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On the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

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Abstract

It is shown that the best constants a and b such that inequalities $\frac{1}{2n+a} \leqslant |\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{1}{k}| < \frac{1}{2n+b}$ hold for every $n \geqslant 1$ are $a = \frac{1}{1-\log 2} - 2 \approx 1.258891$ and $b = 1$.

Keywords: Alternating series; Error term; Harmonic series

1. Introduction

Let $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ be an alternating series such that $0 < a_{k+1} < a_k$ for every $k \geqslant 1$ and $\lim_{k\to\infty} a_k = 0$. It is well known that this series converges (Leibniz's convergence test) and $|\sum_{k=n+1}^{\infty}(-1)^{k-1}a_k| < a_{n+1}$, i.e., the error made by using the sum of the first n terms as an approximation for the sum of the series is less than the first neglected term a_{n+1}.

The first author of the present paper constructed in [5] a class of alternating series for which one has sharper estimates of the error terms than the usual estimate of above. This class includes, for example, the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k} = \log 2, \qquad \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1} = \frac{\pi}{4}$$

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and it is shown in [5] that inequalities

$$\frac{1}{2n+a} < \left| \sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{k} \right| < \frac{1}{2n+b} \tag{1.1}$$

and

$$\frac{1}{4n+c} < \left| \sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} \right| < \frac{1}{4n+d} \tag{1.2}$$

hold for every $n \geqslant 1$, where $a = 2\sqrt{7} - 4 \approx 1.291502$, $b = 1$, $c = 2\sqrt{19} - 8 \approx 0.717797$ and $d = 0$. These improve earlier estimates of Kazarinoff [2] ($a = c = 2$, $b = d = -2$).

A natural question is the following: which are the best constants a and b (the smallest a and the largest b) such that inequalities (1.1) hold for every $n \geqslant 1$ or for every $n \geqslant n_0$, respectively.

The same question can be raised concerning inequalities (1.2) and regarding other special convergent sequences and series. We mention here the following known results. The best constants α and β such that inequalities

$$\frac{e}{2n+\alpha} < e - \left(1 + \frac{1}{n}\right)^n \leqslant \frac{e}{2n+\beta} \tag{1.3}$$

hold for every $n \geqslant 1$ are $\alpha = 11/6$ and $\beta = (4-e)/(e-2)$, see [3].

The inequalities

$$\frac{1}{2n+\delta} \leqslant \sum_{k=1}^{n} \frac{1}{k} - \log n - C < \frac{1}{2n+\eta} \tag{1.4}$$

hold for every $n \geqslant 1$, where C is the Euler constant, $\delta = (2C-1)/(1-C)$, $\eta = 1/3$ and these are the best constants δ and η, cf. [4].

In what follows we consider inequalities (1.1) and present a treatment which furnishes the best constants. This method works also for inequalities (1.4).

2. Results

Consider the sequence $(x_n)_{n \geqslant 1}$ defined by

$$\frac{1}{2n+x_n} = \left| \sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{k} \right| \equiv \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots, \tag{2.1}$$

i.e.

$$x_n = \left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots \right)^{-1} - 2n. \tag{2.2}$$

Theorem. (1) *For every $n \geqslant 1$ we have*

$$1 + \frac{1}{\sqrt{(n+1)^2+1} + n + 1} < x_n < 1 + \frac{1}{\sqrt{n^2+1} + n}. \tag{2.3}$$

(2) *The sequence* $(x_n)_{n\geqslant 1}$ *is strictly decreasing and converges to* 1.

