz $\rho^{-1}\Delta f_j = \lambda_j f_j$, for $j = 1, ..., n$, where

$$f_j = \left(Ae^{2\pi ij\,\frac{1}{n}}, ae^{i\pi ij\,\frac{2}{n}}, Ae^{2\pi ij\,\frac{3}{n}}, ..., A^{2\pi ij\,\frac{n-1}{n}}, ae^{2\pi ij\,\frac{n}{n}}\right)$$

We see for $k = 1, ..., \nu = \frac{n}{2}$

$$\begin{aligned}
M^{-1}\left(ae^{2\pi ij\,\frac{2k}{n}} + ae^{2\pi ij\,\frac{2k-2}{n}} - 2Ae^{2\pi ij\,\frac{2k-1}{n}}\right) &= \lambda Ae^{2\pi ij\,\frac{2k-1}{n}} \\
m^{-1}\left(Ae^{2\pi ij\,\frac{2k+1}{n}} + Ae^{2\pi ij\,\frac{2k-1}{n}} - 2ae^{2\pi ij\,\frac{2k}{n}}\right) &= \lambda ae^{2\pi ij\,\frac{2k}{n}}
\end{aligned}$$

So we write

$$\begin{pmatrix} -2M^{-1} & 2M^{-1}\cos\left(2\pi\,\frac{j}{n}\right) \\ 2m^{-1}\cos\left(2\pi\,\frac{j}{n}\right) & -2m^{-1} \end{pmatrix}\begin{pmatrix} A \\ a \end{pmatrix} = \lambda\begin{pmatrix} A \\ a \end{pmatrix}$$

The eigenvalues are

$$\lambda_{j\pm} = -\frac{M+m}{Mm} \pm \frac{1}{Mm}\left((M-m)^2 + 4Mm\cos^2\left(\pi\,\frac{j}{\nu}\right)\right)^{1/2}$$

and $\forall j = 1, ..., n$

$$0 \;<\; \cos^2\left(\pi\,\frac{j}{\nu}\right) \;\leq\; \cos^2\left(\pi\,\frac{\nu}{\nu}\right) \;=\; 1$$

On the other hand

$$\begin{aligned} 0 \;&<\; (M-m)^2 + 4Mm\cos^2\left(\pi\,\frac{j}{\nu}\right) \\ &\leq\; (M-m)^2 + 4Mm \;=\; (M+m)^2 \end{aligned}$$

so we can write

$$\lambda_{j-} \;<\; -\frac{M+m}{Mm} \;\leq\; \lambda_{j+} \;\leq\; 0$$

and using

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{22}-\lambda \\ -a_{21} \end{pmatrix} \;=\; \lambda \begin{pmatrix} a_{22}-\lambda \\ -a_{21} \end{pmatrix}$$

we get

$$\begin{pmatrix} A \\ a \end{pmatrix} \;=\; \begin{pmatrix} 2+m\lambda_{j\pm} \\ 2\cos\left(\pi\,\frac{j}{\nu}\right) \end{pmatrix}$$

□

Proposition 2.7: Let be

$$\rho \;:=\; \begin{pmatrix} M & & & & 0 \\ & m & & & \\ & & M & & \\ & & & \ddots & \\ 0 & & & & m \end{pmatrix}$$

and Δ the Laplace operator. The matrix $\rho^{-1}\Delta$ is a self adjoint matrix with respect to $\langle v, w\rangle_\rho$ with Eigenvalues $\lambda_{j\pm}$. These eigenvalues satisfy $\forall i, j = 1, ..., n$

$$\lambda_{j-} \;<\; \lambda_{j+} \;\leq\; 0$$

and the eigenvectors are

$$f_{j\pm} \;=\; \left(A_{j\pm}e^{2\pi i j\,\frac{1}{n}}, a_{j\pm}e^{2\pi i j\,\frac{2}{n}}, ...\right)$$

with

$$\begin{aligned} A_{j\pm} \;&=\; \frac{1}{\sqrt{\nu}}\,\frac{2+m\lambda_{j\pm}}{\sqrt{M\left(2+m\lambda_{j\pm}\right)^2 + m\left(2\cos\left(\pi\,\frac{j}{\nu}\right)\right)^2}} \\ a_{j\pm} \;&=\; \frac{2\cos\left(\pi\,\frac{j}{\nu}\right)}{\sqrt{\nu}\sqrt{M\left(2+m\lambda_{j\pm}\right)^2 + m\left(2\cos\left(\pi\,\frac{j}{\nu}\right)\right)^2}} \end{aligned}$$

and $f_{j\pm}$ for $j = 1, ..., \nu$ is an orthonormal basis of $\mathbb{R}^n$ with respect to $\langle v, w\rangle_\rho$. □

Observe that

$$\nu \;=\; 2\left[\frac{\nu}{2}\right] + 1 \qquad \lambda_{j\pm} \;=\; \lambda_{\nu-j\pm}$$

and

$$\begin{aligned} 0 \;&=\; \lambda_{\nu+} \;>\; \lambda_{1+} \;>\; \lambda_{2+} \;>\; ... \;>\; \lambda_{[\frac{\nu}{2}]+} \\ -2\,\frac{M+m}{Mm} \;&=\; \lambda_{\nu-} \;<\; \lambda_{1-} \;<\; \lambda_{2-} \;<\; ... \;<\; \lambda_{[\frac{\nu}{2}]-} \end{aligned}$$

Theorem 2.4: The solution to the equation

$$\rho \frac{\mathrm{d}^2}{\mathrm{d}t^2} u = K\Delta u \qquad u(0) = u_0 \qquad \frac{\mathrm{d}}{\mathrm{d}t} u(0) = v$$

is given by

$$u(t) = \sum_{\substack{j=1,\ldots,\nu \\ \sigma=\pm}} \left(\langle u_0, f_{j\sigma} \rangle_\rho \cos(\omega_{j\sigma} t) + \langle v, f_{j\sigma} \rangle_\rho \frac{\sin(\omega_{j\sigma} t)}{\omega_{j\sigma}} \right) f_{j\sigma}$$

□

2.2 Infinite Fourier Series

Let be $f \in C^\infty_{\text{per}}[0, 2\pi)$. Define the j-th Fourier coefficient of f by $\widehat{f}(j) := \left\langle f, e^{ijx} \right\rangle$, where $j \in \mathbb{Z}$ and

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(x) g^\star(x) \,\mathrm{d}x$$

The (formal) Fourier series associated to f is

$$\sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ijx}$$

Now we could ask

1) Does

$$\sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ijx}$$

converge in any useful sense?

