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Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions

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Dedicated to Professor Hari M Srivastava

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Abstract

In this paper, we introduce and investigate two new subclasses $H_{\sigma}^{\mu}(\lambda, \varphi)$ and $M_{\sigma}^{\nu}(\lambda, \mu, \varphi)$ of Ma-Minda bi-univalent functions defined by using subordination in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. For functions belonging to these new subclasses, we obtain estimates for the initial coefficients $|a_2|$ and $|a_3|$. The results presented in this paper would generalize those in related works of several earlier authors.

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1 Introduction

Let C be a set of complex numbers and let $N = \{1, 2, 3, \dots\} = N_0 \setminus \{0\}$ be a set of positive integers. Let A be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Also, let S denote a subclass of all functions in A which are univalent in D (for details, see [1, 2]).

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk D . However, the famous Koebe one-quarter theorem [1] ensures that the image of the unit disk D under every function $f \in S$ contains a disk of radius $1/4$. Thus, every univalent function $f \in S$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in D)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (1.2)$$

A function $f \in A$ is said to be bi-univalent in D if both f and f^{-1} are univalent in D . Let σ denote the class of bi-univalent functions defined in the unit disk D . In 1967, Lewin [3] first introduced the class σ of bi-univalent functions and showed that $|a_2| \leq 1.51$ for every $f \in \sigma$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \sigma$. Later, Netanyahu [5] proved that $\max_{f \in \sigma} |a_2| = 3/4$. The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$) is still an open problem.

Brannan and Taha [6] (see also [7]) introduced certain subclasses of a bi-univalent function class σ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order α ($0 < \alpha \leq 1$), respectively (see [8]). Thus, following Brannan and Taha [6] (see also [7]), a function $f \in A$ is in the class $S_\sigma^*(\alpha)$ of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$) if both functions f and f^{-1} are strongly starlike functions of order α . The classes $S_\sigma^*(\alpha)$ and $K_\sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α , corresponding (respectively) to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S_\sigma^*(\alpha)$ and $K_\sigma(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [6, 7]).

An analytic function f is subordinate to an analytic function g , written $f \prec g$, if there is an analytic function w with $|w(z)| \leq |z|$ such that $f = (g(w))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(D) \subseteq g(D)$. Ma and Minda [9] unified various subclasses of starlike and convex functions for which either of the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk D , $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps D onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes $S^*(\varphi)$ and $K(\varphi)$ of Ma-Minda starlike and Ma-Minda convex functions are respectively characterized by $zf'(z)/f(z) \prec \varphi(z)$ or $1 + zf''(z)/f'(z) \prec \varphi(z)$. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $S_\sigma^*(\varphi)$ and $K_\sigma(\varphi)$. Recently, Srivastava *et al.* [10], Frasin and Aouf [11] and Caglar *et al.* [12] introduced and investigated various subclasses of bi-univalent functions and found estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these classes. Very recently, Ali *et al.* [13], Kumar *et al.* [14], Srivastava *et al.* [15] and Xu *et al.* [16] unified and extended some related results in [7, 10–12, 17] by generalizing their classes using subordination.

Motivated by Ali *et al.* [13] and Kumar *et al.* [14], we investigate the estimates for the initial coefficients $|a_2|$ and $|a_3|$ of bi-univalent functions of Ma-Minda type belonging to the classes $H_\sigma^\mu(\lambda, \varphi)$ and $M_\sigma^\nu(\lambda, \mu, \varphi)$ defined in Section 2. Our results generalize several well-known results in [10–14] and these are also pointed out.

2 Coefficient estimates

Throughout this paper, we assume that φ is an analytic univalent function with positive real part in D , $\varphi(D)$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$, and $\varphi'(0) > 0$. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \quad (B_1 > 0). \tag{2.1}$$

With this assumption on φ , we now introduce the following subclasses of Ma-Minda bi-univalent functions.

