Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2012,**32B**(2):441–454
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

# COMPLEX RATIONAL TYPE INTERPOLATION AT DISTURBED FEJÉR POINTS ON A CURVE*

*Tu Tianliang* (涂天亮)
*Department of Mathematics and Informatics, North China Institute of Water Conservancy and Hydroelectric Power, Beihuan Roal No. 36, Zhengzhou 450011, China*
*E-mail: tutl@ncwu.edu.cn*
*Mo Jiong* (莫炯)
*Department of Physics, Zhengzhou University, Zhengzhou 450052, China*
*E-mail: mojiong@zzu.edu.cn*

**Abstract** The results of accurate order of uniform approximation and simultaneous approximation in the early work "Jackson Type Theorems on Complex Curves" are improved from Fejér points to disturbed Fejér points in this article. Furthermore, the stability of convergence of $T^*_{n,\varepsilon}(f,z)$ with disturbed sample values $f(z^*_k)+\varepsilon_k$ are also proved in this article.

**Key words** Disturbed Fejér points; complex interpolation; order of approximation; closed Jordan curve; uniform convergence

**2000 MR Subject Classification** 30E05; 30E10; 41A25; 41A28

## 1 Introduction

Let $D$ be a simply connected domain with the boundary to be a closed Jordan curve $\Gamma$ in the complex plane, $z=0\in D$. Suppose that univalent analytic function $z=\psi(w)$ is a conformal mapping from $|w|>1$ onto the complement of $\overline{D}$ such that $\psi(\infty)=\infty,\ \psi'(\infty)=1$. Extend $\psi$ to a continuous function on $|w|\ge 1$. $z_k=\psi(\mathrm{e}^{\frac{2k\pi}{n}\mathrm{i}})$, $k=0,1,\cdots,n-1$, are called Fejér points on $\Gamma$.

Suppose that $\Gamma$ is a smooth chlosed Jordan curve. Denote by $\sigma_1(t)$ the modulus of continuity of $\psi'(w)$ on $|w|=1$. If for $a>0$ the inequality

$$\int_0^a \frac{\sigma_1(t)}{t}dt<+\infty,\quad a>0, \tag{1.1}$$

holds, then $\Gamma$ is said to be of class $\widetilde{J}$.

Recently, the first author of this article, by the help of the $q$-th class of fundamental polynomials of Hermite interpolation, $A_{k,q}(z)$ for $f\in C(\Gamma)$, and Fejér points on $\Gamma$, introduced

*Received April 7, 2009; revised November 28, 2011. Supported by NSF of Henan Province of China (20001110001).

the interpolation operator [1] $T_{n,q}(f,z)$ as well as $T_n(f,z)$, and researched the accuracy of order of approximation by $T_{n,q}(f,z)$ and simultaneous approximation on a curve in complex plane.

In 1969, suppose nonzero derivative $\psi'(w)\in \mathrm{Lip}\alpha$ on $|w|=1$, Thompson [2] obtained the convergence on closed subset of $D$ by the Lagrange interpolation at asymptotic neutral. In 1991–1992, X. C. Shen and B. P. Shuai [3–5], under the boundary condition that $\varGamma=\{|w|=1\}$, $\varGamma\in C^{1+\delta}$, or the second derivative of $\psi(w)$ is continuous on $|w|=1$ respectively, researched the order of approximation by Lagrange interpolation at disturbed nodes on $\varGamma$. In 1993, C. K. Chui, X. C. Shen, and L. F. Zhong [6] studied Lagrange interpolation at disturbed roots of unity. In 2005, Wang Xinghua & Cui Fen [7], for the stable Lagrange numerical differentiation, obtained some results.

It is natural to ask whether the results in [1] still hold after Fejér points in [1] are replaced by disturbed Fejér points. The answer is positive. To this end, there is great need for lots of preparations.

**Definition** $\{z_k^*=\psi(\mathrm{e}^{\frac{2k+t_k}{n}\pi i})\}_0^{n-1}$ with $\sum\limits_{k=0}^{n-1}|t_k|\le 0.7778\cdots<\frac{4}{\pi+2}$ and $t_n=t_0$ are called disturbed Fejér points on $\varGamma$.

Clearly, disturbed Fejér points include Fejér points as its particular case that $\sum\limits_{k=0}^{n-1}|t_k|=0$. Let

$$\omega_n^*(z)=\prod_{k=0}^{n-1}(z-z_k^*),\quad l_k^*(z)=\prod_{\substack{l=0\\ l\neq k}}^{n-1}\frac{z-z_l^*}{z_k^*-z_l^*}=\frac{\omega_n^*(z)}{\omega_n^{*\prime}(z_k^*)(z-z_k^*)},$$

$$A_{k,j}^*(z)=\left(\frac{\omega_n^*(z)}{z-z_k^*}\right)^{q+1}\frac{(z-z_k^*)^j}{j!}\sum_{\nu=0}^{q-j}\alpha_{k,\nu}^*(z-z_k^*)^\nu,\quad j=0,1,\cdots,q,$$

$$\alpha_{k,\nu}^*:=\alpha_{k,\nu}^*(q,n)=\frac{1}{\nu!}\left[\left(\frac{z-z_k^*}{\omega_n^*(z)}\right)^{q+1}\right]_{z_k^*}^{(\nu)}.$$

$$T_{n,q}^*(f,z)=\frac{\sum\limits_{k=0}^{n-1}|A_{k,q}^*(z)|^pf(z_k^*)}{\sum\limits_{k=0}^{n-1}|A_{k,q}^*(z)|^p}=\frac{\sum\limits_{k=0}^{n-1}\left|\left(\frac{1}{\omega_n^{*\prime}(z_k^*)}\right)^q l_k^*(z)\right|^pf(z_k^*)}{\sum\limits_{k=0}^{n-1}\left|\left(\frac{1}{\omega_n^{*\prime}(z_k^*)}\right)^q l_k^*(z)\right|^p},\quad p>2, z\in\varGamma,\tag{1.2}$$

here $T_{n,q}^*(f,z)\in\varGamma$ and $T_{n,q}^*(f,z_k)=f(z_k)$, $k=0,1,\cdots,n-1$. In $T_{n,q}^*(f,z)$, if $q=0$, there is

$$T_n^*(f,z)=\frac{\sum\limits_{k=0}^{n-1}|l_k^*(z)|^pf(z_k^*)}{\sum\limits_{k=0}^{n-1}|l_k^*(z)|^p},\quad p>2, z\in\varGamma,\tag{1.2$'$}$$

$T_n^*(f,z)\in\varGamma$ and $T_n^*(f,z_k)=f(z_k)$, $k=0,1,\cdots,n-1$.

