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An application of averaging operators to multilinearity

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The aim of this note is to give an application of the results on averaging operators, by e.g. Milutin [M], Pełczyński [P], Ditor [Dt], Haydon [Ha], Hoffmann [Ho], to the theory of multilinear operators. Using their results we show that for example a multilinear operator $T: C(K) \to C(K)$ can unexpectedly often be presented as a product of bounded linear operators and the pointwise multiplication operator on $C(K)$ (Theorem 2). The crucial fact used in our construction is that given two compacta, K and S, and a continuous surjection $\varphi: K \to S$ with an averaging operator, the operator φ° from $C(S)$ into $C(K)$ is an isometric embedding, which is also an algebra homomorphism and admits a left inverse. This allows us to transform the multiplication operator of $C(S)$ to that of $C(K)$, see the end of the proof of Theorem 2.

We start with presenting some of our notation; in general we use the terminology of [LT, Sects. II.4.h, i] and [P]. The space of bounded linear operators between the Banach spaces X and Y is denoted by $L(X, Y)$, or by $L(X)$, if $X = Y$. If K is a compact metric space, we denote by $C(K)$ the Banach space of continuous, real or complex valued mappings, endowed with the sup-norm. If K and S are compact metric spaces and $\varphi: K \to S$ is a continuous surjection, we denote by φ° the linear isometry from $C(S)$ into $C(K)$ given by $\varphi^\circ f = f \circ \varphi$. If there exists a (contractive) linear projection from $C(K)$ onto $\varphi^\circ(C(S))$, we say that φ admits a (regular) averaging operator. By Δ we denote the Cantor set which is a compact metric space homeomorphic to the topological product $\prod_{m=1}^{\infty} \{0, 1\}$.

Our first lemma already gives the desired factorization result in a simple case. The use of averaging operators here is not always essential: if K has M disjoint closed subspaces homeomorphic to K, we can replace the spaces K_n in the proof by them. The case of non-metrizable K is discussed in Remark 3.

Lemma 1. *Let $M \in \mathbb{N}$, $M \geqq 2$, let S be a compact space, let K be an uncountable compact metric space and let $(A_n)_{n=1}^N$, $N \in \mathbb{N}$, be a sequence of bounded linear operators $C(S) \to C(K)$. There exist operators $A \in L(C(S), C(K))$, $B \in L(C(K))$*

such that the M-linear operator

$$P:(f_1, \ldots, f_M) \mapsto \sum_{n=1}^{N} \lambda_n \sum_{i=1}^{M} A_n f_i \qquad (1)$$

from $C(S) \times \ldots \times C(S)$ into $C(K)$ can be written in the form

$$P(f_1, \ldots, f_M) = B\left(\prod_{i=1}^{M} Af_i\right). \qquad (2)$$

Proof. We can find a closed subspace of K homeomorphic to Δ, and hence we can choose N disjoint closed subspaces K_n of K homeomorphic to Δ. By [P, Theorem 5.6], for all n there exist continuous surjections $\varphi_n : K_n \to K$ having (regular) averaging operators. Each operator φ_n° has a bounded left inverse, which we denote by $\hat{\varphi}_n$. (So, $\hat{\varphi}_n \in L(C(K_n), C(K))$.)

For all n, let V_n be an open neighbourhood of K_n such that $V_n \cap V_m = \emptyset$ for $n \neq m$. Let F_n be the Borsuk-Kakutani extension operator $C(K_n \cup (K \setminus V_n)) \to C(K)$ and let $E_n : C(K_n) \to C(K)$ be the extension operator $E_n f = F_n \hat{f}$, where $\hat{f}(x) = f(x)$ for $x \in K_n$ and $\hat{f}(x) = 0$ for $x \in K \setminus V_n$. We define

$$Af = \sum_{n=1}^{N} E_n(\varphi_n^\circ A_n f), \qquad Bf = \sum_{n=1}^{N} \lambda_n \hat{\varphi}_n(f|_{K_n}). \qquad (3)$$

The Eq. (2) is a direct consequence of definitions:

$$\begin{aligned} B\left(\prod_{i=1}^{M} Af_i\right) &= \sum_{m=1}^{N} \lambda_m \hat{\varphi}_m \left(\prod_{i=1}^{M} \sum_{n=1}^{N} (E_n(\varphi_n^\circ A_n f_i))|_{K_m}\right) \\ &= \sum_{n=1}^{N} \lambda_n \hat{\varphi}_n \left(\prod_{i=1}^{M} \varphi_n^\circ A_n f_i\right) \\ &= \sum_{n=1}^{N} \lambda_n \hat{\varphi}_n \varphi_n^\circ \prod_{i=1}^{M} A_n f_i = \sum_{n=1}^{N} \lambda_n \prod_{i=1}^{M} A_n f_i . \end{aligned}$$ □

