Induced Choquet Ordered Averaging Operator and Its Application to Group Decision Making

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Yager (*Fuzzy Sets Syst* 2003;137:59–69) extended the idea of order-induced aggregation to the Choquet aggregation and defined a more general type of Choquet integral operator called the induced Choquet ordered averaging (I-COA) operator, which take as their argument pairs, in which one component called order-inducing variable is used to induce an ordering over the second components called argument variable and then aggregated. The aim of this paper is to develop the I-COA operator. Some of its properties are investigated. We show its relationship to the induced-ordered weighted averaging operator. Finally, we provide some I-COA operators to aggregate fuzzy preference relations in group decision-making problems. © 2009 Wiley Periodicals, Inc.

1. INTRODUCTION

Preference relations are the most common representation structures of information used in decision-making problems because they are a useful tool in modeling decision processes. The aggregation of decision makers' preferences consisting of combining the individual preferences into a collective one in such a way that all of the properties contained in all the individual preferences are summarized or reflected, is a necessary and very important task to perform when we want to obtain a final solution of multicriteria decision-making or group decision-making (GDM) problems.[1–3]

In 1988, Yager[4] provided a family of averaging operators called the ordered weighted averaging (OWA) operator where the ordering of the arguments plays a central role in the operation, which is commutative, idempotent, continuous, monotonic, neutral, compensative, and stable for positive linear transformations.[5] As an extension of arithmetic mean and weighted mean, OWA operator has recently been used in a wide range of decision-making[6–8] and other different fields. Later on, Yager and Filev[9,10] provided a generalization of the process used for ordering the

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argument values and introduced a more general type of OWA operator, which they named the induced ordered weighted averaging (IOWA) operator. At present, the induced aggregation operators are an interesting research topic.[11–17] Xu and Da[11] introduced a class of induced ordered weighted geometric averaging operator, which have many desirable properties similar to those of the IOWA operators. Yager[12] developed an induced fuzzy integral aggregation operator, which extends the Sugeno integral aggregation operator by allowing the ordering operation to be based on values other than those being aggregated. Chen and Chen[13] presented some FN-IOWA operators based on fuzzy numbers. Yager[14] suggested a number of applications of the IOWA aggregation operator, and extended the idea of order-induced aggregation to the Choquet aggregation, resulting in what he called the induced Choquet ordered averaging (I-COA) operator. Chiclana and Herrera-Viedma[15] introduced the induced ordered weighted geometric (IOWG) operator and its properties and provided some IOWG operators to aggregate multiplicative preference relations in GDM problems. Xu[16] developed some induced uncertain linguistic OWA operators to GDM with uncertain linguistic information. Chiclana and Herrera-Viedma[17] provided some IOWA operators to aggregate fuzzy preference relations in GDM problems.

Although there has been progress in induced aggregation operators, most of the work is based on the assumption that preferences of decision makers are independent, which is characterized by an independence axiom,[18,19] that is, the operator is based on the implicit assumption that preferences of decision makers are independent of one another; their effects are viewed as additive. Since subjective preference evaluation of human being always shows nonlinearity, there is usually an interaction among preference of decision makers. Therefore, the independence axiom generally cannot be satisfied, and this assumption condition is too strong to match decision-making behavior in the real world. Thus, it is more reasonable and appropriate for us to use a nonadditive measure instead of traditional additive measure operators to approximate people's evaluation processes for decision-making problems. In 1974, Sugeno[20] introduced the concept of fuzzy measure (nonadditive measure), which only makes a monotonicity instead of additivity property. For decision-making problems, it does not need assumption that criteria or preferences are independent of one another and was used as an effective tool for modeling interaction phenomena in decision making.[21–25] As a generalization of the weighted average and the OWA operator,[26–28] Choquet integral with respect to fuzzy measure allows for the inclusion of context-dependent information, such as relevancy, in the aggregation process. Recently, Choquet integral operator has received increasing attention for decision-making problems.[27–32] Central to the implementation of the Choquet integral aggregation operator is an ordering operation based on the arguments to be aggregated. According to the idea of induced aggregation operators, in Ref. 14 Yager defined a more general type of the Choquet integral operator called the I-COA operator. Inspired by this work, in this paper, we further develop the I-COA operator and apply it to deal with group decision making with fuzzy preference relations.