## 2 Disturbed Roots of Unity

We call $w_k^*=\mathrm{e}^{\frac{2k+t_k}{n}\pi i}$ with $\sum\limits_{k=0}^{n-1}|t_k|\le 0.7778\cdots<\frac{4}{\pi+2}$ and $t_n=t_0$ the disturbed roots of unity similar to roots of unity.

In this section, we are going to research the relations between disturbed roots of unity and roots of unity as well as their related quantities.

**Theorem 2.1** Suppose that $\{w_k^*\}_0^{n-1}$ are disturbed roots of unity, then, for $|w|=1$

$$|\omega_n^*(w)| < \left(|\omega_n(w)| + \frac{4\pi}{\pi+2}\right) \mathrm{e}^{\frac{2\pi}{\pi+2}} \le C. \tag{2.1}$$

Here and later on, C denotes a positive constant without reference to its value.

**Proof** Because $\omega_n^*(w\mathrm{e}^{\frac{2\pi}{n}i}) = \omega_n^*(w)$, we can set $|\arg w| \le \frac{\pi}{n}$.

$$\begin{aligned} \omega_n^*(w) &= |w-w_0^*|\prod_{l=1}^{n-1}|w-w_l^*| \le |w-w_0+w_0-w_0^*|\prod_{l=1}^{n-1}|w-w_l+w_l-w_l^*| \\ &\le (|w-w_0|+|w_0-w_0^*|)\prod_{l=1}^{n-1}|w-w_l|\prod_{l=1}^{n-1}\left[1+\left|\frac{w_l-w_l^*}{w-w_l}\right|\right] \\ &\le \left\{|\omega_n(w)| + |w_0-w_0^*|\left|\frac{\omega_n(w)}{w-w_0}\right|\right\}\exp\left\{\sum_{l=1}^{n-1}\left|\frac{w_l-w_l^*}{w-w_l}\right|\right\}. \end{aligned} \tag{2.2}$$

In order to estimate $\sum\limits_{l=1}^{n-1}\left|\frac{w_l-w_l^*}{w-w_l}\right|$, we see clearly that, for $|\arg w| \le \frac{\pi}{n}$ and $l \ne 0$,

$$\begin{aligned} |w-w_l| &\ge \left|2\sin\frac{1}{2n}\pi\right| \ge \left|\frac{2}{\pi}2\frac{\pi}{2n}\right| = \frac{2}{n}, \\ |w_l-w_l^*| &= \left|2\sin\frac{t_l}{2n}\pi\right| \le \frac{\pi}{n}|t_l|. \end{aligned}$$

Thus,

$$\sum_{l=1}^{n-1}\left|\frac{w_l-w_l^*}{w-w_l}\right| < \frac{n}{2}\sum_{l=1}^{n-1}|t_l|\frac{\pi}{n} < \frac{2\pi}{\pi+2} \tag{2.3}$$

At the same time,

$$\left|\frac{\omega_n(w_0)}{w-w_0}\right| \le n, \quad |w_0-w_0^*| = |2\sin\frac{t_0}{2n}\pi| < \frac{\pi}{n}\frac{4}{\pi+2}. \tag{2.4}$$

Uniting (2.3) with (2.4), from (2.2), we obtain

$$|\omega_n^*(w)| \left[|\omega_n(w)| + \frac{4\pi}{\pi+2}\right] \mathrm{e}^{\frac{2\pi}{\pi+2}}.$$

Theorem 2.1 is proved.

**Lemma 2.1** Suppose $a_i \ge 0, \sum\limits_{i=1}^{n} a_i \le S_n < A$ for $n > 0$ uniformly and $A \ge 1$, then, $\prod\limits_{i=1}^{n}(A-a_i) \ge A - S_n$.

**Proof** Clearly, when $n=1$, it is true. Let Lemma 2.1 hold when $n=k$, then we prove that Lemma 2.1 is also true when $n=k+1$.

$$\begin{aligned} \sum_{i=1}^{k+1}(A-a_i) &= \left[\sum_{i=1}^{k}(A-a_i)\right](A-a_{k+1}) \ge (A-S_k)(A-a_{k+1}) \\ &= A^2 - AS_{k+1} + S_k a_{k+1} = A(A-S_{k+1}) + S_k a_{k+1} \ge A - S_{k+1}. \end{aligned}$$

Let $A=1$, the known result [3] is obtained, and from [3] can deduce Lemma 2.1 too.

**Theorem 2.2** For $\{w_k^*\}_0^{n-1}$, there are positive constants $C_1$, $C_2$, such that

$$C_2 n \le \left|\omega_n^{*\prime}(w_k^*)\right| \le C_1 n, \quad k=0,1,\cdots,n-1. \tag{2.5}$$

Because $|\omega_n'(w_k)|=n$, the inequalities (2.5) can be rewritten below.

$$C_2|\omega_n'(w_k)| \le \left|\omega_n^{*\prime}(w_k^*)\right| \le C_1|\omega_n'(w_k)|, \quad k=0,1,\cdots,n-1. \tag{2.5$'$}$$

**Proof** We prove first for the case of $k=0$.

$$\begin{aligned}\left|\omega_n^{*\prime}(w_0^*)\right| &= \prod_{l=1}^{n-1}|w_0^*-w_l^*| = \prod_{l=1}^{n-1}|w_0^*-w_l+w_l-w_l^*| \\ &\le \prod_{l=1}^{n-1}|w_0^*-w_l|\prod_{l=1}^{n-1}\left[1+\left|\frac{w_l-w_l^*}{w_0^*-w_l}\right|\right] \\ &\le \prod_{l=1}^{n-1}|w_0^*-w_l|\exp\left\{\sum_{l=1}^{n-1}\left|\frac{w_l-w_l^*}{w_0^*-w_l}\right|\right\}. \end{aligned}\tag{2.6}$$

We estimate the sum in (2.6), for $l\ne 0$,

$$\begin{aligned}|w_0^*-w_l| &\ge \min\{|w_0^*-w_1|,|w_0^*-w_{n-1}|\} = \min\left\{\left|2\sin\frac{2-t_0}{2n}\pi\right|,\left|2\sin\frac{2+t_0}{2n}\pi\right|\right\} \\ &= \left|2\frac{2}{\pi}\frac{2-|t_0|}{2n}\pi\right| = \frac{2}{n}(2-|t_0|) > \frac{2}{n}\left(2-\frac{4}{\pi+2}\right) = \frac{4\pi}{n(\pi+2)}, \\ |w_l-w_l^*| &= \left|2\sin\frac{t_l}{2n}\pi\right| < \left|2\frac{t_l}{2n}\pi\right| = \frac{\pi}{n}|t_l|,\end{aligned}$$

thus, for any $n>0$, from Definition,

$$S_n := \sum_{l=1}^{n-1}\left|\frac{w_l-w_l^*}{w_0^*-w_l}\right| < \frac{n(\pi+2)}{4\pi}\sum_{l=1}^{n-1}|t_l|\frac{\pi}{n} < \frac{n(\pi+2)}{4\pi}\frac{4}{\pi+2}\frac{\pi}{n} = 1. \tag{2.7}$$