(3) *The best constants* a *and* b *such that*

$$\frac{1}{2n+a} \leqslant \left| \sum_{k=n+1}^{\infty} (-1)^{k-1}\frac{1}{k} \right| < \frac{1}{2n+b} \tag{2.4}$$

holds for every $n \geqslant 1$ *and every* $n \geqslant n_0$ *are* $a = x_1 = \frac{1}{1-\log 2} - 2 \approx 1.258891$, $b = 1$ *and* $a = x_{n_0}$, $b = 1$, *respectively. Then one has equality in the left-hand side inequality for* $n = 1$ *and* $n = n_0$, *respectively, while the right-hand side inequality is strict.*

3. Proofs

Let's examine the sequence $(x_n)_{n\geqslant 1}$. It follows from (1.1) that $1 \leqslant x_n \leqslant 1.291502\ldots$ for every $n \geqslant 1$ and direct computations show (we used the software package MAPLE) that

$$x_1 = \frac{1}{1-\log 2} - 2 \approx 1.258891, \qquad x_2 \approx 1.177398, \qquad x_3 \approx 1.133372,$$
$$x_4 \approx 1.106319, \qquad x_5 \approx 1.088176, \qquad x_{100} \approx 1.004974,$$
$$x_{1000} \approx 1.00050.$$

This suggests that $(x_n)_{n\geqslant 1}$ is strictly decreasing and converging to 1. To show this we need a recurrence relation for $(x_n)_{n\geqslant 1}$.

The identity (2.1) with $n+1$ instead of n yields

$$\frac{1}{2n+2+x_{n+1}} = \frac{1}{n+2} - \frac{1}{n+3} + \frac{1}{n+4} - \cdots,$$

which gives

$$x_{n+1} = \frac{(n+1)(2-x_n)}{x_n+n-1}, \quad n \geqslant 1, \tag{3.1}$$

and its equivalent form

$$x_n = \frac{2n+2-(n-1)x_{n+1}}{x_{n+1}+n+1}, \quad n \geqslant 1. \tag{3.2}$$

Lemma 1. *Let* $t_n = \sqrt{n^2+1} - n$. *For every* $n \geqslant 1$, *the following inequalities are equivalent*:

(i) $x_n > 1 + t_{n+1}$,

(ii) $x_{n+1} < 1 + t_{n+1}$,

(iii) $x_{n+1} < x_n$.

Proof. The recurrence relation (3.1) gives

$$x_n - x_{n+1} = \frac{x_n^2 - 2nx_n + 2n + 2}{x_n + n - 1}$$

and this is positive precisely when $x_n > 1 + t_{n+1}$.

Similarly, (3.2) yields

$$x_{n+1} - x_n = \frac{x_{n+1}^2 + 2nx_{n+1} - 2n - 2}{x_{n+1} + n + 1}$$

and this is negative precisely when $x_{n+1} < 1 + t_{n+1}$. □

Lemma 2. *The inequalities in Lemma* 1 *hold for all* $n \geqslant 1$.

Proof. We prove the first inequality

$$x_n > 1 + t_{n+1}. \tag{3.3}$$

Define r_n by

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + C + r_n, \quad n \geqslant 1.$$

Then

$$x_{2n} = \frac{1}{r_n - r_{2n}} - 4n = 4n\left(\frac{1}{4n(r_n - r_{2n})} - 1\right),$$

$$x_{2n+1} = \frac{1}{\frac{1}{2n+1} - r_n + r_{2n}} - 4n - 2 = 4n\left(\frac{1}{\frac{4n}{2n+1} - 4n(r_n - r_{2n})} - 1\right) - 2, \quad n \geqslant 1.$$

According to (1.4) one has $1/(2n+1) < r_n < 1/(2n)$ for every $n \geqslant 1$, but we need more precise estimates and use, cf. [1, p. 466],

$$r_n = \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4}, \quad 0 < \varepsilon_n < 1,\ n \geqslant 1, \tag{3.4}$$

and obtain

$$4n(r_n - r_{2n}) = 1 - \frac{1}{4n} + \frac{\delta_n}{n^3}, \quad -\frac{1}{480} < \delta_n < \frac{1}{30},\ n \geqslant 1,$$

therefore

$$x_{2n} > 4n\left(\frac{1}{1 - \frac{1}{4n} + \frac{1}{30n^3}} - 1\right) = \frac{4n(15n^2 - 2)}{60n^3 - 15n^2 + 2}.$$