To do this, this paper is set out as follows. In Section 2, we summarize the basic operators used in this study: the OWA, IOWA operators, and Choquet integral. In Section 3, we introduce the I-COA operator. Moreover, some of its properties are investigated. In Section 4, we define two particular cases of I-COA operator used to aggregate fuzzy preference relations: the I-I-COA and P-I-COA operators. Moreover, we present a process for GDM problems with fuzzy preference relations based on the above-mentioned two I-COA operators. Finally, in Section 5 we draw our conclusions.

2. PRELIMINARIES: IOWA AND CHOQUET INTEGRAL OPERATORS

In this section, we start by providing a summary of the concepts of OWA, IOWA operators, and Choquet integral, which will be used throughout the paper.

2.1. The OWA and IOWA Operators

In Ref. 4, Yager provided a definition of the OWA operator as follows:

DEFINITION 1. *An OWA operator of dimension n is a function OWA:* $(R^+)^n \rightarrow R^+$*, which has associated a set of weights or weighting vector* $W = (w_1, \ldots, w_n)$ *with it with* $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$*, and is defined to aggregate a list of values* $\{f_1, \ldots, f_n\}$ *according to the following expression:*

$$\text{OWA}_w(f_1, \ldots, f_n) = \sum_{i=1}^{n} w_i f_{(i)}, \tag{1}$$

where $(\cdot)$: $\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ *is a permutation such that* $f_{(1)} \geq f_{(2)} \geq \cdots \geq f_{(n)}$*, i.e.,* $f_{(i)}$ *the ith largest value in the set* $\{f_1, \ldots, f_n\}$.

Remark 1. For convenience in the following discussion, in this paper we limit domain of the function OWA in R^+.

An issue in the definition of the OWA operator is how to obtain the associated weighting vector. Given a function Q: $[0, 1] \rightarrow [0, 1]$ such that $Q(0) = 0$, $Q(1) = 1$, and if $x > y$ then $Q(x) \geq Q(y)$, an OWA aggregation guided by this function can be obtained as follows[4]:

$$\text{OWA}_Q(f_1, \ldots, f_n) = \sum_{i=1}^{n} w_i f_{(i)},$$

where

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, 2, \ldots, n. \tag{2}$$

The function Q is called basic unit-interval monotone (BUM) functions in Ref. 14 and "is particularly useful in situations in which the imperative guiding the OWA aggregation is expressed linguistically by a quantifier." In Ref. 33, BUM functions are called regular increasing monotone (RIM) quantifiers.

An important feature of the OWA operator is the reordering step. During this step, the arguments are ordered by their values. In Ref. 34, Mitchell and Estrakh described a modified OWA operator in which the input arguments are not rearranged according to their values but rather using a function of the arguments. Inspired by this work, Yager and Filev introduced[9,10] a more general type of OWA operator, called IOWA operator, which is defined as follows:

DEFINITION 2. *An IOWA operator of dimension n is a function IOWA:* $(R^+ \times R^+)^n \to R^+$*, which has associated a set of weights or weighting vector* $W = (w_1, \ldots, w_n)$ *with it with* $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$*, and it is defined to aggregate the set of second arguments of a list of n 2-tuples* $\{\langle u_1, f_1\rangle, \ldots, \langle u_n, f_n\rangle\}$ *according to the following expression:*

$$\mathrm{IOWA}_w(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle) = \sum_{i=1}^{n} w_i f_{(i)}, \tag{3}$$

where u_i *in 2-tuple* $\langle u_i, f_i\rangle$ *is referred to as the order-inducing variable and* f_i *as the argument variable.* $(\cdot)$: $\{1, \ldots, n\} \to \{1, \ldots, n\}$ *is a permutation such that* $u_{(1)} \geq u_{(2)} \geq \cdots \geq u_{(n)}$*, i.e.,* $\langle u_{(i)}, f_{(i)}\rangle$*is the 2-tuple with* $u_{(i)}$*the ith largest value in the set* $\{u_1, \ldots, u_n\}$*. It is assumed that* f_i *is an exact numerical value in* R^+*, whereas* u_i *can be drawn from any positive ordinal space* R^+*.*