At the same time,

$$\prod_{l=1}^{n-1}|w_0^*-w_l| = \left|\frac{w^n-1}{w-1}\right|_{w_0^*} = \left|\frac{\sin\frac{nt_0}{2n}\pi}{\sin\frac{t_0}{2n}\pi}\right|.$$

Because $|t_0| < \sum\limits_{l=1}^{n-1}|t_l| < \frac{4}{\pi+2}$, $\frac{t_0}{2}\pi < \frac{\pi}{2}$, by formula $\frac{2}{\pi}x \le \sin x \le x$ as $0\le x\le \frac{\pi}{2}$, there are

$$\frac{2}{\pi}n \le \prod_{l=1}^{n-1}|w_0^*-w_l| \le \frac{\pi}{2}n. \tag{2.8}$$

Uniting (2.6) with (2.7) and (2.8), we obtain

$$\omega_n^{*\prime}(w_0^*) \le \frac{\pi}{2}n\mathrm{e} =: C_1 n. \tag{2.9}$$

The inequalities in the right of (2.5) for $k = 0$ are proved. We prove the left in (2.5) for $k = 0$ continually. Recalling (2.6), (2.7), and (2.8), and using Lemma 2.1, we have similarly

$$\left|\omega_n^{*\prime}(w_0^*)\right| \geq \prod_{l=1}^{n-1} |w_0^* - w_l^*| \prod_{l=1}^{n-1} \left[1 - \left|\frac{w_l - w_l^*}{w_0^* - w_l}\right|\right] \geq \frac{2}{\pi} n \left[1 - S_n\right] \geq C_2 n > 0. \tag{2.10}$$

The proof of Theorem 2.2 for $k = 0$ is completed.

It remains to us to prove (2.5) for $k \neq 0$. It is similar to (2.6) that

$$\left|\omega_n^{*\prime}(w_k^*)\right| \leq \prod_{\substack{l=0\\l\neq k}}^{n-1} |w_k^* - w_l| \exp\left\{\sum_{\substack{l=0\\l\neq k}}^{n-1} \left|\frac{w_l - w_l^*}{w_k^* - w_l}\right|\right\}. \tag{2.6$'$}$$

We are going to estimate the sum in $(2.6)'$. For $l \in [0, 1, \cdots, k-1, k+1, \cdots, n-1]$, one has

$$|w_k^* - w_l| \geq \left|2\sin\frac{2 - \min\{|t_{k-1}|, |t_{k+1}|\}}{2n}\pi\right| > \left|\frac{2}{n}\left(2 - \sum_{k=0}^{n-1} |t_k|\right)\right| > \frac{2}{n}\left(2 - \frac{4}{\pi+2}\right) > \frac{\pi}{n}\frac{4}{\pi+2},$$

$$|w_l - w_l^*| = \left|2\sin\frac{t_l}{2n}\pi\right| \leq \frac{\pi}{n}|t_l|,$$

thus,

$$S_n{}' := \sum_{\substack{l=0\\l\neq k}}^{n-1} \left|\frac{w_l - w_l^*}{w_k^* - w_l}\right| < \frac{n(\pi+2)}{4\pi} \sum_{\substack{l=0\\l\neq k}}^{n-1} \frac{\pi}{n}|t_l| < \frac{n(\pi+2)}{4\pi}\frac{\pi}{n}\frac{4}{\pi+2} = 1. \tag{2.7$'$}$$

At the same time,

$$\sum_{\substack{l=0\\l\neq k}}^{n-1} |w_k^* - w_l| = \left|\frac{w^n - 1}{w - w_k}\right|_{w_k^*} = \left|\frac{{w_k^*}^n - 1}{w_k^* - w_k}\right| = \left|\frac{\sin\frac{n(2k+t_k)}{2n}\pi}{\sin\frac{t_k}{2n}\pi}\right| = \left|\frac{\sin\frac{nt_k}{2n}\pi}{\sin\frac{t_k}{2n}\pi}\right|.$$

Because $\frac{|t_k|}{2}\pi < \frac{\pi}{2}$, we have

$$\frac{2}{\pi}n \leq \prod_{\substack{l=0\\l\neq k}}^{n-1} |w_k^* - w_l| \leq \frac{\pi}{2}n. \tag{2.8$'$}$$

From $(2.6)'$, using $(2.8)'$ and $(2.7)'$, we obtain

$$|\omega_n^{*\prime}(w_k^*)| \leq \frac{\pi}{2}n\mathrm{e} = C_1 n. \tag{2.9$'$}$$

It is similar to $(2.6)'$ that

$$|\omega_n^{*\prime}(w_k^*)| \geq \prod_{\substack{l=0\\l\neq k}}^{n-1} |w_k^* - w_l| \prod_{\substack{l=0\\l\neq k}}^{n-1} \left[1 - \left|\frac{w_l - w_l^*}{w_k^* - w_l}\right|\right].$$

From the above inequality, noting Lemma 2.1 and $(2.7)'$ as well as $(2.8)'$, we obtain

$$|\omega_n^{*\prime}(w_k^*)| \geq \frac{2}{\pi}n\left[1 - \sum_{\substack{l=0\\l\neq k}}^{n-1} \left|\frac{w_l - w_l^*}{w_k^* - w_l}\right|\right] \geq \frac{2}{\pi}n[1 - S_n{}'] \geq C_2 n > 0. \tag{2.10$'$}$$

Theorem 2.2 is proved.