Definition 2.1 A function $f \in \sigma$ given by (1.1) is said to be in the class $H_\sigma^\mu(\lambda, \varphi)$ if it satisfies

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} < \varphi(z) \quad (\lambda \geq 1, \mu \geq 1, z \in D) \tag{2.2}$$

and

$$(1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} < \varphi(w) \quad (\lambda \geq 1, \mu \geq 1, w \in D), \tag{2.3}$$

where the function g is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \tag{2.4}$$

We note that, for suitable choices λ, μ and φ , the class $H_\sigma^\mu(\lambda, \varphi)$ reduces to the following known classes.

- (1) $H_\sigma^\mu(\lambda, (\frac{1+z}{1-z})^\alpha) = H_\sigma^\mu(\lambda, \alpha)$ ($\lambda \geq 1, 0 < \alpha \leq 1, \mu \geq 0$) (see Caglar *et al.* [12, Definition 2.1]);
- (2) $H_\sigma^\mu(\lambda, \frac{1+(1-2\beta)z}{1-z}) = H_\sigma^\mu(\lambda, \beta)$ ($\lambda \geq 1, 0 \leq \beta < 1, \mu \geq 0$) (see Caglar *et al.* [12, Definition 3.1]);
- (3) $H_\sigma^1(\lambda, \varphi) = H_\sigma(\lambda, \varphi)$ ($\lambda \geq 1$) (see Kumar *et al.* [14, Definition 1.1]);
- (4) $H_\sigma^\mu(1, \varphi) = H_\sigma^\mu(\varphi)$ ($\mu \geq 0$) (see Kumar *et al.* [14, Definition 2.1]);
- (5) $H_\sigma^1(1, \varphi) = H_\sigma(\varphi)$ (see Ali *et al.* [13, p.345]);
- (6) $H_\sigma^1(\lambda, (\frac{1+z}{1-z})^\alpha) = H_\sigma(\lambda, \alpha)$ ($\lambda \geq 1, 0 < \alpha \leq 1$) (see Frasin and Aouf [11, Definition 2.1]);
- (7) $H_\sigma^1(\lambda, \frac{1+(1-2\beta)z}{1-z}) = H_\sigma(\lambda, \beta)$ ($\lambda \geq 1, 0 \leq \beta < 1$) (see Frasin and Aouf [11, Definition 3.1]);
- (8) $H_\sigma^1(1, (\frac{1+z}{1-z})^\alpha) = H_\sigma(\alpha)$ ($0 < \alpha \leq 1$) (see Srivastava *et al.* [10, Definition 1]);
- (9) $H_\sigma^1(1, \frac{1+(1-2\beta)z}{1-z}) = H_\sigma(\beta)$ ($0 \leq \beta < 1$) (see Srivastava *et al.* [10, Definition 2]).

For functions in the class $H_\sigma^\mu(\lambda, \varphi)$, the following estimates are obtained.

Theorem 2.1 Let the function f given by (1.1) be in the class $H_\sigma^\mu(\lambda, \varphi)$, $\lambda \geq 1$ and $\mu \geq 0$. Then

$$|a_2| \leq \min \left\{ \frac{B_1}{\lambda + \mu}, \sqrt{\frac{2(B_1 + |B_2 - B_1|)}{(1 + \mu)(2\lambda + \mu)}} \right\} \tag{2.5}$$

and

$$|a_3| \leq \begin{cases} \min \left\{ \frac{B_1}{2\lambda + \mu} + \frac{B_1^2}{(\lambda + \mu)^2}, \frac{2(B_1 + |B_2 - B_1|)}{(1 + \mu)(2\lambda + \mu)} \right\}, & 0 \leq \mu < 1, \\ \frac{B_1}{2\lambda + \mu} + \frac{2|B_2 - B_1|}{(1 + \mu)(2\lambda + \mu)}, & \mu \geq 1. \end{cases} \tag{2.6}$$

Proof Since $f \in H_\sigma^\mu(\lambda, \varphi)$, there exist two analytic functions $u, v : D \rightarrow D$, with $u(0) = v(0) = 0$, such that

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} = \varphi(u(z)) \tag{2.7}$$

and

$$(1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{z} \right)^{\mu-1} = \varphi(v(w)). \tag{2.8}$$

