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On Multiplication Modules

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Abstract. Let R be a commutative ring with identity and M be a unital R-module. Then M is called a multiplication module provided for every submodule N of M there exists an ideal I of R such that $N = IM$. Our objective is to investigate properties of prime and semiprime submodules of multiplication modules.

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Throughout this paper all rings will be commutative with identity and all modules will be unitary. Let R be a ring and M be a unital R-module. For any submodule N of M, we define $(N : M) = \{r \in R : rM \subseteq N\}$. A submodule N of M is called prime if $N \neq M$ and whenever $r \in R, m \in M$ and $rm \in N$, then $m \in N$ or $r \in (N : M)$. A submodule N of M is called semiprime if $N \neq M$ and whenever $r \in R, m \in M$, and $r^n m \in N$ for some positive integer n, then $rm \in N$. In recent years, prime and semiprime submodules have attracted a good deal of attention; see , for example $[2 - 5]$.

An R-module M is called a multiplication module provided for each submodule N of M there exists an ideal I of R such that $N = IM$. We say that I is a presantation ideal of N. Clearly, every submodule of M has a presantation ideal if and only if M is a multiplication module. Let N and K be submodules of a multiplication with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R.The product N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [6,theorem 3.4], the product of N and K is independent of presentation of N and K. Note that this definition is different from the definition of ordinary ideal multiplication. Indeed, let $R = Z$ be the ring of integers, and let $M = 2Z$ and $N = K = 4Z$. Then NK is $16Z$ by the usual definition and is $8Z$ by the our definition. Moreover, for $a, b \in M$, by ab we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$see,for example,[[7] − [9] .]

The purpose of this paper is to introduce interesting and useful properties of prime and semiprime submodules of multiplication modules.

Theorem 1. *Let M be a non-zero R-module. Then, M is a faithful R-module (not necessarily multiplication) such that every proper submodule is prime if and only if R is a field.*

Proof. $\Leftarrow$: Clear.

$\Rightarrow$: Suppose that R is not field. Note that the zero submodule of M is prime. Then R is a domain and M is a torsion-free R-module. Because R is not field, M is not simple. Let Rm be a proper non-zero submodule of M. Assume that $0 \neq a$ is not invertible element of R. Then Ram is prime so that aM is contained in Ram (which gives contradiction $M = Rm$) or m belongs to Ram (which gives $Ra = R$, a contradiction). ∎

Corollary 2. *Let M be a faithful multiplication R-module. Then, M is simple if and only if every proper submodule of M is prime.*

Proposition 3. *Let M be a multiplication R-module and $N_1, N_2, ..., N_k$ be submodules of M.Let N be prime submodule of M. Then the following statements are equivalent.*

(i) $N_j \subseteq N$ for some j with $1 \leq j \leq k$.

(ii) $\bigcap_{i=1}^{k} N_i \subseteq N$

(iii) $\prod_{i=1}^{k} N_i \subseteq N$

Proof. $(i) \Rightarrow (ii)$: Clear.

$(ii) \Rightarrow (iii)$: Since $\prod_{i=1}^{k} N_i \subseteq \bigcap_{i=1}^{k} N_i$, $\prod_{i=1}^{k} N_i \subseteq N$ by (ii).

$(iii) \Rightarrow (i)$: We have $N_i = I_i M$ for some ideals $I_i \, (1 \leq i \leq k)$ of R. Then $N_1 N_2 ... N_k = I_1 I_2 ... I_k M \subseteq N$ and so $I_1 I_2 ... I_k \subseteq (N : M)$. Since $(N : M)$ is a prime ideal of R, $I_j \subseteq (N : M)$ for some $j \, (1 \leq j \leq k)$ .Therefore, $N_j = I_j M \subseteq N$ for some $j \, (1 \leq j \leq k)$. ∎

Definition 1. *Let M be a multiplication R-module. A nonempty subset S^* of M is said to be multiplicatively closed if $mn \cap S^* \neq \varnothing$ whenever $m, n \in S^*$.*

Proposition 4. *Let M be a multiplication R-module. Then, a proper submodule N of M is prime if and only if $M \backslash N$ is a multiplicatively closed.*

Proof. Let N be a prime submodule of M and let $a, b \in M \backslash N$. Since N is prime, $ab \subsetneq N$. Then $ab \cap (M \backslash N) \neq \varnothing$. Conversely, let $a, b \notin N$. Then $a, b \in M \backslash N$. Since $M \backslash N$ is a multiplicatively set, $ab \cap (M \backslash N) \neq \varnothing$. Therefore, $ab \subsetneq N$ [see, 6] . ∎