**Theorem 2.3** Suppose that $\{w_k^*\}_{k=0}^{n-1}$ are the disturbed roots of unity, then, for $|w|=1$ and $k=0,1,\cdots,n-1$,

$$\mathrm{e}^{-\frac{2\pi}{\pi-2}}\left|\frac{\omega_n(w)}{w-w_0}\right| \le \left|\frac{\omega_n^*(w)}{w-w_0^*}\right| \le \mathrm{e}^{\frac{2\pi}{\pi+2}}\left|\frac{\omega_n(w)}{w-w_0}\right|, \tag{2.11a}$$

$$\left|\frac{\omega_n^*(w)}{w-w_k^*}\right| \le \mathrm{e}^{\frac{2\pi}{\pi+2}}\left[\left|\frac{\omega_n(w)}{w-w_k}\right| + \frac{4\pi}{\pi+2}\left|\frac{1}{w-w_k}\right|\right], \quad k\neq 0. \tag{2.11b}$$

**Proof** First, we consider that, for $k=0,1,\cdots,n-1$,

$$\left|\frac{\omega_n^*(w)}{w-w_k^*}\right| = \left|\prod_{\substack{l=0\\ l\neq k}}^{n-1}(w-w_l^*)\right| = \prod_{\substack{l=0\\ l\neq k}}^{n-1}\left|(w\mathrm{e}^{\frac{2\pi}{n}i}-w_l^*\mathrm{e}^{\frac{2\pi}{n}i})\right|\left|\mathrm{e}^{-\frac{2\pi}{n}i}\right| = \prod_{\substack{l=0\\ l\neq k}}^{n-1}|w\mathrm{e}^{\frac{2\pi}{n}i}-w_l^*\mathrm{e}^{\frac{2\pi}{n}i}|,$$

it is enough to prove this theorem for $|\arg w|\le\frac{\pi}{n}$.

We start to prove the case of $k=0$.

$$\begin{aligned}\left|\frac{\omega_n^*(w)}{w-w_0^*}\right| &= \prod_{j=1}^{n-1}|w-w_j^*| \le \prod_{j=1}^{n-1}|w-w_j|\prod_{j=1}^{n-1}\left[1+\left|\frac{w_j-w_j^*}{w-w_j}\right|\right]\\ &\le \left|\frac{\omega_n(w)}{w-w_0}\right|\exp\left\{\sum_{j=1}^{n-1}\left|\frac{w_j-w_j^*}{w-w_j}\right|\right\}.\end{aligned} \tag{2.12a}$$

It is similar to the proof of (2.3), for $|\arg w|\le\frac{\pi}{n}$ and $j\neq 0$, we have

$$|w-w_j| \ge \left|2\sin\frac{1}{2n}\pi\right| \ge \left|\frac{2}{\pi}2\frac{\pi}{2n}\right| = \frac{2}{n}, \quad |w_j-w_j^*| = |2\sin\frac{t_j}{2n}\pi| < \frac{\pi}{n}|t_j|.$$

Consequently,

$$\sum_{j=1}^{n-1}\left|\frac{w_j-w_j^*}{w-w_j}\right| < \frac{n}{2}\frac{\pi}{n}\sum_{j=1}^{n-1}|t_j| < \frac{2\pi}{\pi+2}. \tag{2.13a}$$

Thus, from (2.12a),

$$\left|\frac{\omega_n^*(w)}{w-w_0^*}\right| \le \left|\frac{\omega_n(w)}{w-w_0}\right|\mathrm{e}^{\frac{2\pi}{\pi+2}}. \tag{2.14a}$$

On the other hand,

$$\begin{aligned}\left|\frac{\omega_n(w)}{w-w_0}\right| &= \prod_{j=1}^{n-1}|w-w_j| \le \prod_{j=1}^{n-1}|w-w_j^*|\prod_{j=1}^{n-1}\left[1+\left|\frac{w_j^*-w_j}{w-w_j^*}\right|\right]\\ &\le \left|\frac{\omega_n^*(w)}{w-w_0^*}\right|\exp\left\{\sum_{j=1}^{n-1}\left|\frac{w_j^*-w_j}{w-w_j^*}\right|\right\}.\end{aligned}$$

It is similar to the proof of (2.13a) that

$$|w-w_j^*| \ge \min\{|w-w_1^*|,|w-w_{n-1}^*|\} \ge \frac{2}{\pi}2\frac{\pi}{2n}\left(2-\sum_{j=0}^{n-1}|t_j|\right) > \frac{2}{n}(2-\frac{4}{\pi+2}) = \frac{4\pi}{n(\pi+2)},$$

$$|w_j^*-w_j| \le 2\sin\frac{|t_j|}{2n}\pi \le \frac{\pi}{n}|t_j|,$$

$$\sum_{j=1}^{n-1}\left|\frac{w_j^*-w_j}{w-w_j^*}\right|<\frac{n}{2}\sum_{j=1}^{n-1}\frac{\pi}{n}|t_j|<\frac{n}{2}\frac{\pi}{n}\frac{4}{\pi+2}=\frac{2\pi}{\pi+2}. \tag{2.13b}$$

Then,

$$\left|\frac{\omega_n(w)}{w-w_0}\right|\le\left|\frac{\omega_n^*(w)}{w-w_0^*}\right|\mathrm{e}^{\frac{2\pi}{\pi+2}},\quad \text{that is,}\quad \mathrm{e}^{-\frac{2\pi}{\pi+2}}\left|\frac{\omega_n(w)}{w-w_0}\right|\le\left|\frac{\omega_n^*(w)}{w-w_0^*}\right|. \tag{2.14b}$$

Uniting (2.14a) with (2.14b), we have

$$\mathrm{e}^{-\frac{2\pi}{\pi+2}}\left|\frac{\omega_n(w)}{w-w_0}\right|\le\left|\frac{\omega_n^*(w)}{w-w_0^*}\right|\le\left|\frac{\omega_n(w)}{w-w_0}\right|\mathrm{e}^{\frac{2\pi}{\pi+2}}.$$

The inequalities (2.11a) are proved. We are going to prove (2.11b) continually.

For $k\neq 0$,

$$\begin{aligned}\left|\frac{\omega_n^*(w)}{w-w_k^*}\right|&=|w-w_0^*|\prod_{\substack{j=1\\ j\neq k}}^{n-1}|w-w_j^*|\le(|w-w_0|+|w_0-w_0^*|)\prod_{\substack{j=1\\ j\neq k}}^{n-1}(|w-w_j|+|w_j-w_j^*|)\\ &\le(|w-w_0|+|w_0-w_0^*|)\prod_{\substack{j=1\\ j\neq k}}^{n-1}|w-w_j|\prod_{\substack{j=1\\ j\neq k}}^{n-1}\left[1+\left|\frac{w_j-w_j^*}{w-w_j}\right|\right]\\ &\le\left\{\left|\frac{\omega_n(w)}{w-w_k}\right|+|w_0-w_0^*|\left|\frac{\omega_n(w)}{(w-w_0)(w-w_k)}\right|\right\}\exp\left\{\sum_{\substack{j=1\\ j\neq k}}^{n-1}\left|\frac{w_j-w_j^*}{w-w_j}\right|\right\}.\end{aligned} \tag{2.12b}$$