Recall that an M-linear form u on $C(K)^M$ is called integral (in the sense of Grothendieck), if there exist an element $v \in C(K^M)^*$ such that $u(f_1, \ldots, f_M) = \left\langle \prod_{i=1}^{M} f_i(x_i), v \right\rangle$ for all $f_i \in C(K)$, where $x_i \in K$. Many concrete M-linear forms, such as

$$(f_1, \ldots, f_M) \mapsto \int_{I^M} k(x_1, \ldots, x_M) \prod_{i=1}^{M} f_i(x_i)\, dx ,$$

where $k \in L_1(I^M)$, are integral on $C(I)^M$. For more details we refer to [K, Chaps. 44 and 45].

Theorem 2. a) *Let K and S be a compact spaces, K metric uncountable, let $M \in \mathbb{N}$, $M > 1$ and let P be a continuous, symmetric M-linear map $C(K) \times \ldots \times C(K) \to C(S)$. Assume that for all $t \in S$ the map $f_1(x_1) f_2(x_2) \ldots f_M(x_M) \mapsto$*

$P(f_1, \ldots, f_M)(t)$, where $f_i \in C(K)$, can be linearly and continuously extended to a linear form $m^{(t)}$ from $C(K^M)$ into $\mathbb{K}$ and that $m^{(t)} \in C(K^M)^*$ depends continuously on t when $C(K^M)^*$ is endowed with the weak-topology. Then there exist linear operators $A \in L(C(K))$, $B \in L(C(K), C(S))$ such that for all $f_1, \ldots, f_M \in C(K)$*

$$P(f_1, \ldots, f_M) = B\left(\prod_{i=1}^{M} Af_i\right). \tag{4}$$

Conversely, if P can be represented in the form (4), then the above assumptions on $m^{(t)}$ hold.

b) *Let K and M be as above, let X be a Banach space and let P be a continuous, symmetric M-linear mapping $C(K) \times \ldots \times C(K) \to X$. Assume that for all $t \in X^*$ the map $f_1(x_1)f_2(x_2)\ldots f_M(x_M) \mapsto \langle P(f_1, \ldots, f_M), t\rangle$, where $f_i \in C(K)$, can be linearly and continuously extended to a linear form $m^{(t)}$ from $C(K^M)$ into $\mathbb{K}$ and that $m^{(t)} \in C(K^M)^*$ depends continuously on t when $C(K^M)^*$ and X^* are both endowed with the weak*-topologies. Then there exist linear operators $A \in L(C(K))$ and $B \in L(C(K), X^{**})$ such that for all $f_1, \ldots, f_M \in C(K)$*

$$P(f_1, \ldots, f_M) = B\left(\prod_{i=1}^{M} Af_i\right). \tag{5}$$

Conversely, if X is reflexive and P can be represented in the form (5), then the above assumptions on $m^{(t)}$ hold.

Proof. 1. We prove the case a) (resp. b)). Let us denote for all $t \in S$ (resp. for $t \in X^*$) by P_t the M-linear form $P_t(f_1, \ldots, f_M) = P(f_1, \ldots, f_M)(t)$ (resp. $P_t(f_1, \ldots, f_M) = \langle P(f_1, \ldots, f_M), t\rangle$) and let $m^{(t)} \in C(K^M)^*$ be as in the assumption, i.e.

$$P_t(f_1, \ldots, f_M) = \langle f_1(x_1)\ldots f_M(x_M), m^{(t)}\rangle \tag{6}$$

for all $f_1, \ldots, f_M \in C(K)$ (similarly in the case b)). Let us introduce the operators $A_i \in L(C(K), C(K^M))$, $i = 1, \ldots, M$,

$$(A_i f)(x_1, \ldots, x_M) = (M!)^{-1/M} f(x_i),$$

where $f_i \in C(K)$, $(x_1, \ldots, x_M) \in K^M$. Using the assumption that P_t is symmetric we can write

$$P_t(f_1, \ldots, f_M) = \left\langle \sum_{\sigma} \prod_{i=1}^{M} A_{\sigma(i)} f_i, m^{(t)} \right\rangle \tag{7}$$

where σ runs over all permutations of $\{1, \ldots, M\}$. Furthermore, by the polarization formula [Dn, Theorem I.1.5], the expression $\sum_{\sigma} \prod_{i=1}^{M} A_{\sigma(i)} f_i$ can be written in the form

$$2^{-M} M!^{-1} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_M \left(\sum_{\sigma} \prod_{i=1}^{M} \left(\sum_{j=1}^{M} \varepsilon_j A_j \right) f_i \right)$$

$$= 2^{-M} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_M \prod_{i=1}^{M} \left(\sum_{j=1}^{M} \varepsilon_j A_j \right) f_i .$$