2) If it does, is

$$f(x) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ijx}$$

Look at

$$|\widehat{f}_{(j)} e^{ijx}| = |\widehat{f}(j)| \, |e^{ijx}| = |\widehat{f}(j)|$$

We can say, that

$$\sum_{j \in \mathbb{Z}} |\widehat{f}(j)|$$

converges, when

$$|\widehat{f}(j)| < \frac{1}{|j|^{1+\varepsilon}}$$

This is sufficient but not necessary. Why should $|\widehat{f}(j)|$ be small for large $|j|$?

$$\begin{aligned} \widehat{f}(j) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ijx} \,\mathrm{d}e \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos jx \,\mathrm{d}x - i \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin jx \,\mathrm{d}x \end{aligned}$$

Remember

$$\begin{aligned} \frac{1}{i} \frac{\mathrm{d}}{\mathrm{d}x} e^{ijx} = j e^{ijx} \quad &\Rightarrow \quad \frac{1}{ij} \frac{\mathrm{d}}{\mathrm{d}x} e^{ijx} = e^{ijx} \\ &\Rightarrow \quad \left(\frac{1}{ij} \frac{\mathrm{d}}{\mathrm{d}x} \right)^\ell e^{ijx} = e^{ijx} \end{aligned}$$

and for $f, g \in C^\infty_{\text{per}}[0, 2\pi)$ we had

$$\begin{aligned} \left\langle f, \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\, g \right\rangle &= \left\langle \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\, f, g \right\rangle \\ \Rightarrow \left\langle f, \left(\frac{1}{ij}\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell e^{ijx} \right\rangle &= \frac{1}{j^\ell}\left\langle f, \left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell e^{ijx} \right\rangle \\ &= \frac{1}{j^\ell}\left\langle \left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell f, e^{ijx} \right\rangle \end{aligned}$$

So we can write

$$\begin{aligned} |\widehat{f}(j)| &= \frac{1}{j^\ell}\left|\left\langle \left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell f, e^{ijx} \right\rangle\right| \\ &\leq \frac{1}{j^\ell}\left\| \left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell f \right\|_2 \|e^{ijx}\|_2 \\ &= \frac{1}{j^\ell}\left\| \left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell f \right\|_2 \end{aligned}$$

where

$$\|f\|_2 \;:=\; \sqrt{\langle f, f\rangle}$$

Proposition 2.8: Let be $f \in C^\ell_{[0,2\pi)}(\mathbb{R})$.

$$\Rightarrow |\widehat{f}(j)| \;\leq\; \frac{1}{j^\ell}\left\| \left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell f \right\|_2 \stackrel{j\to\infty}{\longrightarrow} 0$$

□

Remark: One could ask, why the operator $\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}$ exploits the oscillation. We had

$$\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\, e^{ijx} \;=\; je^{ijx}$$

so the operator is self adjoint

$$\Rightarrow \left\langle e^{ijx}, e^{ikx} \right\rangle \;=\; \delta_{jk}$$

and we write

$$\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\, f_j \;=\; \lambda_j f_j \quad\Rightarrow\quad \langle f_j, f_k\rangle \;=\; 0, \quad \lambda_j \;\neq\; \lambda_k$$

□

Remark: We have Parseval's identity

$$\|f\|_2^2 \;=\; \sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2$$

this means, that if $\|f\|_2 < \infty$, the infinite series is finite. Suppose $|f(x)|^2$ is Riemann integrable

$$\Rightarrow \|f\|_2^2 \;=\; \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,\mathrm{d}x \;<\; \infty$$

□

We showed, that for $f \in C^\ell_{[0,2\pi)}(\mathbb{R})$

$$|\widehat{f}(j)| \;\leq\; \frac{1}{j^\ell}\left\| \left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell f \right\|_2$$

Example 2.6:

$$S(x) = \begin{cases} 1 & 0 \le x < \pi \\ -1 & \pi \le x < 2\pi \end{cases}$$

$$\boxtimes$$

$$\widehat{S}(0) = \left\langle S(x), e^{i0x} \right\rangle = \langle S(x), 1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} S(x)\,\mathrm{d}x = 0$$

Claim for $j \neq 0$

$$\widehat{S}(j) = \begin{cases} 0 & j \text{ even} \\ \frac{2}{\pi i j} & j \text{ odd} \end{cases}$$

This converges like $\frac{1}{j}$, even if there is not really a derivative. But there is derivatives if you take away the jumping points. $\square$

Example 2.7: The tent function

$$t(x) := \int_0^x S(y)\,\mathrm{d}y = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi \end{cases}$$

$$\boxtimes$$

We get

$$\widehat{t}(0) = \frac{\pi}{2}$$

$$\widehat{t}(j) = \begin{cases} 0 & j \text{ even} \\ \frac{2}{\pi (ij)^2} & j \text{ odd} \end{cases}$$

$\square$

Example 2.8: Fix $n \ge 1$

$$\delta_n := \begin{cases} n & 0 \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} \le x \le 2\pi \end{cases}$$

$$\boxtimes$$

$$\int_0^{2\pi} \delta_n(x)\,\mathrm{d}x = 1$$

This is a periodic δ-Funktion.

$$\widehat{\delta_n}(0) = \frac{1}{2\pi}$$

$$\widehat{\delta_n}(j) = \frac{1}{2\pi}\,\frac{n}{ij}\left(1 - \cos\left(j\,\frac{1}{n}\right) + i\sin\left(j\,\frac{1}{n}\right)\right)$$

One has to wait, till j gets bigger then n. $\square$

Theorem 2.5: Isoperimetric Inequality
Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ for $0 \le t \le 2\pi$ be a simple closed plane curve such that $\gamma_1(t), \gamma_2(t) \in C^1_{\text{per}}[0, 2\pi)$. Suppose that length$(\gamma) = 2\pi$, i.e.

$$\text{length}(\gamma) = \int_0^{2\pi} \left| \frac{\mathrm{d}}{\mathrm{d}t}\,\gamma(t) \right| \mathrm{d}t = \int_0^{2\pi} \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\gamma_1(t) \right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\gamma_2(t) \right)^2 \right)^{1/2} \mathrm{d}t$$

Then area$(\gamma) \le \pi$ with equality if γ is a circle of radius 1.

$$\boxtimes$$

$\square$

Proof: Define

$$\alpha(s) \;:=\; \int_0^s \left|\frac{\mathrm{d}}{\mathrm{d}t}\,\gamma(t)\right| \mathrm{d}t \quad\Rightarrow\quad \alpha(2\pi) \;=\; 2\pi \;=\; \mathrm{length}(\gamma)$$

where $\alpha : [0, 2\pi] \to [0, 2\pi]$ is monotone increasing. Now parametrize by arclength using $\alpha^{-1} : [0, 2\pi] \to [0, 2\pi]$ and defining

$$\overline{\gamma}(\theta) \;:=\; \gamma(\alpha^{-1}(\theta))$$

Claim 1: $\forall 0 \le \theta \le 2\pi$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\,\overline{\gamma}_1(\theta)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\,\overline{\gamma}_2(\theta)\right)^2 \;=\; 1$$

(QED)

Theorem 2.6: Green

$$\int_{\partial\Omega} f\,\mathrm{d}x + g\,\mathrm{d}y \;=\; \int_{\Omega} \left(-\frac{\partial}{\partial y}\,f + \frac{\partial}{\partial x}\,g\right) \mathrm{d}x\mathrm{d}y$$

□

Look at a general complex way-integral

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z \;&=\; \int_\gamma (u(x,y) + iv(x,y))\,(\mathrm{d}x + i\mathrm{d}y) \\
&=\; \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y \\
\int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y \;&=\; \int_\Omega -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \;=\; 0 \\
\int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y \;&=\; \int_\gamma -\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \;=\; 0
\end{aligned}$$

It follows

$$\frac{\partial u}{\partial y} \;=\; -\frac{\partial v}{\partial x} \qquad \frac{\partial v}{\partial y} \;=\; \frac{\partial u}{\partial x}$$

they are equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}z}\,f(z) \;=\; \lim_{h\to 0}\frac{f(z+h) - f(z)}{h}$$

2.3 Some Integrals

Proposition 2.9:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x \;=\; \sqrt{2\pi}$$

□

Proof:

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x\right)^2 &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} e^{-\frac{1}{2}\left(x^2+y^2\right)}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{1}{2}r^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\pi \int_0^{\infty} e^{-\frac{1}{2}r^2} r\,\mathrm{d}r \\ &= 2\pi \left(-e^{-\frac{1}{2}r^2}\right)\Big|_0^{\infty} \\ &= 2\pi\end{aligned}$$

(QED)

Proposition 2.10: For $\lambda > 0$, we have

$$\int_{-\infty}^{\infty} e^{-\frac{\lambda}{2}x^2}\,\mathrm{d}x = \sqrt{\frac{2\pi}{\lambda}}$$

□

Proof: Substitute $y = \sqrt{\lambda}\,x$

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\frac{\lambda}{2}x^2}\,\mathrm{d}x &= \frac{1}{\sqrt{\lambda}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,\mathrm{d}y \\ &= \frac{\sqrt{2\pi}}{\sqrt{\lambda}}\end{aligned}$$

(QED)

Definition 2.7: For $s > 0$ define the *Gamma-function*

$$\Gamma(s) := \int_0^{\infty} e^{-t}t^{s-1}\,\mathrm{d}t$$

□

Remark: We know

$$\begin{aligned}\Gamma(s+1) &= s\Gamma(s) \\ \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}\end{aligned}$$

□

Proposition 2.11: $\forall \lambda > 0,\ \ell \in \mathbb{N}$

$$\int_{-\infty}^{\infty} x^{2\ell+1} e^{-\frac{\lambda}{2}x^2}\,\mathrm{d}x = 0$$

$$\int_{-\infty}^{\infty} x^{2\ell} e^{-\frac{\lambda}{2}x^2}\,\mathrm{d}x = \sqrt{2\pi}\,\frac{(2\ell)!}{2^\ell \ell!}\,\lambda^{-\frac{2\ell+1}{2}}$$

□

Proof: The first integral is 0, because $x^{2\ell+1}e^{-\frac{\lambda}{2}x^2}$ is an odd function.

$$\begin{aligned}\int_{-\infty}^{\infty} x^{2\ell} e^{-\frac{\lambda}{2}x^2}\,\mathrm{d}x &= 2\int_{0}^{\infty} x^{2\ell} e^{-\frac{\lambda}{2}x^2}\,\mathrm{d}x \\ &= 2\left(\frac{2}{\lambda}\right)^{\ell}\int_0^\infty e^{-t}t^\ell\,\frac{1}{\lambda}\left(\frac{2}{\lambda}\right)^{-1/2} t^{-1/2}\,\mathrm{d}t\end{aligned}$$

where we made the substitution

$$\frac{\lambda}{2}x^2 = t \quad\Rightarrow\quad x^{2\ell} = \left(\frac{2t}{\lambda}\right)^{\ell} \quad\Rightarrow\quad x = \left(\frac{2}{\lambda}\right)^{1/2} t^{1/2}$$

$$\mathrm{d}t = \lambda x\,\mathrm{d}x \qquad \mathrm{d}x = \frac{1}{\lambda}\left(\frac{2}{\lambda}\right)^{-1/2} t^{-1/2}\,\mathrm{d}t$$

then we get

$$\begin{aligned}\int_{-\infty}^{\infty} x^{2\ell} e^{-\frac{\lambda}{2}x^2}\,\mathrm{d}x &= \left(\frac{2}{\lambda}\right)^{\ell+\frac{1}{2}}\int_0^\infty e^{-t}t^{\ell-\frac{1}{2}}\,\mathrm{d}t \\ &= \left(\frac{2}{\lambda}\right)^{\ell+\frac{1}{2}}\int_0^\infty e^{-t}t^{\ell+\frac{1}{2}-1}\,\mathrm{d}t \\ &= \left(\frac{2}{\lambda}\right)^{\ell+\frac{1}{2}}\Gamma\left(\ell+\frac{1}{2}\right) \\ &= 2^{\ell+\frac{1}{2}}\prod_{k=0}^{\ell-1}\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\lambda^{-\frac{2\ell+1}{2}} \\ &= \sqrt{2\pi}\prod_{k=0}^{\ell-1}(2k+1)\,\lambda^{-\frac{2\ell+1}{2}} \\ &= \sqrt{2\pi}\,\frac{(2\ell)!}{2\cdot 4\cdot\ldots\cdot 2\ell}\,\lambda^{-\frac{2\ell+1}{2}} \\ &= \sqrt{2\pi}\,\frac{(2\ell)!}{2^\ell \ell!}\,\lambda^{-\frac{2\ell+1}{2}}\end{aligned}$$

We could also use

$$x^{2\ell}e^{-\frac{\lambda}{2}x^2} = (-2)^\ell\,\frac{\partial^\ell}{\partial\lambda^\ell}\,e^{-\frac{\lambda}{2}x^2}$$

to calculate

$$\begin{aligned}\int_{-\infty}^{\infty} x^{2\ell} e^{-\frac{\lambda}{2}x^2}\,\mathrm{d}x &= (-2)^\ell \int_{-\infty}^{\infty} \frac{\partial^\ell}{\partial\lambda^\ell}\, e^{-\frac{\lambda}{2}x^2}\,\mathrm{d}x \\ &= (-2)^\ell\, \frac{\partial^\ell}{\partial\lambda^\ell} \int_{-\infty}^{\infty} e^{-\frac{\lambda}{2}x^2}\,\mathrm{d}x \\ &= (-2)^\ell\, \frac{\partial^\ell}{\partial\lambda^\ell} \sqrt{\frac{2\pi}{\lambda}} \\ &= \sqrt{2\pi}\,(-2)^\ell\, \frac{\mathrm{d}^\ell}{\mathrm{d}\lambda^\ell}\, \lambda^{-1/2}\end{aligned}$$

Claim:

$$\sqrt{2\pi}\,(-2)^\ell\, \frac{\mathrm{d}^\ell}{\mathrm{d}\lambda^\ell}\, \lambda^{-1/2} = \sqrt{2\pi}\, \frac{(2\ell)!}{2^\ell \ell!}\, \lambda^{-\frac{2\ell+1}{2}}$$

(QED)

Exercise 2.5: Prove

$$\sqrt{2\pi}\,(-2)^\ell\, \frac{\mathrm{d}^\ell}{\mathrm{d}\lambda^\ell}\, \lambda^{-1/2} = \sqrt{2\pi}\, \frac{(2\ell)!}{2^\ell \ell!}\, \lambda^{-\frac{2\ell+1}{2}}$$

□

Proposition 2.12: $\forall k \in \mathbb{R}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-ikx}\,\mathrm{d}x = e^{-\frac{1}{2}k^2}$$

□

Proof:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-ikx}\,\mathrm{d}x &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \sum_{j=0}^{\infty} \frac{(-ikx)^j}{j!}\,\mathrm{d}x \\ &= \sum_{j=0}^{\infty} \frac{(-ik)^j}{j!} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} x^j\,\mathrm{d}x \\ &= \sum_{\ell=0}^{\infty} \frac{(-ik)^{2\ell}}{(2\ell)!} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} x^{2\ell}\,\mathrm{d}x \\ &= \sum_{\ell=0}^{\infty} \frac{(-ik)^{2\ell}}{(2\ell)!} \sqrt{2\pi}\, \frac{(2\ell)!}{2^\ell \ell!} \\ &= \sqrt{2\pi} \sum_{\ell=0}^{\infty} \left(-k^2\right)^\ell \frac{1}{2^\ell \ell!} \\ &= \sqrt{2\pi}\, e^{-\frac{1}{2}k^2}\end{aligned}$$

or

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-ikx}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2+2ikx\right)}\,\mathrm{d}x$$
$$= e^{-\frac{1}{2}k^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2+2ikx-k^2\right)}\,\mathrm{d}x$$
$$= e^{-\frac{1}{2}k^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+ik)^2}\,\mathrm{d}x$$
$$= e^{-\frac{1}{2}k^2}$$

because using complex analysis, we have $\forall k \in \mathbb{R}$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+ik)^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x = \sqrt{2\pi}$$