Now, we estimate $\sum_{j=1,j\neq k}^{n-1}\left|\frac{w_j-w_j^*}{w-w_j}\right|$. Noting $|\arg w|\le\frac{\pi}{n}$ and $j\in[1,\cdots,k-1,k+1,\cdots,n-1]$, we have

$$|w-w_j|\ge\min_{\substack{|\arg w|\le\frac{\pi}{n}\\ j\neq 0,k}}|w-w_j|\ge|2\sin\frac{\pi}{2n}|\ge|2\frac{2}{\pi}\frac{\pi}{2n}|=\frac{2}{n},$$

$$|w_j-w_j^*|\le|2\sin\frac{t_j}{2n}\pi|\le\frac{\pi}{n}|t_j|.$$

Thus

$$\sum_{\substack{j=1\\ j\neq k}}^{n-1}\left|\frac{w_j-w_j^*}{w-w_j}\right|\le\sum_{j=1}^{n-1}\left|\frac{w_j-w_j^*}{w-w_j}\right|<\frac{n}{2}\frac{\pi}{n}\sum_{j=1}^{n-1}|t_j|<\frac{2\pi}{\pi+2}.$$

On the other hand,

$$|w_0-w_0^*|=|2\sin\frac{t_0}{2n}\pi|<\frac{\pi}{n}|t_0|<\frac{4}{\pi+2}\frac{\pi}{n},\qquad\left|\frac{\omega_n(w)}{w-w_0}\right|\le n,$$

thus

$$|w_0-w_0^*|\left|\frac{\omega_n(w)}{(w-w_0)(w-w_k)}\right|\le\frac{4\pi}{\pi+2}\left|\frac{1}{w-w_k}\right|.$$

From (2.12b), for $k\neq 0$, we have

$$\left|\frac{\omega_n^*(w)}{w-w_k^*}\right|\le\left\{\left|\frac{\omega_n(w)}{w-w_k}\right|+\frac{4\pi}{\pi+2}\left|\frac{1}{w-w_k}\right|\right\}\mathrm{e}^{\frac{2\pi}{\pi+2}}.$$

Theorem 2.3 is proved.

**Corollary 2.1** Suppose that $\{w_k^*\}_{k=0}^{n-1}$ are the disturbed roots of unity, then for $|w|=1$ and $k=0,1,\cdots,n-1$,

$$C_4\,|l_0(w)| \le |l_0^*(w)| \le C_3|l_0(w)|, \tag{2.15a}$$

$$|l_k^*(w)| \le C_3|l_k(w)| + C_5\left|\frac{1}{n(w-w_k)}\right|, \quad k\ne 0. \tag{2.15b}$$

In fact, from $|l_k^*(w)| = \left|\frac{\omega_n^*(w)}{\omega_n^{*\prime}(w_k^*)(w-w_k^*)}\right|$, by use of Theorems 2.2 and 2.3, the corollary holds.

Taking note of $|l_k(w)|\le 1$ for $k=0,1,\cdots,n-1$, and $\frac{1}{n|w-w_k|}<1$ for $k\ne 0$ and $|\arg w|\le\frac{\pi}{n}$, from (2.15a) and (2.15b), we obtain

$$|l_k^*(w)| = O(1) \quad k=0,1,\cdots,n-1. \tag{2.15}$$

**Corollary 2.2** Suppose that assumptions are the same as those in Corollary 2.1, then for $k=0,1,\cdots,n-1$, and $|w|=1$

$$\sum_{k=0}^{n-1}|l_k^*(w)|^r = \begin{cases} O(1), & r>1; \quad (2.16a)\\ O(\ln n), & r=1. \quad (2.16b)\end{cases}$$

**Proof** By use of the easily proved inequality:

$$|a\pm b|^r \le 2^r(|a|^r+|b|^r), \quad r>0, \tag{2.17}$$

and (2.15a) as well as (2.15b), one has

$$\begin{aligned}\sum_{k=0}^{n-1}|l_k^*(w)|^r &= |l_0^*(w)|^r + \sum_{k=1}^{n-1}|l_k^*(w)|^r\\ &\le C\,|l_0(w)|^r + C\left\{\sum_{k=1}^{n-1}|l_k(w)|^r + \sum_{k=1}^{n-1}\left|\frac{1}{n(w-w_k)}\right|^r\right\},\end{aligned}$$

in which $\sum\limits_{k=0}^{n-1}|l_k^*(w)|^r$ is a function with period $\frac{2\pi}{n}$. We may set $|\arg(w)|\le\frac{\pi}{n}$. Estimate the last term below

$$\begin{aligned}\sum_{k=1}^{n-1}\left|\frac{1}{n(w-w_k)}\right|^r &= 2\sum_{k=1}^{[\frac{n-1}{2}]+1}\left|\frac{1}{2n\sin\frac{2k-1}{2n}\pi}\right|^r \le C\sum_{k=1}^{[\frac{n-1}{2}]+1}\left|\frac{1}{2k-1}\right|^r\\ &\le C\left[1+\int_1^{[\frac{n-1}{2}]+1}\frac{dx}{2x-1}\right] = \begin{cases}O(\ln n), & r=1,\\ O(1), & r>1.\end{cases}\end{aligned} \tag{2.18a}$$

All the same

$$\sum_{k=1}^{n-1}|l_k(w)|^r \le 2\sum_{k=1}^{[\frac{n-1}{2}]+1}\left|\frac{1}{2n\sin\frac{2k-1}{2n}\pi}\right|^r = \begin{cases}O(\ln n), & r=1,\\ O(1), & r>1.\end{cases} \tag{2.18b}$$

In addition, $|l_0(w)|\le 1$. Consequently, Corollary 2.2 is proved.