Theorem 5. *Let M be a multiplication R-module. Let A be a submodule of M and let S^* be a multiplicatively closed set in M such that $A \cap S^*$ is empty. Then there is a submodule N of M which is maximal with respect to the properties that $A \subseteq N$ and $N \cap S^*$ are empty. Furthermore, N is prime submodule of M.*

Proof. Let H be the set of all submodules B of M such that $A \subseteq B$ and $B \cap S^*$ is empty; H is not empty since $A \in H$. By Zorn's Lemma, H has a maximal element N. To show that N is prime, suppose that it is not and let $ab \subseteq N$, $a \notin N$, $b \notin N$ where $a, b \in M$. Then $N \subsetneq N + Ra$ and $N \subsetneq N + Rb$ so there are elements $s, t \in S^*$ such that $s \in N + Ra$ and $t \in N + Rb$. Hence $st \cap S^* \neq \varnothing$ and $st \subseteq (N + Ra)(N + Rb) \subseteq N$,which contradicts the fact that $N \cap S^*$ is empty. ∎

Definition 2. *Let M be a multiplication module. A zero divisor in M is an element $0_M \neq a \in M$ for which there exists $b \in M$ with $b \neq 0_M$ such that $ab = RaRb = 0_M$.*

Theorem 6. *Let M be a multiplication R-module. Let N be a submodule of M such that $N \neq M$. Then, N is prime if and only if M/N has no zero divisor.*

Proof. Suppose that N be a prime submodule of M. Since M is a multiplication module, M/N is a multiplication module[*see* 6, *Theorem* 3.21] .Let $a \in M$ be such that the element $0_{M/N} \neq \overline{a} = a + N$ in M/N is zero divisor, so that there exists $b \in M$ such that $0_{M/N} \neq \overline{b} = b + N$ and $\overline{a}\overline{b} = 0_{M/N}$. Let I and J be presentation ideals $\overline{a}$ and $\overline{b}$,respectively. Then $\overline{a}\overline{b} = (IJ)\, M/N = N$ and so $ab \subseteq N$. Since N is prime, $a \in N$ or $b \in N$. This is a contradiction. Conversely, let M/N has no zero divisor. Let $ab \subseteq N$ where $a, b \in M$. Then $\overline{a}\overline{b} = 0_{M/N}$. Since M/N has no zero divisor, $\overline{a} = 0_{M/N}$ or $\overline{b} = 0_{M/N}$. Therefore, $a \in N$ or $b \in N$. ∎

Definition 3. *Let M be a multiplication R-module and let N be a submodule of M. Then*

(i) N is called nilpotent if $N^k = 0$ for some positive integer k, where N^k means the product of N, k times;

(ii) An element m of M is called nilpotent if $m^k = 0$ for some positive integer k.

The set of all nilpotent elements of M is denoted by N_M.

Definition 4. *Let M be an R-module and N be a submodule of M. Then, the radical of N denoted by $M - rad(N)$ or $r(N)$ is defined to be intersection of all prime submodules of M containing N. If N is not contained in any prime submodule of M, then $M - rad(N) = M$.*

Theorem 7. [6, *Theorem* 3.13] *Let N be a submodule of a multiplication R-module M.Then $M - rad(N) = \{m \in M : m^k \subseteq N \text{ for some } k > 0\}$.*

Corollary 8. *Let M be a multiplication R-module. Then N_M is the intersection of all prime submodules of M ($r(0) = N_M$).*

Definition 5. *A proper submodule N of a module M over a commutative ring R is said to be weakly prime submodule if whenever $0 \neq rm \in N$, for some $r \in R, m \in M$, then $m \in N$ or $r \in (N : M)$.*

Clearly, every prime submodule of a module is weakly prime submodule. However, since 0 is always weakly prime(by definition), a weakly prime submodule need not be prime.