We show that the latter fraction is $> 1 + t_{2n+1}$. This is equivalent to

$$\sqrt{(2n+1)^2 + 1} + (2n+1) > \left(\frac{4n(15n^2 - 2)}{60n^3 - 15n^2 + 2} - 1\right)^{-1},$$

and to

$$\sqrt{(2n+1)^2 + 1} - (2n+1) > \left(\frac{4n(15n^2 - 2)}{60n^3 - 15n^2 + 2} - 1\right)^{-1} - 4n - 2,$$

which is

$$\sqrt{(2n+1)^2 + 1} - (2n+1) > -\frac{13n^2 - 24n - 6}{15n^2 - 8n - 2}. \tag{3.5}$$

The left-hand side of (3.5) is positive for every $n \geqslant 1$ and its right-hand side is negative for every $n \geqslant 3$. Hence (3.5) holds for every $n \geqslant 3$. It follows that (3.5) holds for every even $n \geqslant 6$.

(3.4) is not sufficiently sharp to obtain $x_{2n+1} > 1 + t_{2n+2}$. The estimate

$$r_n = \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{240n^4} - \frac{\eta_n}{252n^6}, \quad 0 < \eta_n < 1,\ n \geqslant 1, \tag{3.6}$$

cf. [1, p. 466] (here one can found also a more general estimate for r_n, involving the Bernoulli numbers) yields

$$4n(r_n - r_{2n}) = 1 - \frac{1}{4n} + \frac{1}{32n^3} + \frac{\varphi_n}{63n^5}, \quad -1 < \varphi_n < \frac{1}{64},\ n \geqslant 1,$$

and we conclude that

$$\begin{aligned} x_{2n+1} &> 4n\left(\frac{1}{1 - \frac{1}{4n} - \frac{1}{32n^3} + \frac{1}{63n^5}} - 1\right) - 2 \\ &= \frac{2(2016n^6 - 252n^4 + 252n^3 - 65n^2 - 128n - 32)}{4032n^6 - 1008n^5 + 504n^4 - 126n^3 - 63n^2 + 64n + 32} \equiv F(n). \end{aligned}$$

We show that $F(n) > 1 + t_{2n+2}$. This is equivalent to

$$\sqrt{(2n+2)^2+1} + (2n+2) > \big(F(n) - 1\big)^{-1},$$

and to

$$\sqrt{(2n+2)^2+1} - (2n+2) > \big(F(n) - 1\big)^{-1} - 4n - 4,$$

which is

$$\begin{aligned} &\sqrt{(2n+2)^2+1} - (2n+2) \\ &\quad > -\frac{1008n^5 - 2016n^4 + 2378n^3 - 1485n^2 - 1728n - 416}{1008n^5 - 1008n^4 + 630n^3 - 67n^2 - 320n - 96}. \end{aligned} \tag{3.7}$$

The left-hand side of (3.7) is positive for every $n \geqslant 1$ and its right-hand side is negative for every $n \geqslant 2$. Hence (3.7) holds for every $n \geqslant 2$ and (3.3) holds for every odd $n \geqslant 5$.

Now direct computations show that (3.3) is valid for $1 \leqslant n \leqslant 4$ and the proof of Lemma 2 is complete. □

Proof of the theorem. Statements (1) and (2) follow at once from Lemmas 1 and 2. We conclude that the best constants a and b are $a = \sup_{n\geqslant 1} x_n = x_1 = \frac{1}{1-\log 2} - 2$, $b = \inf_{n\geqslant 1} x_n = 1$ and $a = \sup_{n\geqslant n_0} x_n = x_{n_0}$, $b = \inf_{n\geqslant n_0} x_n = 1$, respectively. □

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