⊠

With Cauchy (there comes in complex analysis), we know

$$\int_{\gamma} e^{-\frac{1}{2}z^2}\,\mathrm{d}z$$
$$= \int_{-R}^{R} e^{-\frac{1}{2}x^2}\,\mathrm{d}x + \underbrace{\int_{R}^{R+ik} e^{-\frac{1}{2}z^2}\,\mathrm{d}z}_{\to 0} + \int_{R+ik}^{-R+ik} e^{-\frac{1}{2}z^2}\,\mathrm{d}z + \underbrace{\int_{-R+ik}^{-R} e^{-\frac{1}{2}z^2}\,\mathrm{d}z}_{\to 0}$$
$$\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x - \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+ik)^2}\,\mathrm{d}x = 0$$

(QED)

Definition 2.8: Let $f(x)$ be a complex valued function with

$$\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x < \infty$$

then

$$\widehat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x$$

is the *Fourier transform* of $f(x)$. □

Remark:

$$|\widehat{f}(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|f(x)e^{-ikx}\right|\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x < \infty$$

□

Remark: Se can easyly derive, that

$$\widehat{f+g}(k) = \widehat{f}(k) + \widehat{g}(k)$$
$$\widehat{\lambda f}(k) = \lambda \widehat{f}(k)$$

so $\widehat{\cdot}$ is a linear operator and Proposition 2.12 becomes

$$\widehat{e^{-\frac{1}{2}x^2}}(k) = e^{-\frac{1}{2}k^2}$$

so $e^{-\frac{1}{2}x^2}$ is an eigenfunction of $\widehat{\cdot}$ with eigenvalue 1. □

Proposition 2.13: Let be $A \in M_n(\mathbb{R})$, $A^T = A$ and $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ the strictly positive eigenvalues of A.

$$\Rightarrow \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\langle x, Ax\rangle}\,\mathrm{d}x = (2\pi)^{n/2} \frac{1}{\sqrt{\det A}}$$

□

Proof: There is a $C \in SO(n)$ such that $CAC^{-1} = \mathrm{diag}(\lambda_1, ..., \lambda_n)$

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\langle x, Ax\rangle}\,\mathrm{d}x &= \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\left\langle x, C^{-1}\mathrm{diag}(\lambda_1,...,\lambda_n)Cx\right\rangle}\,\mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\left\langle x, C^{T}\mathrm{diag}(\lambda_1,...,\lambda_n)Cx\right\rangle}\,\mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Cx, \mathrm{diag}(\lambda_1,...,\lambda_n)Cx\rangle}\,\mathrm{d}x
\end{aligned}$$

change variables $x = C^{-1}y$

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\langle x, Ax\rangle}\,\mathrm{d}x &= \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\langle y, \mathrm{diag}(\lambda_1,...,\lambda_n)y\rangle}\,\mathrm{d}C^{-1}y \\
&= \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\langle y, \mathrm{diag}(\lambda_1,...,\lambda_n)y\rangle}\,\big(\det(C^{-1})\big)\,\mathrm{d}y \\
&= \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\langle y, \mathrm{diag}(\lambda_1,...,\lambda_n)y\rangle}\,\mathrm{d}y \\
&= \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\sum_{j=1}^n \lambda_j y_j^2}\,\mathrm{d}y \\
&= \int\limits_{-\infty}^{\infty} \mathrm{d}y_1 ... \int\limits_{-\infty}^{\infty} \mathrm{d}y_n \prod_{j=1}^n e^{-\frac{1}{2}\lambda_j y_j^2} \\
&= \prod_{j=1}^n \int\limits_{-\infty}^{\infty} e^{-\frac{\lambda_j}{2}y_j^2}\,\mathrm{d}y_j = \prod_{j=1}^n \sqrt{\frac{2\pi}{\lambda_j}} \\
&= \frac{(2\pi)^{n/2}}{\sqrt{\prod_{j=1}^n \lambda_j}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}
\end{aligned}$$

(QED)

2.4 Plane waves and the Fourier Transform

Definition 2.9: For each $k \in \mathbb{R}^n$, $f(x) = e^{i\langle k, x\rangle}$ is called a *plane wave*. □

Proposition 2.14: Set $P_s = \{x \in \mathbb{R}^n \mid \langle k', x\rangle = s\}$, where $k' = k/|k|$, $|k| = \rho$, $k = \rho k'$. P_0 is the orthogonal complement of k.

$$\Rightarrow P_s \quad = \quad P_0 + sk' \quad := \quad \{y + sk' \mid y \in P_0\}$$

and

$$e^{i\langle k,n\rangle}\Big|_{k\in P_s} = e^{i\rho s} = \cos(\rho s) + i\sin(\rho s)$$

$\boxtimes$

□

Proof: Take $z \in P_s$ and set $y = z - s\cdot k'$. Then

$$\begin{aligned} \langle y, k'\rangle &= \langle z - s\cdot k', k'\rangle = \langle z, k'\rangle - s\langle k', k'\rangle \\ &= s - s = 0 \end{aligned}$$

$\Leftrightarrow y \in P_0$. And

$$e^{i\langle k,z\rangle} \quad = \quad e^{i\left\langle \rho k', y + s\cdot k'\right\rangle} \quad = \quad e^{i\rho s\left\langle k',k'\right\rangle} \quad = \quad e^{i\rho s}$$

(QED)

Remark: The sum

$$\sum_{j=1}^{N} c_j e^{i\langle k_j, x\rangle}$$

is a finite linear combination of plane waves. Then we say

$$\int_{\mathbb{R}^n} g(k) e^{i\langle k,x\rangle}\,\mathrm{d}k$$

is the most general linear combination of plane waves. □

Definition 2.10: Let $f(x)$ be a complex valued function on $\mathbb{R}^n$ with

$$\int_{\mathbb{R}^n} |f(x)|\,\mathrm{d}x \quad < \quad \infty$$

The *Fourier Transform* of f is given by

$$\widehat{f}(k) \quad = \quad \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} f(x) e^{-i\langle k,x\rangle}\,\mathrm{d}x$$

□

Lemma 2.1: $\widehat{f}(k)$ is uniformly bounded on $\mathbb{R}^n$ □

Proof:

$$\begin{aligned} \left|\widehat{f}(k)\right| &= \left|\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} f(x) e^{-i\langle k,x\rangle}\,\mathrm{d}x\right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} |f(x)|\,\mathrm{d}x \\ &\leq \infty \end{aligned}$$

(QED)

Theorem 2.7:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}^n} \widehat{f}(k) e^{i\langle k,x\rangle} \, \mathrm{d}k$$

This is not completely true. We need some additional assumptions (Fourier representation for f). □