Theorem 9. *Let N be a weakly prime submodule of M. If $(N : M) N \neq 0$, then N is a prime submodule of M.*

Proof. Suppose that $(N : M) N \neq 0$; we show that N is prime. Let $rm \in N$. If $rm \neq 0$, then N weakly prime gives $m \in N$ or $r \in (N : M)$. So assume that $rm = 0$.First suppose that $rN \neq 0$, say $rn_0 \neq 0$ where $n_0 \in N$. Then $0 \neq rn_0 = r(m + n_0) \in N$, so $r \in (N : M)$ or $m + n_0 \in N$. Hence $r \in (N : M)$ or $m \in N$.So we can assume that $rN = 0$. Next suppose that $m(N : M) \neq 0, mk_0 \neq 0$ where $k_0 \in (N : M)$.Then $0 \neq mk_0 = (r + k_0) m \in N$. Therefore $m \in N$ or $r + k_0 \in (N : M)$. Then $m \in N$ or $r \in (N : M)$. So we can assume that $m(N : M) = 0$.

Since $(N : M) N \neq 0$, there exists $k \in (N : M)$ and $n \in N$ with $kn \neq 0$. Then $0 \neq kn = (r + k)(m + n) \in N$; $r + k \in (N : M)$ or $m + n \in N$. Hence $r \in (N : M)$ or $m \in N$. So N is prime submodule of M. ∎

Compare the following Theorem with Theorem 1 in [1].

Theorem 10. *Let M be a multiplication R-module.Let N be a weakly prime submodule of M.If N is not prime, $N^2 = 0$.*

Proof. Since M is a multiplication module, $N = (N : M) M$. Therefore, $N^2 = (N : M)^2 M = (N : M) N = 0$ by Theorem 9 . ∎

Corollary 11. *Let M be a multiplication R-module and N be weakly prime submodule of M. Then $N \subseteq r(0)$ or $r(0) \subseteq N$.*

Proof. If N is a prime submodule, $r(0) \subseteq N$ by Corollary 8. If N is not prime submodule, $N \subseteq r(0)$ by Theorem 10. ∎

Recall that an ideal I in a commutative ring R is called semiprime if $r^n \in I$ for some $n \in Z^+$ implies that $r \in I$. It is well known that an ideal I is semiprime if and only if $I = \sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in Z^+\}$. A submodule N of M is called semiprime if $r^n m \in N$ for some $n \in Z^+$ implies that $rm \in N$. It is clear that N is semiprime if and only if $I^n V \subseteq N$ for some $n \in Z^+$ implies that $IV \subseteq N$.

Theorem 12. *Let N be a proper submodule of a multiplication module M. Then N is semiprime if and only if $U^n \subseteq N$ implies that $U \subseteq N$ for each submodule U of M.*

Proof. Let N be a semiprime submodule and $U^n \subseteq N$ for some submodule U of M. Suppose that I be a presentation of U. Then $U^n = I^n M \subseteq N$. Therefore, $I^n \subseteq (N : M)$. Since N is a semiprime submodule of M, $(N : M)$ is a semiprime ideal of R. Therefore, $I \subseteq (N : M)$ and so $U = IM \subseteq N$.

Conversely, let $U^n \subseteq N$ for some $n \in Z^+$ implies that $U \subseteq N$ for any submodule U of M.Let $I^n V \subseteq N$ for some ideal I of R and a submodule V of M. Suppose that J be a presentation of V. Therefore, $I^n V = I^n JM \subseteq N$ and so $(IV)^n = (IJ)^n M \subseteq N$. Then $IV \subseteq N$. ∎

Corollary 13. *Let N be a proper submodule of M. Then N is semiprime if and only if $m^n \subseteq N$ for some $n \in Z^+$ implies that $m \in N$ for every $m \in M$.*

Proof. Let N be a semiprime submodule. It is clear that $m^n \subseteq N$ for some $n \in Z^+$ implies that $m \in N$ for every $m \in M$. Conversely, let $U \nsubseteq N$.Thus, there is $u \in U \backslash N$. Then $u^n \nsubseteq N$. Therefore $U^n \nsubseteq N$. Thus, N is semiprime. ∎

If N is a submodule of M such that N is an intersection of prime submodules of M, then N is semiprime submodule of M. We don't know if the converse is true in general, but it is true but it is true in the following special case(see, 3)

Theorem 14. *Let M be a multiplication R-module. Then, N is a semiprime submodule of M if and only if $r(N) = N$.*

Proof. It is clear that $N \subseteq r(N)$ for any submodule N of M. Let N be a semiprime submodule of M. Let $m \in r(N)$. Then $m^k \subseteq N$ for some $k \in Z^+$. Since N is semiprime, $m \in N$.Therefore, $N = r(N)$.

Conversely, let $N = r(N)$.Let $m^n \subseteq N$ for some $n \in Z^+$. Since $N = r(N)$, $m \in N$.Therefore, N is semiprime submodule of M. ∎

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