(We take in the polarization formula $n = M$, $E = L(C(K), C(K^M))$, $x_i = A_i$ and for F we take the space of continuous M-linear mappings $C(K)^M \to C(K^M)$ and for L we take the symmetric M-linear map

$$L(B_1, \ldots, B_M)(f_i, \ldots, f_M) \mapsto \sum_\sigma \prod_{i=1}^M B_{\sigma(i)} f_i ,$$

where $B_i \in L(C(K), C(K^M))$, $f_i \in C(K)$ and σ runs over all permutations of $\{1, \ldots, M\}$.) Hence, we can now apply Lemma 1 to the mapping $(f_1, \ldots, f_M) \mapsto \sum_\sigma \prod_{i=1}^M A_{\sigma(i)} f_i$ to find operators $A_0 \in L(C(K), C(K^M))$ and $B_1 \in L(C(K^M))$ satisfying

$$B_1((A_0 f_1) \ldots (A_0 f_M)) = \sum_\sigma \prod_{i=1}^M A_{\sigma(i)} f_i . \tag{8}$$

By assumption, $m^{(t)}$ depends continuously on t when $C(K^M)^*$ is endowed with the weak* topology. The compactness of S (resp. the closed unit ball U of X^* with the weak*-topology) implies that for a fixed $f \in C(K^M)$ the set $\{\langle f, m^{(t)} \rangle \mid t \in S \text{ (resp. } t \in U)\}$ is bounded. Hence, by the Banach-Steinhaus theorem, the set $\{m^{(t)} \mid t \in S \text{ (resp. } t \in U)\} \subset C(K^M)^*$ is bounded (with respect to the norm). So, the mapping B_2,

$$(B_2 f)(t) := \langle f, m^{(t)} \rangle$$

where $f \in C(K^M)$, $t \in S$ (resp. $t \in X^*$), is an element of $L(C(K^M), C(S))$ (resp. $L(C(K^M), X^{**})$). We define $B_0 = B_2 B_1$. We then have for all $f_i \in C(K)$, $t \in S$ (resp. $t \in X^*$) by (8) and (7)

$$\begin{aligned}(B_0((A_0 f_1) \ldots (A_0 f_M)))(t) &= \left(B_2 \left(\sum_\sigma \prod_{i=1}^M A_{\sigma(i)} f_i \right) \right)(t) \\ &= \left\langle \sum_\sigma \prod_{i=1}^M A_{\sigma(i)} f_i , m^{(t)} \right\rangle \\ &= P(f_1, \ldots, f_M)(t) . \end{aligned} \tag{9}$$

Let H be a subspace of K homeomorphic to Δ. Let us denote by $E: C(H) \to C(K)$ the Borsuk-Kakutani extension operator, and by $R: C(K) \to C(H)$ the restriction operator. By [P, Theorem 5.6], there exists a continuous surjection $\varphi: H \to K^M$ having a regular averaging operator. Denoting a bounded left inverse of φ° by $(\varphi^\circ)^{-1}$ we set

$$B = B_0 \varphi^{\circ -1} R , \qquad A = E \varphi^\circ A_0 .$$

This choice and (9) imply

$$P(f_1, \ldots, f_M) = B_0((A_0 f_1) \ldots (A_0 f_M)) = B((A f_1) \ldots (A f_M)) . \tag{10}$$

2. We prove the converse in the case a). The case b) is analogous. So, assume that P has a representation (4). We use the canonical isometry $\psi: C(K) \hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon C(K) \to C(K^M)$, see [K, 44.7(3)]. The diagonal map $\delta: C(K^M) \to C(K)$, $\delta: f(x_1, \ldots, x_M) \mapsto f(x, \ldots, x)$ is clearly bounded. It is now easy to see using [K, 44.4(1)], that

$$m^{(t)}: h \mapsto (B \delta \psi (A \otimes \ldots \otimes A) \psi^{-1} h)(t), \qquad h \in C(K^M) \tag{11}$$

is a bounded linear form on $C(K^M)$ and that it in fact defines a continuous extension of the map $f_1(x_1)\dots f_M(x_M) \mapsto P(f_1, \dots, f_M)(t)$. The fact that $m^{(t)}$ depends continuously on t when $C(K^M)^*$ is endowed with the weak*-topology is clear since $B \in L(C(K))$ and $\delta\psi(A \otimes \dots \otimes A)\psi^{-1}h \in C(K)$ for all $h \in C(K^M)$. □

Remarks. 1. We see that an M-homogeneous polynomial (in the sense of [Dn]) $T: C(K) \to C(K)$ has a particularly simple representation $T(f) = B((Af)^M)$, $A, B \in L(C(K))$, if K and the associated M-linear operator satisfy the hypothesis of Theorem 2.