In the above definition, the reordering of the set of values to aggregate, $\{f_1, \ldots, f_n\}$, is induced by the reordering of the set of values $\{u_1, \ldots, u_n\}$ associated with them, which is based on their magnitude. The main difference between the OWA operator and the IOWA operator resides in the reordering step of the argument variable. In the case of OWA operator, this reordering is based on the magnitude of the values to be aggregated, whereas in the case of IOWA operator an order-inducing variable is used as the criterion to induce that reordering. Another important distinction between OWA and IOWA aggregation processes arises when there is a tie in the ordering operation. Ties in the arguments to be aggregated do not present a problem in OWA aggregation processes; the same result is obtained no matter the order in which the tied arguments are placed. This is not the case for IOWA aggregation processes.[10]

2.2. Fuzzy Measure and Choquet Integral

DEFINITION 3. *Let a universal set* $X = \{x_1, x_2, \ldots, x_n\}$ *and* $P(X)$ *be the power set of* X*. A fuzzy measure on* X *is a set function* m: $P(X) \to [0, 1]$*, satisfies the following conditions:*

(1) $m(\phi) = 0$, $m(X) = 1$ (*boundary conditions*) and
(2) If $A, B \in P(X)$ and $A \subseteq B$ then $m(A) \leq m(B)$ (*monotonicity*).

If the universal set X is infinite, it is necessary to add an extra axiom of continuity.[35,36] However, in actual practice, it is enough to consider the finite universal set. $\mu(S)$ can be viewed as the grade of subjective importance of set S. Thus, in addition to the usual weight on element $x_i \in X$ taken separately, weights on any combination of element $x_i \in X$ are also defined. To determine fuzzy measures on $X = \{x_1, x_2, \ldots, x_n\}$, we generally need to find $2^n - 2$ values, only values $m(\phi)$ and $m(X)$ are always equal to 0 and 1, respectively. So the evaluation model obtained becomes quite complex. To avoid the problems with computational complexity, λ-fuzzy measure g, a special kind of fuzzy measure defined on $P(X)$ and satisfying the finite λ-rule,[20] is proposed by Sugeno, which satisfies the following additional property:

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)\, g(B), \tag{4}$$

where $\lambda > -1$ for all $A, B \in P(X)$ and $A \cap B = \phi$.

In Equation 4, $\lambda = 0$ indicates that the λ-fuzzy measure g is additive and there is no interaction between A and B. $\lambda \neq 0$ indicates that the λ-fuzzy measure g is nonadditive, and there is interaction between A and B. If $\lambda > 0$, then $g(A \cup B) > g(A) + g(B)$, which implies that the set $\{A, B\}$ has multiplicative effect. If $\lambda < 0$, then $g(A \cup B) < g(A) + g(B)$, which implies that the set $\{A, B\}$ has substitutive effect. By parameter λ, the interaction between sets or elements of set can be represented.

If X is a finite set, then $\cup_{i=1}^{n} x_i = X$. The λ-fuzzy measure g satisfies the following Equation 5:

$$g(X) = g\left(\bigcup_{i=1}^{n} x_i\right) = \begin{cases} \dfrac{1}{\lambda}\left(\prod_{i=1}^{n}[1 + \lambda g(x_i)] - 1\right) & \text{if } \lambda \neq 0, \\ \sum_{i=1}^{n} g(x_i) & \text{if } \lambda = 0, \end{cases} \tag{5}$$

where $x_i \cap x_j = \phi$ for all $i, j = 1, 2, \ldots, n$ and $i \neq j$. It can be noted that $g(x_i)$ for a subset with a single element x_i is called a fuzzy density and can be denoted as $g_i = g(x_i)$.