Define the functions p and q by

$$\begin{aligned} p(z) &= \frac{1 + u(z)}{1 - u(z)} = 1 + p_1z + p_2z^2 + \dots \quad \text{and} \\ q(z) &= \frac{1 + v(z)}{1 - v(z)} = 1 + q_1z + q_2z^2 + \dots, \end{aligned} \tag{2.9}$$

or, equivalently,

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left(p_1z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \dots \right) \tag{2.10}$$

and

$$v(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left(q_1z + \left(q_2 - \frac{q_1^2}{2} \right) z^2 + \dots \right). \tag{2.11}$$

It is clear that p and q are analytic in D and $p(0) = q(0) = 1$. Since $u, v : D \rightarrow D$, the functions p and q have positive real part in D , and hence $|p_i| \leq 2$ and $|q_i| \leq 2$ ($i = 1, 2, \dots$). By virtue of (2.7), (2.8), (2.10) and (2.11), we have

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} = \varphi \left(\frac{p(z) - 1}{p(z) + 1} \right) \tag{2.12}$$

and

$$(1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{z} \right)^{\mu-1} = \varphi \left(\frac{q(w) - 1}{q(w) + 1} \right). \tag{2.13}$$

Using (2.10), (2.11), together with (2.1), we easily obtain

$$\varphi \left(\frac{p(z) - 1}{p(z) + 1} \right) = 1 + \frac{1}{2} B_1 p_1 z + \left(\frac{1}{2} B_1 \left(p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} B_2 p_1^2 \right) z^2 + \dots \tag{2.14}$$

and

$$\varphi \left(\frac{q(w) - 1}{q(w) + 1} \right) = 1 + \frac{1}{2} B_1 q_1 w + \left(\frac{1}{2} B_1 \left(q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} B_2 q_1^2 \right) w^2 + \dots. \tag{2.15}$$

Since $f \in \sigma$ has the Maclaurin series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion given by (1.2). Also, since

$$\begin{aligned} f'(z) &= 1 + 2a_2z + 3a_3z^2 + \dots \quad \text{and} \\ g'(w) &= 1 - 2a_2w + 3(2a_2 - a_3)w^2 - \dots, \end{aligned}$$

it follows from (2.12)-(2.15) that

$$(\lambda + \mu)a_2 = \frac{1}{2}B_1p_1, \tag{2.16}$$

$$(2\lambda + \mu)a_3 + \frac{(\mu - 1)(2\lambda + \mu)}{2}a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2, \tag{2.17}$$

$$-(\lambda + \mu)a_2 = \frac{1}{2}B_1q_1 \tag{2.18}$$

and

$$-(2\lambda + \mu)a_3 + \frac{(3 + \mu)(2\lambda + \mu)}{2}a_2^2 = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2q_1^2. \tag{2.19}$$

From (2.16) and (2.18), we get

$$p_1 = -q_1 \tag{2.20}$$

and

$$8(\lambda + \mu)^2a_2^2 = B_1^2(p_1^2 + q_1^2). \tag{2.21}$$

Also, from (2.17) and (2.19), we obtain

$$(1 + \mu)(2\lambda + \mu)a_2^2 = \frac{1}{2}B_1(p_2 + q_2) + \frac{1}{4}(B_2 - B_1)(p_1^2 + q_1^2),$$

or

$$a_2^2 = \frac{2B_1(p_2 + q_2) + (B_2 - B_1)(p_1^2 + q_1^2)}{4(1 + \mu)(2\lambda + \mu)}. \tag{2.22}$$

Since $|p_i| \leq 2$ and $|q_i| \leq 2$ ($i = 1, 2$), it follows from (2.21) and (2.22) that

$$|a_2| \leq \frac{B_1}{\lambda + \mu} \tag{2.23}$$

and

$$|a_2| \leq \sqrt{\frac{2(B_1 + |B_2 - B_1|)}{(1 + \mu)(2\lambda + \mu)}}, \tag{2.24}$$

which yields the desired estimate on $|a_2|$ as asserted in (2.5).