Exercise 2.6: Look at the interval $[-L, L]$ and define the funktions

$$\exp\left(\frac{ik\pi}{L} x\right)$$

for $k \in \mathbb{Z}$ and

$$f(x) = \sum_{k\in\mathbb{Z}} ... \cdot \exp\left(\frac{ik\pi}{L} x\right)$$

□

Definition 2.11: The *Laplace Operator* is defined by

$$\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

□

Proposition 2.15:

$$\Delta e^{i\langle k,x\rangle} = -\|k\|^2 e^{i\langle k,x\rangle}$$

□

Definition 2.12: The *scalar product* of two functions is defined by

$$\langle \phi, \psi \rangle_{\mathbb{R}^n} = \int\limits_{\mathbb{R}^n} \phi(x) \cdot \overline{\psi(x)} \, \mathrm{d}x$$

□

Proposition 2.16:

$$\langle \Delta\phi, \psi \rangle = \langle \phi, \Delta\psi \rangle$$

□

Proof: For $n = 1$, assume, ϕ, ψ are smooth and $\phi(\pm\infty) = \psi(\pm\infty) = 0$

$$\begin{aligned}
\langle \Delta\phi, \psi \rangle &= \int\limits_{-\infty}^{\infty} \phi'' \cdot \overline{\psi} \, \mathrm{d}x \\
&= \phi'(+\infty) \cdot \phi(+\infty) - \phi'(-\infty) \cdot \psi(-\infty) - \int\limits_{-\infty}^{\infty} \phi' \cdot \overline{\psi}' \, \mathrm{d}x \\
&= -\int\limits_{-\infty}^{\infty} \phi' \cdot \overline{\psi}' \, \mathrm{d}x = \int\limits_{-\infty}^{\infty} \phi \cdot \overline{\psi}'' \, \mathrm{d}x \\
&= \langle \phi, \Delta\psi \rangle
\end{aligned}$$

(QED)

Remark: $f(x) = e^{i\langle k,x\rangle}$, $\lambda = -\|k\|^2$, $\Delta f = \lambda f$

$$\int\limits_{\sqrt{-\lambda}}^{n-1} = \{k \in \mathbb{R}^n : \|k\|^2 = -\lambda\}$$

□

Definition 2.13: Define the set

$$L^1(\mathbb{R}^1) := \left\{ f(x) \mid \int\limits_{-\infty}^{\infty} |f(x)| \,\mathrm{d}x < \infty \right\}$$

where $f(x)$ is a complex valued function of x. □

Definition 2.14: For each $f \in L^1(\mathbb{R}^1)$, the Fourier transform $\widehat{f}(k)$ is given by

$$\widehat{f}(k) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{-ikx} \,\mathrm{d}x$$

□

Proposition 2.17: If $f \in L^1(\mathbb{R}^1)$, then $\widehat{f}(k)$ is uniformly continuous on $\mathbb{R}^1$ and

$$|\widehat{f}(k)| \leq \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} |f(x)| \,\mathrm{d}x$$

□

Exercise 2.7: Prove Proposition 2.17. □

Remark: We've already seen, that

$$\widehat{\left(e^{-\frac{1}{2}x^2}\right)}(k) = e^{-\frac{1}{2}k^2}$$

□

Proposition 2.18:

$$\widehat{\left(\frac{e^{-\frac{1}{2\lambda}x^2}}{\sqrt{2\pi\lambda}}\right)}(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda}{2}k^2}$$

□

Proof:

$$\begin{aligned}
\widehat{\left(\frac{e^{-\frac{1}{2\lambda}x^2}}{\sqrt{2\pi\lambda}}\right)}(k) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{-\frac{1}{2\lambda}x^2}}{\sqrt{2\pi\lambda}} e^{-ikx} \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi\lambda}} e^{-ik\sqrt{\lambda}\,y} \sqrt{\lambda} \,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} e^{-i(k\sqrt{\lambda})y} \,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda}{2}k^2}
\end{aligned}$$

(QED)

Remark: We want to draw a picture of the function

$$f(x) = \frac{e^{-\frac{1}{2\lambda}x^2}}{\sqrt{\lambda}}$$

$\boxtimes$

$\frac{1}{\lambda}$ is the decay rate. For $0 < x < \sqrt{\lambda}$, we have $0 < x^2 < \lambda$ and $0 < x^2/\lambda < 1$. On the other hand

$$x > \sqrt{\lambda} \Rightarrow \frac{x^2}{\lambda} > 1$$

We can observe, that the Fourier transform of a localized function is an extended function. □

Remark: We can write

$$\begin{aligned} e^{-\frac{1}{2}k^2} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-ikx} \, \mathrm{d}x \\ e^{-\frac{1}{2}x^2} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}k^2} e^{-ikx} \, \mathrm{d}k \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}p^2} e^{ipx} \, \mathrm{d}p \\ e^{-\frac{1}{2}x^2} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}k^2} e^{ikx} \, \mathrm{d}k \end{aligned}$$

where e^{ikx} is a plane wave. Again, the most general linear combination of 1-dimensional plane waves is

$$\int_{-\infty}^{\infty} c(k) e^{ikx} \, \mathrm{d}k$$

The question would now be, which functions we can write in that way. □

Theorem 2.8: Suppose $f(x)$ is continuous, bounded and belongs to $L^1(\mathbb{R}^1)$. If $\widehat{f}(k) \in L^1(\mathbb{R}^1)$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k) e^{ikx} \, \mathrm{d}k$$

□

Remark: This is not the best possible theorem. □

Definition 2.15: For $g(k) \in L^1(\mathbb{R}^1)$ define

$$\overset{\vee}{g}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx} \, \mathrm{d}k$$

$\Rightarrow \overset{\vee}{\cdot} = \widehat{\cdot}^{-1}$ □

Proof:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(k)e^{ikx}\,\mathrm{d}k &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(k)e^{ikx}\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}e^{-\frac{\varepsilon^2}{2}k^2}\,\mathrm{d}k\\
&= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(k)e^{ikx}e^{-\frac{\varepsilon^2}{2}k^2}\,\mathrm{d}k\\
&= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)e^{-iky}\,\mathrm{d}y\,e^{ikx}e^{-\frac{\varepsilon^2}{2}k^2}\,\mathrm{d}k\\
&\overset{\text{Fubini}}{=} \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ik(y-x)}e^{-\frac{\varepsilon^2}{2}k^2}\,\mathrm{d}k\,\mathrm{d}y\\
&= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)\,\frac{1}{\varepsilon}\,e^{-\frac{1}{2\varepsilon^2}(x-y)^2}\,\mathrm{d}y\\
&= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(a+x)\,\frac{1}{\varepsilon}\,e^{-\frac{1}{2\varepsilon^2}a^2}\,\mathrm{d}a\\
&= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{-\infty}^{\infty}f(x+\varepsilon b)\frac{e^{-\frac{1}{2}b^2}}{\sqrt{2\pi}}\,\mathrm{d}b\\
&\overset{|f|\leq\infty}{=} \int_{-\infty}^{\infty}\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}f(x+\varepsilon b)\,\frac{e^{-\frac{1}{2}b^2}}{\sqrt{2\pi}}\,\mathrm{d}b\\
&= f(x)\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}b^2}}{\sqrt{2\pi}}\,\mathrm{d}b\\
&= f(x)
\end{aligned}$$

Note, that we could use Fubini only, because we added $e^{-\frac{\varepsilon^2}{2}k^2}$.