Especially for every subset$A \in P(X)$, we have

$$g(A) = \begin{cases} \dfrac{1}{\lambda}\left(\prod_{i \in A}[1 + \lambda g_i] - 1\right) & \text{if } \lambda \neq 0, \\ \sum_{i \in A} g_i & \text{if } \lambda = 0. \end{cases} \tag{6}$$

Based on Equation 5, the value of λ can be uniquely determined from $g(X) = 1$, which is equivalent to solving

$$\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda g_i). \tag{7}$$

DEFINITION 4. *Let f be a positive real-valued function $f: X \to R^+$ and m be a fuzzy measure on X. The (discrete) Choquet integral of f with respect to m is*

$$C_m(f_1, \ldots, f_n) = \sum_{i=1}^{n} f_{(i)}\big[m\big(A_{(i)}\big) - m\big(A_{(i-1)}\big)\big], \tag{8}$$

where $(\cdot)$ indicates a permutation on X such that $f_{(1)} \geq f_{(2)} \geq \cdots \geq f_{(n)}$, i.e., $f_{(i)}$ is the ith largest value in the set $\{f_1, \ldots, f_n\}$. Therefore, $A_{(i)} = \{x_{(1)}, \ldots, x_{(i)}\}$ when $i \geq 1$ and $A_{(0)} = \phi$.

3. THE INDUCED CHOQUET ORDERED AVERAGING OPERATOR AND ITS PROPERTIES

In this section, we introduce the I-COA operator and study its properties.

3.1. The Induced Choquet Ordered Averaging Operator

As a flexible aggregation operator, the Choquet integral with respect to fuzzy measures is often used in information fusion and decision making, where interaction between criteria or preferences is considered. Central to this aggregation operator is also the step of ordering the arguments. This produces the ordered argument vector and effects the determination of the fuzzy measures. The ordering is based on the value of the arguments f_i. Inspired by ideal of the IOWA operator, Yager[14] considered a more general policy toward to ordering of the arguments and the formulation of the ordered argument vector. This generalization will lead to a more general type of Choquet integral operator, named the induced Choquet ordered averaging (I-COA) operator, which is defined as follows:

DEFINITION 5. *Let f be a positive real-valued function on X, and m be a fuzzy measure on X. An induced Choquet ordered averaging operator of dimension n is a function I-COA: $(R^+ \times R^+)^n \to R^+$, which is defined to aggregate the set of second arguments of a list of n 2-tuples $\{\langle u_1, f_1\rangle, \ldots, \langle u_n, f_n\rangle\}$ according to the following expression:*

$$\text{I-COA}_m(\langle u_1,\ f_1\rangle, \langle u_2,\ f_2\rangle, \ldots, \langle u_n,\ f_n\rangle) = \sum_{i=1}^{n} f_{(i)}\left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right], \tag{9}$$

where $(\cdot)$: $\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ is a permutation such that $u_{(1)} \geq u_{(2)} \geq \cdots \geq u_{(n)}$, *i.e.*, $\langle u_{(i)}, f_{(i)} \rangle$ *is the 2-tuple with* $u_{(i)}$ *the ith largest value in the set* $\{u_1, \ldots, u_n\}$, $A_{(i)} = \{x_{(1)}, \ldots, x_{(i)}\}$ *when* $i \geq 1$ *and* $A_{(0)} = \phi$.

Remark 2. For the 2-tuple $\langle u_i, f_i \rangle$ in I-COA operator, we shall refer to u_i as the order-inducing variable and to f_i as the argument variable. Then we order the objects by the order-inducing variable and aggregate the argument variables. It is also emphasized that the inducing ordering is used to generate the fuzzy measures. It is assumed that f_i is an exact numerical value in R^+ whereas u_i can be drawn from any positive ordinal space R^+.

Similar to the above-mentioned relation between OWA and IOWA operators, the main difference between the Choquet integral and the I-COA operator resides in the reordering step of the argument variable. In the case of Choquet integral, this reordering is based on the magnitude of the values to be aggregated, whereas in the case of the I-COA operator an order-inducing variable is used as the criterion to induce that reordering. Obviously, an immediate consequence of Definition 5 is that if the order-inducing variable is the argument variable then the I-COA operator is reduced to the Choquet integral operator.