## 3 Disturbed Fejér Points

It is similar to Fejér points that $\{z_k^*\}_{k=0}^{n-1}=\{\psi(\mathrm{e}^{\frac{2k+t_k}{n}\pi i})\}_{k=0}^{n-1}$ are called disturbed Fejér points on $\Gamma$, in which

$$\sum_{k=0}^{n-1}|t_k|\le 0.7778\cdots<\frac{4}{\pi+2},\quad t_n=t_0. \tag{3.1}$$

**Theorem 3.1** Suppose that $\Gamma$ is a closed smooth Jordan curve, then, $\psi'(w)$ is continuous and does not vanish on $|w|\ge 1$, and for positive constants $A_1>A_2$,

$$A_2\le|\psi'(w)|\le A_1,\quad |w|\ge 1; \tag{3.2}$$

$$A_2\le\left|\frac{\psi(w)-\psi(u)}{w-u}\right|\le A_1,\quad |w|\ge 1,|u|\ge 1. \tag{3.3}$$

Moreover, if $\Gamma\in\widetilde{J}$, then for a certain constant $A>0$,

$$\int_{|u|=1}\left|\frac{1}{u-w}-\frac{\psi'(u)}{\psi(u)-\psi(w)}\right||\mathrm{d}u|<A<+\infty,\quad |w|=1. \tag{3.4}$$

**Proof** From Section 1, we know that $\psi(w)$ is continuous on $|w|\ge 1$. Hence, for $z\in\Gamma$, $z(\theta)=\psi(\mathrm{e}^{\mathrm{i}\theta})=u(\theta)+\mathrm{i}v(\theta)$ is continuous on $[0,2\pi]$. By virtue of the smoothness of $\Gamma$, it is well known that $z'(\theta)=u'(\theta)+\mathrm{i}v'(\theta)$ is continuous and does not vanish on $[0,2\pi]$. On the other hand, $\psi(w)$ is univalent analytic in $|w|>1$, that is, $\psi'(w)$ is continuous and does not vanish in $|w|>1$. Considering linear derivative and domain derivative, from Walsh–Sewell theory [8], we know that $\psi'(w)$ is continuous and does not vanish on $|w|\ge 1$. Thereby, (3.2) and (3.3) are obtained directly.

Owing to

$$|\psi(u)-\psi(w)-\psi'(u)(u-w)|=\int_w^u[\psi'(t)-\psi(u)]dt\le C\sigma_1(|u-w|)|u-w|,$$

by use of (3.3) and assumption condition $\Gamma\in\widetilde{J}$, one has at once

$$\int_{|w|=1}\left|\frac{1}{u-w}-\frac{\psi'(u)}{\psi(u)-\psi(w)}\right||du|<A<+\infty,\quad |w|=1.$$

**Lemma 3.1** [9, Lemma 3.2] Suppose that the continuous function $F(\theta)$ with period $2\pi$ has total variation $\bigvee_0^{2\pi}(F)\le A<+\infty$, then

$$\left|\frac{1}{2\pi}\int_{\frac{t_0}{n}\pi}^{\frac{2n+t_n}{n}\pi}F(\theta)\mathrm{d}\theta-\frac{1}{n}\sum_{k=1}^{n}F(\frac{2k+t_k}{n}\pi)\right|\le\frac{C_A}{n},$$

where $C_A$ is a positive constant dependent only on $A$.

**Theorem 3.2** [9, Lemma 3.3] Suppose $\Gamma\in\widetilde{J}$ and

$$\Omega_n^*(w)=\frac{\omega_n^*(z)}{\omega_n^*(w)}. \tag{3.5}$$

Then

$$\mathrm{e}^{-C_A}<|\Omega_n^*(w)|<\mathrm{e}^{C_A}. \tag{3.6}$$

**Corollary 3.1** Under the assumptions of Theorem 3.2, the following inequalities hold.

$$\text{I}\quad \mathrm{e}^{-2C_A} \le \left|\frac{\Omega_n^*(w)}{\Omega_n^*(w_k^*)}\right| \le \mathrm{e}^{2C_A}. \tag{3.7}$$

$$\text{II}\quad |\omega_n^*(z)| = |\Omega_n^*(w)\omega_n^*(w)| \le C. \quad \text{(see (3.5), (3.6) and Theorem 2.1)} \tag{3.8}$$

$$\text{III}\quad {\omega_n^*}'(z_k^*) = \Omega_n^*(w_k^*){\omega_n^*}'(w_k^*)\frac{1}{\psi'(w_k^*)}. \tag{3.9}$$

Thereby, utilizing (3.6), Theorem 2.2, and inequality (3.2), one obtains at once

$$A_7 n \le |{\omega_n^*}'(z_k^*)| \le A_6 n, \quad A_6 > A_7 \text{ — positive constants} \tag{3.10}$$

**Theorem 3.3** Suppose $\Gamma \in \widetilde{J}$, then

$$\max_{z\in\overline{D}} \sum_{k=0}^{n-1} |l_k^*(z)|^r = \begin{cases} O(1), & r > 1; & (3.11\text{a}) \\ O(\ln n), & r = 1. & (3.11\text{b}) \end{cases}$$

**Proof** Using (3.5), (3.9), (3.2), (3.3), and (3.7), we have

$$\begin{aligned}\sum_{k=0}^{n-1} |l_k^*(z)|^r &= \sum_{k=0}^{n-1} \left|\frac{\omega_n^*(z)}{{\omega_n^*}'(z_k^*)(z-z_k^*)}\right|^r \le C\sum_{k=0}^{n-1}\left|\frac{\omega_n^*(w)\Omega_n^*(w)\psi'(w_k^*)}{\Omega_k^*(w_k^*){\omega_n^*}'(w_k^*)(w-w_k^*)}\right|^r \\ &\le C\sum_{k=0}^{n-1}\left|\frac{\omega_n^*(w)}{{\omega_n^*}'(w_k^*)(w-w_k^*)}\right|^r \le C\sum_{k=0}^{n-1}|l_k^*(w)|^r. \end{aligned} \tag{3.12}$$

By Corollary 2.2, Theorem 3.3 is proved.

**Theorem 3.4** Suppose $\Gamma \in \widetilde{J}$ and $r > 1$, then,

$$\sum_{k=0}^{n-1} |l_k^*(z)|^r \ge C_r > 0.$$

**Proof** It is similar to the proof of Theorem 3.3 that

$$\sum_{k=0}^{n-1} |l_k^*(z)|^r \ge C\sum_{k=0}^{n-1}|l_k^*(w)|^r.$$

From periodicity and (2.15a)

$$\begin{aligned}\min_{z\in\Gamma}\sum_{k=0}^{n-1}|l_k^*(z)|^r &\ge \min_{|w|=1} C\sum_{k=0}^{n-1}|l_k^*(w)|^r = \min_{|\arg w|\le\frac{\pi}{n}} C\sum_{k=0}^{n-1}|l_k^*(w)|^r \ge \min_{|\arg w|\le\frac{\pi}{n}} C\,|l_0^*(w)|^r \\ &\ge \min_{|\arg w|\le\frac{\pi}{n}} C_4 C|l_0(w)|^r \ge C_r(\frac{2}{\pi})^r, C_r\text{-positive constant dependent on } r.\end{aligned}$$

Theorem 3.4 is proved.