Next, in order to find the bound on $|a_3|$, by subtracting (2.19) from (2.17), we get

$$2(2\lambda + \mu)(a_3 - a_2^2) = \frac{1}{2}B_1(p_2 - q_2) + \frac{1}{4}(B_2 - B_1)(p_1^2 - q_1^2). \tag{2.25}$$

Using (2.20) and (2.21) in (2.25), we have

$$a_3 = \frac{1}{4(2\lambda + \mu)}B_1(p_2 - q_2) + \frac{1}{4(\lambda + \mu)^2}B_1^2p_1^2,$$

which evidently yields

$$|a_3| \leq \frac{B_1}{2\lambda + \mu} + \frac{B_1^2}{(\lambda + \mu)^2}. \tag{2.26}$$

On the other hand, by using (2.20) and (2.22) in (2.25), we obtain

$$a_3 = \frac{B_1[(\mu + 3)p_2 + (1 - \mu)q_2] + (B_2 - B_1)(p_1^2 + q_1^2)}{4(1 + \mu)(2\lambda + \mu)}, \tag{2.27}$$

and applying $|p_i| \leq 2$ and $|q_i| \leq 2$ ($i = 1, 2$) for (2.27), we get

$$|a_3| \leq \frac{B_1}{2(2\lambda + \mu)} \left[\frac{\mu + 3}{1 + \mu} + \frac{|1 - \mu|}{1 + \mu} \right] + \frac{2|B_2 - B_1|}{(1 + \mu)(2\lambda + \mu)}. \tag{2.28}$$

Now, we consider the bounds on $|a_3|$ according to μ .

Case 1. If $0 \leq \mu < 1$, then from (2.28)

$$|a_3| \leq \frac{2(B_1 + |B_2 - B_1|)}{(1 + \mu)(2\lambda + \mu)}. \tag{2.29}$$

Case 2. If $\mu \geq 1$, then from (2.28)

$$|a_3| \leq \frac{B_1}{2\lambda + \mu} + \frac{2|B_2 - B_1|}{(1 + \mu)(2\lambda + \mu)}. \tag{2.30}$$

Thus, from (2.26), (2.29) and (2.30), we obtain the desired estimate on $|a_3|$ given in (2.6). This completes the proof of Theorem 2.1. \square

Putting $\mu = 1$ and $\lambda = \mu = 1$ in Theorem 2.1, we respectively get the following Corollaries 2.1 and 2.2.

Corollary 2.1 *If $f \in H_\sigma(\lambda, \varphi)$ ($\lambda \geq 1$), then*

$$|a_2| \leq \min \left\{ \frac{B_1}{\lambda + 1}, \sqrt{\frac{B_1 + |B_2 - B_1|}{2\lambda + 1}} \right\}$$

and

$$|a_3| \leq \begin{cases} \min \left\{ \frac{B_1}{2\lambda + 1} + \frac{B_1^2}{(\lambda + 1)^2}, \frac{B_1 + |B_2 - B_1|}{2\lambda + 1} \right\}, & 0 \leq \mu < 1, \\ \frac{B_1 + |B_2 - B_1|}{2\lambda + 1}, & \mu \geq 1. \end{cases}$$

Corollary 2.2 *If $f \in H_\sigma(\varphi)$, then*

$$|a_2| \leq \min \left\{ \frac{B_1}{2}, \sqrt{\frac{B_1 + |B_2 - B_1|}{3}} \right\}$$

and

$$|a_3| \leq \begin{cases} \min \left\{ \frac{B_1}{3} + \frac{B_1^2}{4}, \frac{B_1 + |B_2 - B_1|}{3} \right\}, & 0 \leq \mu < 1, \\ \frac{B_1 + |B_2 - B_1|}{3}, & \mu \geq 1. \end{cases}$$

Remark 2.1 The estimates of the coefficients $|a_2|$ and $|a_3|$ of Corollaries 2.1 and 2.2 are the improvement of the estimates obtained in [14, Theorem 2.1] and [13, Theorem 2.1], respectively.

Remark 2.2 If we set

$$\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + \dots \quad (0 \leq \beta < 1)$$

in Corollaries 2.1 and 2.2, the results obtained improve the results in [11, Theorem 3.2, inequalities (3.3) and (3.4)] and [10, Theorem 2, inequality (3.3)], respectively.