A physicist might prove this in the following way

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(k)e^{ikx}\,\mathrm{d}k &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)e^{-iky}\,\mathrm{d}y\,e^{ikx}\,\mathrm{d}k\\
&= \int_{-\infty}^{\infty}f(y)\,\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik(x-y)}\,\mathrm{d}k\,\mathrm{d}y\\
&= \int_{-\infty}^{\infty}f(y)\delta(x-y)\,\mathrm{d}y\\
&= f(x)
\end{aligned}$$

(QED)

Definition 2.16: Take $f \in L^1(\mathbb{R}^n)$. For $k \in \mathbb{R}^n$

$$\widehat{f}(k) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-i\langle k,x\rangle}\,\mathrm{d}x$$

□

Proposition 2.19: Suppose $\phi, \psi \in L^1(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} \widehat{\phi + \psi}(k) &= \widehat{\phi}(k) + \widehat{\psi}(k) \\ \widehat{\lambda\phi}(k) &= \lambda\widehat{\phi}(k) \end{aligned}$$

So, $\widehat{\cdot}$ is a linear transformation. □

Remark: $L^1(\mathbb{R}^n)$ is a complex vector space. □

Exercise 2.8: Prove Proposition 2.19. □

Example 2.9: Define the function

$$S_a(x) = \begin{cases} 1 & -a \leq x < a \\ 0 & \text{otherwise} \end{cases}$$

This function is in $L^1(\mathbb{R}^1)$, because

$$\int_{-\infty}^{\infty} |S_a(x)| \, \mathrm{d}x = \int_{-a}^{a} 1 \, \mathrm{d}x = 2a < \infty$$

The Fourier transform is

$$\begin{aligned} \widehat{S_a}(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_a(x) e^{-ikx} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{-ik} e^{-ikx} \Big|_{-a}^{a} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{-ik} \left(e^{-ika} - e^{ika} \right) \\ &= \frac{2}{\sqrt{2\pi}} \frac{\sin ak}{k} \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin ak}{k} \end{aligned}$$

□

Exercise 2.9: Show

$$\int_{-\infty}^{\infty} \left| \frac{\sin ka}{k} \right| \mathrm{d}k \geq \text{const} \cdot \sum_{\ell=1}^{\infty} \frac{1}{\ell} = \infty$$

which means, that

$$\widehat{S_a}(k) = \sqrt{\frac{2}{\pi}} \frac{\sin ka}{k} \notin L^1(\mathbb{R}^1)$$

□

Remark: For $m > 0$ and $x \in \mathbb{R}^3$ define

$$\frac{e^{-m|x|}}{|x|}$$

For $m = 0$, this would be the Coulomb-Potential, but like this, it's called the *Yukawa-Potential*. m usually means the mass of a particle. We want to do the Fourier Transform. We first check

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{e^{-m|x|}}{|x|}\,\mathrm{d}x &= \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{e^{-mr}}{r} \sin\theta\, r^2 \,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r \\
&= 2\pi \int_0^\infty \frac{e^{-mr}}{r} r^2 \,\mathrm{d}r \int_0^\pi \sin\theta\,\mathrm{d}\theta \\
&= -4\pi \frac{\mathrm{d}}{\mathrm{d}m} \int_0^\infty e^{-mr}\,\mathrm{d}r \\
&= -4\pi \, \frac{\mathrm{d}}{\mathrm{d}m} \left. \frac{e^{-mr}}{-m} \right|_0^\infty \\
&= -4\pi \frac{\mathrm{d}}{\mathrm{d}m} \frac{1}{m} \\
&= 4\pi\, m^{-2}
\end{aligned}$$

So, this function is in $L^1(\mathbb{R}^3)$ and we can do the Fourier Transform. For this, we know $\langle k, x\rangle = \langle Rk, Rx\rangle \; \forall R \in O(3),\; k, x \in \mathbb{R}^3$ and for each $k \in \mathbb{R}^3$, there is an $R \in \mathrm{SO}(3)$, such that $Rk = (0, 0, |k|)$.

$$\begin{aligned}
\widehat{\frac{e^{-m|x|}}{|x|}}(k) &= \int_{\mathbb{R}^3} \frac{e^{-m|x|}}{|x|} e^{-i\langle k,x\rangle}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{e^{-m|R^{-1}x|}}{|R^{-1}x|} e^{-i\left\langle k, R^{-1}x\right\rangle}\,\mathrm{d}R^{-1}x \\
&= \int_{\mathbb{R}^3} \frac{e^{-m|x|}}{|x|} e^{-i\left\langle k, R^{-1}x\right\rangle} \det R^{-1}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{e^{-m|x|}}{|x|} e^{-i\left\langle (R^{-1})^T k, x\right\rangle}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{e^{-m|x|}}{|x|} e^{-i\langle (0,0,|k|), x\rangle}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{e^{-m|x|}}{|x|} e^{-i|k|x_3}\,\mathrm{d}x
\end{aligned}$$

So we get

$$
\begin{aligned}
\widehat{\frac{e^{-m|x|}}{|x|}}(k) &= \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} \frac{e^{-m|x|}}{|x|}\, e^{-i|k|x_3}\,\mathrm{d}x \\
&= \frac{1}{(2\pi)^{3/2}} \int\limits_0^\infty \int\limits_0^\pi \int\limits_0^{2\pi} \frac{e^{-mr}}{r}\, e^{-i|k|r\cos\theta} r^2 \sin\theta \,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \frac{2\pi}{(2\pi)^{3/2}} \int\limits_0^\infty \int\limits_0^\pi \frac{e^{-mr}}{r}\, e^{-i|k|r\cos\theta} \sin\theta\, r^2 \,\mathrm{d}\theta\,\mathrm{d}r \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty e^{-mr} r \int\limits_0^\pi e^{-i|k|r\cos\theta} \sin\theta \,\mathrm{d}\theta\,\mathrm{d}r \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty e^{-mr} r\ \left.\frac{e^{-i|k|r\cos\theta}}{i|k|r}\right|_0^\pi \mathrm{d}r \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty e^{-mr} r \left(\frac{e^{i|k|r} - e^{-i|k|r}}{i|k|r}\right) \mathrm{d}r \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty \frac{e^{(i|k|-m)r}}{i|k|} - \frac{e^{-(i|k|+m)r}}{i|k|}\,\mathrm{d}r \\
&= \frac{1}{\sqrt{2\pi}} \left.\frac{e^{(i|k|-m)r}}{i|k|(i|k|-m)}\right|_0^\infty + \frac{1}{\sqrt{2\pi}} \left.\frac{e^{-(i|k|+m)r}}{i|k|(i|k|+m)}\right|_0^\infty \\
&= -\frac{1}{\sqrt{2\pi}} \frac{1}{i|k|(i|k|-m)} - \frac{1}{\sqrt{2\pi}} \frac{1}{i|k|(i|k|+m)} \\
&= -\frac{1}{\sqrt{2\pi}} \frac{1}{i|k|} \left(\frac{1}{i|k|-m} + \frac{1}{i|k|+m}\right) \\
&= -\frac{1}{\sqrt{2\pi}} \frac{1}{i|k|} \frac{i|k|+m+i|k|-m}{-(|k|^2+m^2)} \\
&= \sqrt{\frac{2}{\pi}}\, \frac{1}{|k|^2+m^2}
\end{aligned}
$$

This function has lots of derivatives and is not in $L^1(\mathbb{R}^3)$. For $m = 0$, we would get something different, because it then wouldn't be differentable at 0. □

Remark: The definition of the Fourier Transform might, in some cases, be different, i.e. the 2π could also stay within the expression of the plane wave. □