Example 1. Let m be a fuzzy measure on the space $A = \{A_1, A_2, A_3, A_4\}$ in which

$$m(\phi) = 0, \quad m(\{A_1\}) = 0.3, \quad m(\{A_2\}) = 0.3, \quad m(\{A_3\}) = 0.4, \quad m(\{A_4\}) = 0.1,$$

$$m(\{A_1, A_2\}) = 0.6, \quad m(\{A_1, A_3\}) = 0.5, \quad m(\{A_1, A_4\}) = 0.4, \quad m(\{A_2, A_3\}) = 0.5,$$

$$m(\{A_2, A_4\}) = 0.5, \quad m(\{A_3, A_4\}) = 0.6, \quad m(\{A_1, A_2, A_3\}) = 0.7,$$

$$m(\{A_1, A_2, A_4\}) = 0.8,$$

$$m(\{A_1, A_3, A_4\}) = 0.7, \quad m(\{A_2, A_3, A_4\}) = 0.9, \quad m(\{A_1, A_2, A_3, A_4\}) = 1.0.$$

Assume we have four pairs: $A_1 = \langle 6, 0 \rangle$, $A_2 = \langle 7, 0.2 \rangle$, $A_3 = \langle 2, 0.9 \rangle$, and $A_4 = \langle 6, 1 \rangle$ in which the first component is the order-induced variable. It is easily see that there exists a tie between $\langle 6, 0 \rangle$ and $\langle 6, 1 \rangle$ with respect to the order-inducing variable. On the basis of the order-induced variables, we have two possible order sequences as follows:

$$\langle 7, 0.2 \rangle = A_2, \quad \langle 6, 1 \rangle = A_4, \quad \langle 6, 0 \rangle = A_1, \quad \langle 2, 0.9 \rangle = A_3;$$

$$\langle 7, 0.2 \rangle = A_2, \quad \langle 6, 0 \rangle = A_1, \quad \langle 6, 1 \rangle = A_4, \quad \langle 2, 0.9 \rangle = A_3.$$

For the first case, according to Ref. 14, we have

$$\text{I-COA}_m(A_1, A_2, A_3, A_4) = 0.3 \times 0.2 + 0.2 \times 1 + 0.3 \times 0 + 0.2 \times 0.9 = 0.44.$$

For the second case according to Definition 5, we have

$$\text{I-COA}_m(A_1, A_2, A_3, A_4) = 0.3 \times 0.2 + 0.3 \times 0 + 0.2 \times 1 + 0.2 \times 0.9 = 0.44.$$

If $m(\{A_1, A_2, A_4\}) = 0.75$, then for the first case, $\text{I-COA}_m(A_1, A_2, A_3, A_4) = 0.485$. For the second case, $\text{I-COA}_m(A_1, A_2, A_3, A_4) = 0.435$.

From the above analysis, it is seen that if there are some ties between $\langle u_i, f_i\rangle$ and $\langle u_j, f_j\rangle$ with respect to order-inducing variables in I-COA operator aggregation processes, different result can be obtained when different order in which the tied arguments are placed, which is same with the IOWA and the induced Sugeno integral operator.[12] Yager[10] provided a procedure to deal with "ties" with respect to the ordering induced by the application of the IOWA operator. Since in the IOWA operator the weighting vector is unalterable even if the order-inducing variables change. However, for the induced Choquet ordered averaging operator, the inducing ordering variable is used to generate the fuzzy measures. If the inducing ordering variable changes, the fuzzy measures will correspondingly change.