Definition 2.2 Let $\gamma \in C^* = C \setminus \{0\}$, $\lambda \geq 0$ and $\mu \geq 0$. A function $f \in \sigma$ given by (1.1) is said to be in the class $M_\sigma^\gamma(\lambda, \mu, \varphi)$, if the following subordinations hold:

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z) + (2\lambda\mu + \lambda - \mu)z^2f''(z) + \lambda\mu z^3f'''(z)}{(1 - \lambda + \mu)f(z) + (\lambda - \mu)zf'(z) + \lambda\mu z^2f''(z)} - 1 \right) \prec \varphi(z)$$

and

$$1 + \frac{1}{\gamma} \left(\frac{wg'(w) + (2\lambda\mu + \lambda - \mu)w^2g''(w) + \lambda\mu w^3g'''(w)}{(1 - \lambda + \mu)g(w) + (\lambda - \mu)wg'(w) + \lambda\mu w^2g''(w)} - 1 \right) \prec \varphi(w),$$

where the function g is defined by (2.4).

We note that, by choosing appropriate values for λ, μ, γ and φ , the class $M_\sigma^\gamma(\lambda, \mu, \varphi)$ reduces to several earlier known classes.

- (1) $M_\sigma^\gamma(\lambda, 0, \varphi) = N_{\sigma, \gamma}^\lambda(\varphi)$ ($\lambda \geq 0, \gamma \in C^*$) (see Kumar *et al.* [14, Definition 2.2]);
- (2) $M_\sigma^1(0, 0, \frac{1+(1-2\beta)z}{1-z}) = S_\sigma^*(\beta)$ ($0 \leq \beta < 1$) (see Brannan and Taha [6, Definition 3.1]);
- (3) $M_\sigma^1(1, 0, \frac{1+(1-2\beta)z}{1-z}) = K_\sigma(\beta)$ ($0 \leq \beta < 1$) (see Brannan and Taha [6, Definition 4.1]);
- (4) $M_\sigma^1(0, 0, (\frac{1+z}{1-z})^\alpha) = S_\sigma^*(\alpha)$ ($0 < \alpha \leq 1$) (see Taha [7]).

For functions in the class $M_\sigma^\gamma(\lambda, \mu, \varphi)$, the following estimates are derived.

Theorem 2.2 Let $\gamma \in C^*, \lambda \geq 0$ and $\mu \geq 0$. If $f \in M_\sigma^\gamma(\lambda, \mu, \varphi)$, then

$$|a_2| \leq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|[2(6\lambda\mu + 2\lambda - 2\mu + 1) - (2\lambda\mu + \lambda - \mu + 1)^2]B_1^2\gamma + 2(2\lambda\mu + \lambda - \mu + 1)^2(B_1 - B_2)|}} \quad (2.31)$$

and

$$|a_3| \leq \frac{|\gamma|(|B_1 + |B_2 - B_1||)}{|2(6\lambda\mu + 2\lambda - 2\mu + 1) - (2\lambda\mu + \lambda - \mu + 1)^2|}. \quad (2.32)$$

Proof If $f \in M_\sigma^\gamma(\lambda, \mu, \varphi)$, then there are analytic functions $u, v : D \rightarrow D$, with $u(0) = v(0) = 0$, satisfying

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z) + (2\lambda\mu + \lambda - \mu)z^2f''(z) + \lambda\mu z^3f'''(z)}{(1 - \lambda + \mu)f(z) + (\lambda - \mu)zf'(z) + \lambda\mu z^2f''(z)} - 1 \right) = \varphi(u(z)) \quad (2.33)$$

and

$$1 + \frac{1}{\gamma} \left(\frac{wg'(w) + (2\lambda\mu + \lambda - \mu)w^2g''(w) + \lambda\mu w^3g'''(w)}{(1 - \lambda + \mu)g(w) + (\lambda - \mu)wg'(w) + \lambda\mu w^2g''(w)} - 1 \right) = \varphi(v(w)). \tag{2.34}$$