Remark: Look at the integral

$$
\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{ik(x-y)}\,\mathrm{d}k = \delta(x-y) \tag{2.2}
$$

A physicist would define the δ-Function with

$$
\int\limits_{-\infty}^{\infty} \delta(x-y) f(y)\,\mathrm{d}y = f(x)
$$

Define

$$
\delta_R(x-y) := \frac{1}{\sqrt{2\pi}} \int\limits_{-R}^{R} e^{ik(x-y)}\,\mathrm{d}k = \sqrt{\frac{2}{\pi}}\, \frac{\sin\left(R(x-y)\right)}{(x-y)}
$$

where we want

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \delta_R(x-y) f(y)\,\mathrm{d}y \;=\; f(x)$$

□

Proposition 2.20: If f ist a "nice function"

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \delta_R(x-y) f(y)\,\mathrm{d}y \;=\; f(x)$$

This is, what equation (2.2) means. □

Proof:

$$\begin{aligned}\lim_{R\to\infty} \int_{-\infty}^{\infty} \delta_R(x-y) f(y)\,\mathrm{d}y &= \lim_{R\to\infty} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\,\frac{\sin R(x-y)}{(x-y)}\, f(y)\,\mathrm{d}y \\ &= \lim_{R\to\infty} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\,\frac{\sin Rt}{t}\, f(x+t)\,\mathrm{d}t \\ &= \lim_{R\to\infty} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\,\frac{\sin s}{s}\, f\left(x+\frac{1}{R}\,s\right)\,\mathrm{d}s\end{aligned}$$

(QED)

Exercise 2.10: Finish the proof above. □

Remark: We'd like to find a vector space V of functions, such that $\hat{\cdot} : V \to V$ is bijective. □

Definition 2.17: We call $\alpha \in \mathbb{N}_0^n$ a *multi index*, so $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j \in \mathbb{N}_0$. By definition, we set

$$|\alpha| \;:=\; \alpha_1 + ... + \alpha_n \qquad x^\alpha \;:=\; x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$$

which is a monomial of degree $|\alpha|$. Therefore, we write

$$\partial^\alpha \;=\; \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

□

Definition 2.18: Define the *(Laurent) Schwartz space* by

$$\mathcal{S}(\mathbb{R}^n) \;:=\; \left\{ f(x) \in C^\infty(\mathbb{R}^n) \mid \sup_{x\in\mathbb{R}^n} \left| x^\beta \partial^\alpha f(x) \right| < \infty \;\; \forall \alpha, \beta \in \mathbb{N}_0^n \right\}$$

□

Theorem 2.9: $\hat{\cdot}$ is a bijective linear map on $\mathcal{S}(\mathbb{R}^n)$. Moreover unhat $\overset{\vee}{\cdot}$ is defined by

$$\overset{\vee}{g}(x) \;:=\; \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(x) e^{i\langle k,x\rangle}\,\mathrm{d}x$$

and it satisfies

$$\hat{\cdot} \circ \overset{\vee}{\cdot} \;=\; \overset{\vee}{\cdot} \circ \hat{\cdot} \;=\; \mathbf{1}$$

□

Remark: We have

$$\widehat{\cdot} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

Next step, we want to construct a basis of eigenfunctions of $\widehat{\cdot}$. □

Lemma 2.2: Let be $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$, $k \in \mathbb{N}$.

$\Rightarrow$ There is a constant $c_{\alpha,k} = c_{\alpha,k}(\varphi)$, such that

$$|\partial^\alpha \varphi(x)| \ \leq \ c_{\alpha,k} \frac{1}{(1+|x|)^k}$$

where $|x| = \sqrt{x_1^2 + ... + x_n^2}$. □

Proof: We know

$$|x^\beta \partial^\alpha \varphi(x)| \ \leq \ c(\alpha, \beta)$$

We can write

$$(1+|x|)^2 \ \leq \ 2(1+|x|^2) \quad \Leftrightarrow \quad (|x|-1)^2 \ \geq \ 0$$

which is always true. Then we write

$$\begin{aligned}(1+|x|)^k &= \left((1+|x|)^2\right)^{k/2} \ \leq \ 2^{k/2}(1+|x|^2)^{k/2} \\ &= 2^{k/2}(1+x_1^2+...+x_n^2)^{k/2}\end{aligned}$$

Look at the functions $f_k(x) := x^{k/2}$. For $k \geq 2$, this function is konvex

$$\Rightarrow f_k\left(\frac{y_1 + ... + y_n}{n}\right) \ \leq \ \frac{f_k(y_1) + ... + f_k(y_n)}{n}$$

Therefore, we can estimate

$$\begin{aligned}|(1+|x|)^k \partial^\alpha \varphi(x)| &\leq 2^{k/2}|(1+x_1+...+x_n)^{k/2}\partial^\alpha\varphi(x)| \\ &= 2^{k/2}|f_k(1+x_1+x_2+...+x_n)\partial^\alpha\varphi(x)| \\ &= 2^{k/2}\left|f_k\left(\frac{n+nx_1+nx_2+...+nx_n}{n}\right)\partial^\alpha\varphi(x)\right| \\ &\leq 2^{k/2}\left|\frac{f_k(n)+f_k(nx_1)+...+f_k(nx_n)}{n}\partial^\alpha\varphi(x)\right| \\ &= 2^{k/2}\left|n^{\frac{k}{2}-1}\left(1+x_1^{k/2}+...+x_n^{k/2}\right)\partial^\alpha\varphi(x)\right| \\ &\leq (2n)^{k/2}\left(|\partial^\alpha\varphi| + \sum_{i=1}^n \left|x_i^{k/2}\partial^\alpha\varphi(x)\right|\right) \\ &\leq (2n)^{k/2}\left(|\partial^\alpha\varphi| + \sum_{i=1}^n \begin{cases} \left|x_i^{\lceil \frac{k}{2}\rceil}\partial^\alpha\varphi(x)\right|, & |x_i| \geq 1 \\ \left|x_i^{\lfloor \frac{k}{2}\rfloor}\partial^\alpha\varphi(x)\right|, & |x_i| < 1 \end{cases}\right) \\ &\leq c(\alpha, k)\end{aligned}$$

and so we get

$$|\partial^\alpha\varphi| \ \leq \ c_{\alpha,k}\frac{1}{(1+|x|)^k}$$

(QED)

Remark:

$$\begin{aligned} f \text{ smooth} \quad &\leftrightarrow \quad \widehat{f} \text{ goes to zero quickly} \\ f \text{ goes to zero quickly} \quad &\leftrightarrow \quad \widehat{f} \text{ is smooth}\end{aligned}$$

□

Proposition 2.21: Let be $f \in \mathcal{S}(\mathbb{R}^n)$. Then

i) $\widehat{\partial_x^\alpha f}(k) = i^{|\alpha|} k^\alpha \widehat{f}(k)$ where

$$\partial_x^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

ii) $\widehat{x^\alpha f}(k) = i^{|\alpha|} \partial_k^\alpha \widehat{f}(k)$ where

$$\partial_k^\alpha = \frac{\partial^\alpha}{\partial k_1^{\alpha_1} ... \partial k_n^{\alpha_n}}$$

□

Remark: Set $F := \widehat{\cdot}$, then we can write this as

$$F \partial_x^\alpha = i^{|\alpha|} k^\alpha F \quad \Rightarrow \quad F \partial_x^\alpha F^{-1} = i^{|\alpha|} k^\alpha$$