## 4 Main Results and Its Proofs

**Basic Theorem** Suppose that $\Gamma \in \widetilde{J}$, $f \in C(\Gamma)$, and $\{z_k^*\}$ are the disturbed Fejér points, and for $z \in \Gamma$ and $p > 2$, set

$$T_{n,q}^*(f,z) = \frac{\sum\limits_{k=0}^{n-1}\left|A_{k,q}^*(z)\right|^p f(z_k^*)}{\sum\limits_{k=0}^{n-1}\left|A_{k,q}^*(z)\right|^p} = \frac{\sum\limits_{k=0}^{n-1}\left|\left(\frac{1}{{\omega_n^*}'(z_k^*)}\right)^q l_k^*(z)\right|^p f(z_k^*)}{\sum\limits_{k=0}^{n-1}\left|\left(\frac{1}{{\omega_n^*}'(z_k^*)}\right)^q l_k^*(z)\right|^p},$$

which is a continuous function on $\Gamma$, and $T^*_{n,q}(z_k) = f(z^*_k)$, $k = 0, \cdots, n-1$. Then

$$\max_{z\in\Gamma} |T^*_{n,q}(f,z) - f(z)| = O\left(\omega(f, \frac{1}{n})\right).$$

**Proof**

$$\Delta^*_{p,q} := \left|T^*_{n,q}(f,z) - f(z)\right| = \left|\frac{\sum\limits_{k=0}^{n-1}\left|A^*_{k,q}(z)\right|^p f(z^*_k)}{\sum\limits_{k=0}^{n-1}\left|A^*_{k,q}(z)\right|^p} - f(z)\right|$$
$$\leq \frac{\sum\limits_{k=0}^{n-1}\left|\left(\frac{\omega^*_n(z)}{\omega^{*\prime}_n(z^*_k)}\right)^q l^*_k(z)\right|^p |f(z^*_k) - f(z)|}{\sum\limits_{k=0}^{n-1}\left|\left(\frac{\omega^*_n(z)}{\omega^{*\prime}_n(z^*_k)}\right)^q l^*_k(z)\right|^p},$$

using (3.10), we have

$$\Delta^*_{p,q} \leq C\frac{\sum\limits_{k=0}^{n-1}|l^*_k(z)|^p\,|f(z^*_k) - f(z)|}{\sum\limits_{k=0}^{n-1}|l^*_k(z)|^p}$$
$$\leq C\frac{\sum\limits_{k=0}^{n-1}|l^*_k(z)|^p\,\{|f(z^*_k) - f(z_k)| + |f(z_k) - f(z)|\}}{\sum\limits_{k=0}^{n-1}|l^*_k(z)|^p}$$
$$\leq C\frac{\sum\limits_{k=0}^{n-1}|l^*_k(z)|^p\,\{\omega(f, |z^*_k - z_k|) + \omega(f, n|z_k - z|\frac{1}{n})\}}{\sum\limits_{k=0}^{n-1}|l^*_k(z)|^p}. \tag{4.1}$$

Because

$$|z^*_k - z_k| = C|2\sin\frac{t_k}{2n}\pi| \leq C\frac{\pi}{n}|t_k| \leq C\frac{4\pi}{n(\pi+2)},$$
$$\omega(f, |z^*_k - z_k|) \leq \left(1 + \frac{4C\pi}{\pi+2}\right)\omega(f, \frac{1}{n})$$

and

$$\omega(f, n|z - z_k|\frac{1}{n}) \leq (1 + n|z - z_k|)\omega(f, \frac{1}{n}),$$

then

$$\Delta^*_{p,q} = O(\omega(f, \frac{1}{n})) + O(\omega(f, \frac{1}{n}))\frac{n\sum\limits_{k=0}^{n-1}|l^*_k(z)|^p|z - z_k|}{\sum\limits_{k=0}^{n-1}|l^*_k(z)|^p}. \tag{4.2}$$

By using (3.8), (3.10), and (3.11a), there are

$$n\sum_{k=0}^{n-1}|l^*_k(z)|^p|z - z_k| = n\sum_{k=0}^{n-1}|l^*_k(z)|^{p-1}\left|\frac{\omega^*_n(z)}{\omega^{*\prime}_n(z^*_k)}\right| = O(1). \tag{4.3}$$

From Theorem 3.4,

$$\sum_{k=0}^{n-1}|l_k^*(z)|^p \geq C_p > 0. \tag{4.4}$$

Uniting (4.2)–(4.4), we obtain

$$\Delta_{p,q}^* = O(\omega(f,\frac{1}{n})).$$

Basic Theorem is proved.

In Basic Theorem, let $q=0$, then $A_{p,q}^*(z)=l_k^*(z)$, and the following theorem is obtained.

**Theorem 4.1** Suppose that $\Gamma \in \widetilde{J}$, $f \in C(\Gamma)$, and $\{z_k^*\}_0^{n-1}$ are the disturbed Fejér points on $\Gamma$, and for $p>2, z\in\Gamma$, set

$$T_n^*(f,z) = \frac{\sum\limits_{k=0}^{n-1}|l_k^*(z)|^p f(z_k^*)}{\sum\limits_{k=0}^{n-1}|l_k^*(z)|^p},$$

then,

$$\max_{z\in\Gamma}|T_n^*(f,z)-f(z)| = O\left(\omega(f,\frac{1}{n})\right),$$

in which $T_n^*(f,z)$ is a continuous function on $\Gamma$, and $T_n^*(f,z)=f(z_k^*), k=0,\cdots,n-1$.

Let $\sum\limits_{k=0}^{n-1}|t_k|=0$, then disturbed Fejér points becomes usual Fejér points. Clearly, Theorems 2.1 and 2.2 in [1] are the particular cases of Basic Theorem and Theorem 4.1 in this article. Because we have already proved in [1] that the orders of approximation in Theorems 2.1 and 2.2 are accurate, naturally the orders of approximation in Basic Theorem and Theorem 4.1 in this article, generally, cannot be improved again.