Let p and q be defined as in (2.8), then it is clear from (2.33), (2.34), (2.9) and (2.10) that

$$\begin{aligned} 1 + \frac{1}{\gamma} \left(\frac{zf'(z) + (2\lambda\mu + \lambda - \mu)z^2f''(z) + \lambda\mu z^3f'''(z)}{(1 - \lambda + \mu)f(z) + (\lambda - \mu)zf'(z) + \lambda\mu z^2f''(z)} - 1 \right) \\ = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right) \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} 1 + \frac{1}{\gamma} \left(\frac{wg'(w) + (2\lambda\mu + \lambda - \mu)w^2g''(w) + \lambda\mu w^3g'''(w)}{(1 - \lambda + \mu)g(w) + (\lambda - \mu)wg'(w) + \lambda\mu w^2g''(w)} - 1 \right) \\ = \varphi\left(\frac{q(w) - 1}{q(w) + 1}\right). \end{aligned} \tag{2.36}$$

It follows from (2.35), (2.36), (2.14) and (2.15) that

$$(2\lambda\mu + \lambda - \mu + 1)a_2 = \frac{1}{2}B_1p_1\gamma, \tag{2.37}$$

$$\begin{aligned} -(2\lambda\mu + \lambda - \mu + 1)^2a_2^2 + 2(6\lambda\mu + 2\lambda - 2\mu + 1)a_3 \\ = \gamma \left[\frac{1}{2}B_1 \left(p_2 - \frac{1}{2}p_1^2 \right) + \frac{1}{4}B_2p_1^2 \right], \end{aligned} \tag{2.38}$$

$$-(2\lambda\mu + \lambda - \mu + 1)a_2 = \frac{1}{2}B_1q_1\gamma \tag{2.39}$$

and

$$\begin{aligned} [4(6\lambda\mu + 2\lambda - 2\mu + 1) - (2\lambda\mu + \lambda - \mu + 1)^2]a_2^2 - 2(6\lambda\mu + 2\lambda - 2\mu + 1)a_3 \\ = \gamma \left[\frac{1}{2}B_1 \left(q_2 - \frac{1}{2}q_1^2 \right) + \frac{1}{4}B_2q_1^2 \right]. \end{aligned} \tag{2.40}$$

Equations (2.37) and (2.39) yield

$$p_1 = -q_1 \tag{2.41}$$

and

$$8(2\lambda\mu + \lambda - \mu + 1)^2a_2^2 = B_1^2\gamma^2(p_1^2 + q_1^2). \tag{2.42}$$

From (2.38), (2.40), (2.41) and (2.42), it follows that

$$a_2^2 = \frac{\gamma^2 B_1^3 (p_2 + q_2)}{4[(2(6\lambda\mu + 2\lambda - 2\mu + 1) - (2\lambda\mu + \lambda - \mu + 1)^2)B_1^2\gamma + (2\lambda\mu + \lambda - \mu + 1)^2(B_1 - B_2)]}$$

which yields the desired estimate on $|a_2|$ as described in (2.31).

Similarly, it can be obtained from (2.38), (2.40) and (2.41) that

$$a_3 = \frac{\gamma B_1 [p_2(4(6\lambda\mu + 2\lambda - 2\mu + 1) - (2\lambda\mu + \lambda - \mu + 1)^2) + q_2(2\lambda\mu + \lambda - \mu + 1)^2]}{8[2(6\lambda\mu + 2\lambda - 2\mu + 1) - (2\lambda\mu + \lambda - \mu + 1)^2](6\lambda\mu + 2\lambda - 2\mu + 1)} + \frac{2\gamma(B_2 - B_1)(6\lambda\mu + 2\lambda - 2\mu + 1)p_1^2}{8[2(6\lambda\mu + 2\lambda - 2\mu + 1) - (2\lambda\mu + \lambda - \mu + 1)^2](6\lambda\mu + 2\lambda - 2\mu + 1)}$$

which easily leads to the desired estimate (2.32) on $|a_3|$. □

Taking $\mu = 0$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.3 [14, Theorem 2.3] *If $f \in N_{\sigma,\gamma}^\lambda(\varphi)$, then*

$$|a_2| \leq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|(1 + 2\lambda - \lambda^2)B_1^2\gamma + (1 + \lambda)^2(B_1 - B_2)|}} \quad \text{and} \quad |a_3| \leq \frac{|\gamma|(B_1 + |B_2 - B_1|)}{|1 + 2\lambda - \lambda^2|}.$$

Further, for $\gamma = 1$, putting $\lambda = 0$ and $\lambda = 1$ in Corollary 2.3, respectively, we have the following Corollaries 2.4 and 2.5.