2. The above considerations can immediately be generalized to commutative C^*-algebras using the Gelfand transform. I do not know, if anything similar is available in the non-commutative case.

3. There are many difficulties in trying to prove our result for general non-metrizable K, and we can only give some sufficient conditions. In Lemma 1 it would be enough, if for example K were an almost Milutin space, i.e. a space such that there would exist a continuous surjection from $\{0,1\}^{\mathbf{m}}$ ($\mathbf{m}$ = the topological weight of K) onto K with an averaging operator, and if K contained N disjoint closed subspaces K_n homeomorphic to $\{0,1\}^{\mathbf{m}}$ such that for all n there would exist a bounded linear extension operator $E_n: C(K_n) \to C(K)$ with $E_n f$ and $E_m g$ having disjoint supports for all $f \in C(K_n)$, $g \in C(K_m)$, where $m \neq n$. The papers [P], [Dt] and [Ha] contain examples of non-metrizable Milutin spaces. Another possibility is that K satisfies the remark just before the Lemma, but even in this case one has to assume the existence of suitable extension operators.

Concerning the proof of Theorem 2, the crucial thing is that in (10) we need something like an algebra homomorphism $C(K^M) \to C(K)$ which has a bounded left inverse. This is directly available, if there exists a continuous surjection $\varphi: K \to K^M$ having an averaging operator. It is also sufficient that K and hence K^M are almost Milutin spaces and K has a subspace H homeomorphic to $\{0,1\}^{\mathbf{m}}$ such that there exists a bounded linear extension operator $C(H) \to C(K)$.

4. Having in mind the application presented in the previous remark we show that if a compact K is the union of the almost Milutin spaces K_j, $j = 1,2$ and if there exists a bounded linear extension operator $E: C(K_1 \cap K_2) \to C(K_2)$, then K is an almost Milutin space.

Let $\mathbf{m}(j)$ be the topological weight of K_j and let $\varphi_j : D_j := \{0,1\}^{\mathbf{m}(j)} \to K_j$ be for $j = 1,2$ a continuous surjection with an averaging operator. Define $\hat{K}$ be the disjoint union of the spaces K_j and $D := \{0,1\} \times D_1 \times D_2$. Define $\psi: D \to \hat{K}$ by $\psi(x) = \psi_j(\varrho_j(x))$, if the first coordinate of x equals $j-1$, where ϱ_j is the canonical projection from D onto D_j. It is clear that φ is a continuous surjection. If Q_j denotes a bounded projection from $C(D_j)$ onto $\varphi_j^\circ C(K_j)$, the operator

$$f(x) \mapsto (Q_j f|_{D_j}) \circ \varrho_j(x), \quad \text{for} \quad x \in \{j-1\} \times D_1 \times D_2$$

is easily seen to be a bounded projection from $C(D)$ onto $\psi^\circ C(\hat{K})$.

We define the map $\iota: \hat{K} \to K$, $\iota(x) = \iota_j(x)$ for $x \in K_j$, where $\iota_j: K_j \to K$ is the inclusion. We define the operator Q on $C(\hat{K})$ by $(Qf)(x) = f(x)$, if $x \in K_1 \subset \hat{K}$ and $(Qf)(x) = E(f \circ \iota^* - f)(x) + f(x)$, if $x \in K_2 \subset \hat{K}$, where ι^* is a map from $K_1 \cap K_2 \subset K_2 \subset \hat{K}$ into $K_1 \subset \hat{K}$, $\iota^*(y)$ denotes the inverse image in $K_1 \subset \hat{K}$ (with respect to ι) of the element $\iota_2(y)$. It is easily verified that Q is a bounded projection from $C(\hat{K})$ onto $\iota^\circ C(K)$; note that $g \circ \iota \circ \iota^*(x) - g \circ \iota(x) = 0$ for $g \in C(K)$, $x \in K_1 \cap K_2 \subset K_2 \subset \hat{K}$.

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