Remark 3. In this paper, we will focus on the aggregation of numerical preferences, that is why we assume that the nature of the first argument of the I-COA operator is also numeric, although it could be linguistic.[9,12,14,16]

3.2. Properties of the I-COA Operator

PROPOSITION 1. *The induced Choquet ordered averaging (I-COA) operator is a monotonically increasing function with respect to argument variable.*

This property can be easily obtained by taking partial derivative of the I-COA operator

$$\frac{\partial \text{I-COA}_m(\langle u_1, f_1\rangle, \ldots, \langle u_n, f_n\rangle)}{\partial f_{(i)}} = m\big(A_{(i)}\big) - m\big(A_{(i-1)}\big) \geq 0.$$

According to the above analysis, suppose two sets with two tuples $(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle)$ *and* $(\langle u_1, f_1'\rangle, \langle u_2, f_2'\rangle, \ldots, \langle u_n, f_n'\rangle)$ *such that* $f_i \leq f_i', \forall i$. *Then* $f_{(i)} \leq f_{(i)}'$, *so*

$$\text{I-COA}_m(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle) \leq \text{I-COA}_m(\langle u_1, f_1'\rangle, \langle u_2, f_2'\rangle, \ldots, \langle u_n, f_n'\rangle).$$

This property is based on the assumption that the order-inducing values are unchanged. From the Definition 5, if this is not the case, then monotonicity does not necessarily hold.

PROPOSITION 2. *Let* $f_{\min} = \min(f_1, \ldots, f_n)$, $f_{\max} = \max(f_1, \ldots, f_n)$, *then*

$$f_{\min} \leq \text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) \leq f_{\max}.$$

where $(\cdot)$ *is the permutation such that* $\langle u_{(i)}, f_{(i)}\rangle$*is the 2-tuple with* $u_{(i)}$*the ith largest value in the set* $\{u_1, \ldots, u_n\}$, $A_{(i)} = \{x_{(1)}, \ldots, x_{(i)}\}$.

Proof. Let $(\cdot)$ be the permutation such that $\langle u_{(i)}, f_{(i)}\rangle$is the 2-tuple with $u_{(i)}$the ith largest value in the set $\{u_1, \ldots, u_n\}$, then $f_{\min} \leq f_{(i)} \leq f_{\max}$. So we have

$$\sum_{i=1}^{n} f_{\min}\left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right] \leq \sum_{i=1}^{n} f_{(i)}\left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right]$$

$$\leq \sum_{i=1}^{n} f_{\max}\left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right].$$

Since

$$\sum_{i=1}^{n}\left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right] = 1,$$

thus

$$f_{\min} \leq \text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) \leq f_{\max}.$$

■

According to Proposition 2, if $f_i = f, \forall i = 1, \ldots, n$, then $f_{\min} = f_{\max} = f$, so we immediately have the following conclusion:

PROPOSITION 3. *If* $f_i = f, \forall i = 1, \ldots, n$, *then*

$$\text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = f.$$

PROPOSITION 4. *Let* π *be a permutation of the set* $\{1, \ldots, n\}$ *and* $(\langle u_{\pi(1)}, f_{\pi(1)}\rangle, \langle u_{\pi(2)}, f_{\pi(2)}\rangle, \ldots, \langle u_{\pi(n)}, f_{\pi(n)}\rangle)$ *is a reordering of the set with two tuples* $(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle)$, then

$$\text{I-COA}_m(\langle u_{\pi(1)}, f_{\pi(1)}\rangle, \langle u_{\pi(2)}, f_{\pi(2)}\rangle, \ldots, \langle u_{\pi(n)}, f_{\pi(n)}\rangle)$$

$$= \text{I-COA}_m(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle).$$

Proof. Since the sets of order-inducing values $(u_1, \ldots, u_n)$ and $(u_{\pi(1)}, \ldots, u_{\pi(n)})$ have the same elements, the ordering of them from highest to lowest is unique. So $\{u_{(1)}, \ldots, u_{(n)}\}$ and $\{u_{\pi_1(\pi(1))}, \ldots, u_{\pi_1(\pi(n))}\}$, $(\cdot)$ and π_1 being two permutations such that $u_{(i)} \geq u_{(i+1)}$, $A_{(i)} = \{(1), \ldots, (i)\}$, and $u_{\pi_1(\pi(i))} \geq u_{\pi_1(\pi(i+1))}$, $B_{\pi_1(\pi(i))} = \{\pi_1(\pi(1)), \ldots, \pi_1(\pi(i))\}$, $\forall i = 1, \ldots, n-1$, respectively, are the same sets, i.e., $(\cdot) = \pi_1 {}^\circ \pi$, so,