Corollary 2.4 [13, Corollary 2.1] *If $f \in M_\sigma^1(0, 0, \varphi) = ST_\sigma(\varphi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 + B_1 - B_2|}} \quad \text{and} \quad |a_3| \leq B_1 + |B_2 - B_1|.$$

Corollary 2.5 [13, Corollary 2.2] *If $f \in M_\sigma^1(1, 0, \varphi) = CV_\sigma(\varphi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2|B_1^2 + 2B_1 - 2B_2|}} \quad \text{and} \quad |a_3| \leq \frac{1}{2}(B_1 + |B_2 - B_1|).$$

Remark 2.3 If we set

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots \quad (0 < \alpha \leq 1)$$

and

$$\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)^2 z^2 + \dots \quad (0 \leq \beta < 1)$$

in Corollaries 2.4 and 2.5, we obtain the results of Brannan and Taha [6, Theorems 2.1, 3.1 and 4.1, respectively].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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References

1. Duren, PL: Univalent Functions. Grundlehren der Mathematischen Wissenschaften, Band 259. Springer, Berlin (1983)
2. Srivastava, HM, Owa, S (eds.): Current Topics in Analytic Function Theory. World Scientific, Singapore (1992)
3. Lewin, M: On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* **18**, 63-68 (1967)
4. Brannan, DA, Clunie, JG (eds.): Aspects of contemporary complex analysis. In: Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham, July 1-20, 1979. Academic Press, London (1980)
5. Netanyahu, E: The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Arch. Ration. Mech. Anal.* **32**, 100-112 (1969)
6. Brannan, DA, Taha, TS: On some classes of bi-univalent functions. In: Mazhar, SM, Hamoui, A, Faour, NS (eds.) *Mathematical Analysis and Its Applications*, Kuwait, February 18-21, 1985. KFAS Proceedings Series, vol. 3, pp. 53-60. Pergamon, Oxford, (1988) see also *Stud. Univ. Babeş-Bolyai, Math.* **31**(2), 70-77 (1986)
7. Taha, TS: Topics in Univalent Function Theory. Ph.D. thesis, University of London (1981)
8. Brannan, DA, Clunie, J, Kirwan, WE: Coefficient estimates for a class of star-like functions. *Can. J. Math.* **22**, 476-485 (1970)
9. Ma, WC, Minda, D: A unified treatment of some special classes of univalent functions. In: Proceedings of the Conference on Complex Analysis, Tianjin, 1992. *Conf. Proc. Lecture Notes Anal.*, vol. I, pp. 157-169. International Press, Cambridge (1994)
10. Srivastava, HM, Mishra, AK, Gochhayat, P: Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.* **23**, 1188-1192 (2010)
11. Frasin, BA, Aouf, MK: New subclasses of bi-univalent functions. *Appl. Math. Lett.* **24**, 1569-1573 (2011)
12. Caglar, M, Orhan, H, Yagmur, N: Coefficient bounds for new subclasses of bi-univalent functions. *Filomat* (2013, in press)
13. Ali, RM, Lee, SK, Ravichandran, V, Supramaniam, S: Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. *Appl. Math. Lett.* **25**, 344-351 (2012)
14. Kumar, SS, Kumar, V, Ravichandran, V: Estimates for the initial coefficients of bi-univalent functions. Preprint
15. Srivastava, HM, Bulut, S, Çağlar, M, Yağmur, N: Coefficient estimates for a general subclass of analytic and bi-univalent functions. *Filomat* **27**, 831-842 (2013)
16. Xu, Q-H, Xiao, H-G, Srivastava, HM: A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. *Appl. Math. Comput.* **218**, 11461-11465 (2012)
17. Xu, Q-H, Gui, Y-C, Srivastava, HM: Coefficient estimates for a certain subclass of analytic and bi-univalent functions. *Appl. Math. Lett.* **25**, 990-994 (2012)

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