□

Proof: First we calculate

$$\begin{aligned}
\widehat{\partial_{x_1} f}(k) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \partial_{x_1} f(x) e^{-i\langle k,x\rangle} \, \mathrm{d}x \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^1} \partial_{x_1} f(x_1, \overline{x}) e^{-ik_1 x_1} \, \mathrm{d}x_1 \, e^{-i\langle \overline{k}, \overline{x}\rangle} \, \mathrm{d}\overline{x}
\end{aligned}$$

where $\overline{x} = (x_2, ..., x_n)$, $\overline{k} = (k_2, ..., k_n)$ and

$$\begin{aligned}
\int_{\mathbb{R}^1} \partial_{x_1} f(x_1, \overline{x}) e^{-ik_1 x_1} \, \mathrm{d}x_1 &= \lim_{x_1 \to \infty} f(x_1, \overline{x}) e^{-ik_1 x_1} - \lim_{x_1 \to \infty} f(-x_1, \overline{x}) e^{ik_1 x_1} \\
&\quad - \int f(x)(-ik_1) e^{-ik_1 x_1} \, \mathrm{d}x_1 \\
&= ik_1 \int_{\mathbb{R}^1} f(x) e^{-ik_1 x_1} \, \mathrm{d}x_1
\end{aligned}$$

Doing this recursively, we get

$$\int_{\mathbb{R}^1} \partial_{x_1}^{\alpha_1} f(x_1, \overline{x}) e^{-ik_1 x_1} \, \mathrm{d}x_1 = i^{\alpha_1} k_1^{\alpha_1} \int_{\mathbb{R}^1} f(x) e^{-ik_1 x_1} \, \mathrm{d}x_1$$

And therefore

$$\begin{aligned}
\widehat{\partial_x^\alpha f}(k) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \partial_x^\alpha f(x) e^{-i\langle k,x\rangle} \, \mathrm{d}x \\
&= i^{|\alpha|} k^\alpha \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle k,x\rangle} \, \mathrm{d}x \\
&= i^{|\alpha|} k^\alpha \widehat{f}(k)
\end{aligned}$$

For equation ii), we first do the following

$$\begin{aligned}
\int_{\mathbb{R}^1} x_1 f(x_1, \overline{x}) e^{-ik_1 x_1} \, \mathrm{d}x_1 &= i \int_{\mathbb{R}^1} f(x_1, \overline{x}) \, \partial_{k_1} e^{-ik_1 x_1} \, \mathrm{d}x_1 \\
&= i \, \partial_{k_1} \int_{\mathbb{R}^1} f(x_1, \overline{x}) e^{-ik_1 x_1} \, \mathrm{d}x_1
\end{aligned}$$

and then

$$\begin{aligned}\widehat{x^\alpha f}(k) &= \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} x^\alpha f(x) e^{-i\langle k,x\rangle}\,\mathrm{d}x \\ &= i^{|\alpha|}\partial_k^\alpha \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} f(x) e^{-i\langle k,x\rangle}\,\mathrm{d}x \\ &= i^{|\alpha|}\partial_k^\alpha \widehat{f}(k)\end{aligned}$$

(QED)

Lemma 2.3: If you have a good function $g(x,k) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, then

$$\partial_k^\alpha \int\limits_{\mathbb{R}^n} g(x,k)\,\mathrm{d}x = \int\limits_{\mathbb{R}^n} \partial_k^\alpha g(x,k)\,\mathrm{d}x$$

□

2.5 Eigenvalues of $\widehat{\cdot}$

Write $F := \widehat{\cdot}$, then 1 is an eigenvalue of F and $e^{-\frac{1}{2}|x|^2}$ is its eigenvector, so

$$F e^{-\frac{1}{2}|x|^2} = e^{-\frac{1}{2}|k|^2}$$

Proposition 2.22:

$$F^4 = \mathbf{1}$$

□

Corollary 2.1:

$$\operatorname{spec}(F) \subset \{1, i, -1, -i\}$$

□

Proof:

$$\begin{aligned} F\varphi = \lambda\varphi \;&\Rightarrow\; F^2\varphi = \lambda F\varphi = \lambda^2\varphi \\ &\Rightarrow\; F^4\varphi = \lambda^4\varphi \stackrel{!}{=} \varphi \\ &\Rightarrow\; \lambda^4 = 1 \end{aligned}$$

(QED)

A Proposals to solve the exercices

A.1 Kapitel 1

Exercice 2.2: We have

$$F_{jk} = \frac{1}{\sqrt{n}} e^{-2\pi i \frac{jk}{n}}$$

Set $B := F^2$

$$\begin{aligned}
\Rightarrow B_{\mu\nu} &= \sum_{j=1}^{n} F_{\mu j} F_{j\nu} = \frac{1}{n} \sum_{j=1}^{n} e^{-2\pi i \frac{j(\mu+\nu)}{n}} \\
&= \frac{1}{n} \sum_{j=1}^{n} \left(e^{-2\pi i \frac{\mu+\nu}{n}} \right)^j \\
&= \frac{1}{n} \sum_{j=0}^{n-1} \left(e^{-2\pi i \frac{\mu+\nu}{n}} \right)^j
\end{aligned}$$

If $\mu + \nu = n$ or $\mu = \nu = n$, we get

$$B_{\mu\nu} = \frac{1}{n} \sum_{j=0}^{n-1} \left(e^0 \right)^j = 1$$

and if $\mu + \nu \neq kn \ \forall k \in \mathbb{Z}$, we get

$$B_{\mu\nu} = \frac{1}{n} \sum_{j=0}^{n-1} \left(e^{-2\pi i \frac{\mu+\nu}{n}} \right)^j = \frac{1 - \left(e^{-2\pi i \frac{\mu+\nu}{n}} \right)^n}{1 - e^{-2\pi i \frac{\mu+\nu}{n}}} = 0$$

So B looks like this

$$B = \begin{pmatrix}
0 & \dots & \dots & 0 & 1 & 0 \\
\vdots & & 0 & 1 & 0 & \vdots \\
\vdots & \iddots & \iddots & \iddots & \vdots & \vdots \\
0 & 1 & 0 & & \vdots & \vdots \\
1 & 0 & \dots & \dots & 0 & 0 \\
0 & \dots & \dots & \dots & 0 & 1
\end{pmatrix} \qquad \det B = \pm 1$$

B is real, symmetric and orthogonal, which means that

$$F^4 = B^2 = \mathbf{1}$$

Exercice 2.5:

$$\begin{aligned}
\sqrt{2\pi}\,(-2)^\ell \frac{\mathrm{d}^\ell}{\mathrm{d}\lambda^\ell} \lambda^{-1/2} &= -\sqrt{2\pi}\,(-2)^\ell \frac{1}{2} \frac{\mathrm{d}^{\ell-1}}{\mathrm{d}\lambda^{\ell-1}} \\
&= (-1)^\ell \sqrt{2\pi}\,(-2)^\ell \frac{1 \cdot 3 \cdot \ldots \cdot (2\ell-1)}{2^\ell} \lambda^{-\frac{2\ell+1}{2}} \\
&= \sqrt{2\pi} \frac{(2\ell)!}{2 \cdot 4 \cdot \ldots \cdot 2\ell} \lambda^{-\frac{2\ell+1}{2}} \\
&= \sqrt{2\pi} \frac{(2\ell)!}{2^\ell \ell!} \lambda^{-\frac{2\ell+1}{2}} \lambda^{-3/2}
\end{aligned}$$

Stichwortverzeichnis