By same method of Theorem 6.2 in [1], we obtain

**Theorem 4.2** Suppose that $\Gamma \in \widetilde{J}$, $f \in C^q(\Gamma)$, and $\{z_k^* = \psi(\mathrm{e}^{\frac{2k+t_k}{n}\pi i})\}_0^{n-1}$ are the disturbed Fejér points on $\Gamma$, then, for $0\leq j\leq q$,

$$\max_{z\in\Gamma}|f^{(q-j)}(z)-\tau_{n,j}^*(z)| = O\left(\frac{1}{n^j}\omega(f^{(q)},\frac{1}{n})\right), \tag{4.5}$$

in which

$$\begin{aligned}
&I_{n,0}^*(z) \overset{\text{def}}{=} 0, \quad \tau_{n,0}^*(z) = I_{n,0}^*(z)+T_n^*(f^{(q)}-I_{n,0}^*,z) = T_n^*(f^{(q)},z)\\
&I_{n,1}^{*'}(z) = \tau_{n,0}^*(z) = T_n^*(f^{(q)},z), \quad \tau_{n,1}^* = I_{n,1}^*(z)+T_n^*(f^{(q-1)}-I_{n,1}^*,z),\\
&I_{n,j}^{*'}(z) = \tau_{n,j-1}^*(z), \quad \tau_{n,j}^*(z) = I_{n,j}^*(z)+T_n^*(f^{(q-j)}-I_{n,j}^*,z),\\
&\cdots\cdots
\end{aligned}$$

and $\tau_{n,j}^*(z)$ is continuous on $\Gamma$, $\tau_{n,j}^*(z_k)=f^{(q-j)}(z_k)$, $k=0,1.\cdots,n-1$.

In (4.5), let $j=q$, then

$$\max_{z\in\Gamma}|f(z)-\tau_{n,q}^*(z)| = O\left(\frac{1}{n^q}\omega(f^{(q)},\frac{1}{n})\right). \tag{4.6}$$

In (4.6), let $q=0$, then $\max_{z\in\Gamma}|f(z)-\tau_{n,0}^*(z)| = O\left(\omega(f,\frac{1}{n})\right)$ and $\tau_{n,0}^*(z)=T_n^*(f,z)$.

Clearly, Theorem 4.1 is similar to Jackson's Theorem 1 [10, p.117], and the inequality (4.6) is similar to Jackson's Theorem 2 [10, p.121].

## 5 Disturbed Boundary Data

If the boundary data $\{f(z_k^*)\}_0^{n-1}$ have disturbances too, for example, truncation [7], then, we have

**Theorem 5.1** Suppose $\Gamma \in \widetilde{J}$, $f \in C(\Gamma)$, and $f_\varepsilon(z_k^*) = f(z_k^*) + \varepsilon_k$. Set

$$T_{n,\varepsilon}^*(f,z) = \sum_{k=0}^{n-1} |l_k^*(z)|^p f_\varepsilon(z_k^*) \Big/ \sum_{k=0}^{n-1} |l_k^*(z)|^p, \quad p > 2, z \in \Gamma,$$

then

$$\max_{z\in\Gamma} |T_{n,\varepsilon}^*(f,z) - f(z)| = O\left(\omega(f, \frac{1}{n})\right) + \sum_{k=0}^{n} |\varepsilon_k|.$$

**Proof**

$$\begin{aligned}
\max_{z\in\Gamma} |T_{n,\varepsilon}^*(f,z) - f(z)| &= \max_{z\in\Gamma} \left| \frac{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p f_\varepsilon(z_k^*)}{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p} - f(z) \right| \\
&\le \max_{z\in\Gamma} \frac{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p |f_\varepsilon(z_k^*) - f(z)|}{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p} \\
&\le \max_{z\in\Gamma} \frac{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p |f(z_k^*) - f(z)|}{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p} + \max_{z\in\Gamma} \frac{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p |\varepsilon_k|}{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p} \\
&= \mathrm{I} + \mathrm{II}. \qquad (5.1)
\end{aligned}$$

First, from the estimate of $\Delta_{p,q}^*$ in Basic Theorem, we see

$$\mathrm{I} = \max_{z\in\Gamma} \frac{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p |f(z_k^*) - f(z)|}{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p} = O(\omega(f, \frac{1}{n})). \qquad (5.2)$$

Next,

$$\mathrm{II} = \max_{z\in\Gamma} \frac{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p |\varepsilon_k|}{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p} \le \max_{z\in\Gamma} \frac{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p \sum\limits_{k=0}^{n-1} |\varepsilon_k|}{\sum\limits_{k=0}^{n-1} |l_k^*(z)|^p} \le \sum_{k=0}^{n-1} |\varepsilon_k|. \qquad (5.3)$$

By uniting (5.1)–(5.3), Theorem 5.1 is proved.

**Corollary 5.1** Suppose $\Gamma \in \widetilde{J}$, $f \in C(\Gamma)$, and the disturbed Fejér points $\{z_k^*\}_0^{n-1}$ are given, then for $\lim\limits_{n\to\infty} \max\limits_{z\in\Gamma} |T_{n,\varepsilon}^*(f,z) - f(z)| = 0$, the sufficient and necessary condition is $\sum\limits_{k=0}^{n-1} |\varepsilon_k| = o(1)$.

In sum, this article extends all the results in reference [1] from Fejér points to disturbed Fejér points on a closed Jordan curve. Furthermore, the stable uniform convergence of $T_{n,\varepsilon}^*(f,z)$ with disturbed boundary data $f(z_k^*) + \varepsilon_k$ is also proved in this article.

## References

[1] Tu Tianliang. Jackson type theorems on complex curves. Sci in China Ser A-Math, 2009, **52**(3): 493–506

[2] Thompson M. Complex polynomial interpolation to continuous boundary data. Proc Amer Math Soc, 1969, **20**: 327–332

[3] Shen Xiechang, Shuai Binpeng. Order of approximation by Lagrange interpolating polynomials at nearly roots of unity. J of Math (PRC), 1991, **11**(3): 287–297 (in Chinese)

[4] Shen Xiechang, Shuai Binpeng. Order of approximation by Lagrange interpolating polynomials at nearly Fejér points. J of Math (PRC), 1992, **12**(1): 1–10 (in Chinese)

[5] Shen Xiechang, Shuai Binpeng. Uniform convergence of Lagrange interpolating polynomials at asymptotic Fejér points in Jordan domain. Approx Theory & its Appl, 1991, **7**(1): 39–55

[6] Chui C K, Shen X C, Zhong L F. On Lagrange interpolation at disturbed roots of unity. Trans Amer Nath Soc, 1993, **336**(2): 817–830

[7] Wang Xinghua, Cui Feng. Stable Lagrange numerical differentiation with the highest order of approximation. Sci in China: Series A Mathematics, 2006, **44**(2): 225–232

[8] Lee K P, Liu H J, Qiu H J. Theory of Moden Function (in Chinese). Wuhan: Wuhan University Press, 1983: 242–244

[9] Tu TianLiang. Simultaneous approximation by complex Hermite interpolation at disturbed Fejér points. SCIENTIA SINICA Mathematia, 2010, **40**(4): 349–366 (in Chinese)

[10] Natanson N P. Constructive Theory of Functions (in Russian). M-L: GIT-TL, 1949: 117; 121