$$A_{(i)} = B_{\pi_1(\pi(i))}.$$

$$\begin{aligned}
&\text{I-COA}_m(\langle u_{\pi(1)}, f_{\pi(1)}\rangle, \langle u_{\pi(2)}, f_{\pi(2)}\rangle, \ldots, \langle u_{\pi(n)}, f_{\pi(n)}\rangle) \\
&= \sum_{i=1}^{n} f_{\pi_1(\pi(i))} \left[m\left(B_{(i)}\right) - m\left(B_{(i-1)}\right)\right] \\
&= \sum_{i=1}^{n} f_{(i)} \left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right] \\
&= \text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right).
\end{aligned}$$

■

Proposition 5. *For $\forall r \geq 0$, $t > 0$,*

$$\begin{aligned}
&\text{I-COA}_m\left(\langle u_1, tf_1 + r\rangle, \langle u_2, tf_2 + r\rangle, \ldots, \langle u_n, tf_n + r\rangle\right) \\
&\quad = t\text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) + r.
\end{aligned}$$

where $(\cdot)$: $\{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that $u_{(1)} \geq u_{(2)} \geq \cdots \geq u_{(n)}$, i.e., $\langle u_{(i)}, f_{(i)}\rangle$ is the 2-tuple with $u_{(i)}$ the ith largest value in the set $\{u_1, \ldots, u_n\}$, $A_{(i)} = \{x_{(1)}, \ldots, x_{(i)}\}$ when $i \geq 1$ and $A_{(0)} = \phi$.

Proof. Let $(\cdot)$ be the permutation such that $\langle u_{(i)}, f_{(i)}\rangle$ is the 2-tuple with $u_{(i)}$ the ith largest value in the set $\{u_1, \ldots, u_n\}$, then for $\forall r \geq 0, t > 0$, $\langle u_{(i)}, tf_{(i)} + r\rangle$ is also the 2-tuple with $u_{(i)}$ the ith largest value in the set $\{u_1, \ldots, u_n\}$, according to Definition 5, we have

$$\begin{aligned}
&\text{I-COA}_m\left(\langle u_1, tf_1 + r\rangle, \langle u_2, tf_2 + r\rangle, \ldots, \langle u_n, tf_n + r\rangle\right) \\
&= \sum_{i=1}^{n} \left(tf_{(i)} + r\right) \left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right]
\end{aligned}$$

$$= t\sum_{i=1}^{n} f_{(i)}\left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right] + r\sum_{i=1}^{n}\left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right]$$

$$= t\text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) + r.$$

■

According to Definition 5, it is easily obtained the following proposition:

PROPOSITION 6. *(1) If $m(A) = 1$ for any $A \in P(X)$, $A \neq \phi$, then*

$$\text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = \max\left(f_1, \ldots, f_n\right).$$

(2) If $\mu(A) = 0$ for any $A \in P(X)$ and $A \neq X$, then $\text{I-COA}_m(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle) = \min(f_1, \ldots, f_n)$.

(3) For any A, $B \in P(X)$ such that $|A| = |B|$, if $m(A) = m(B)$ and $g\{x_{(1)}, \ldots, x_{(i)}\} = i/n$, $1 \leq i \leq n$, then

$$\text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = \frac{1}{n}\sum_{i=1}^{n} f_i,$$

i.e., the I-COA operator is reduced to an arithmetic average operator.

From the above analysis, it is easily seen that the induced Choquet ordered averaging operator possesses many desirable properties similar to those of the Choquet integral operator.

PROPOSITION 7. *For λ-fuzzy measure g on X, if $\lambda \neq 0$, then*

$$\text{I-COA}_g\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = \sum_{i=1}^{n} f_{(i)} \cdot g_{(i)} \prod_{j=1}^{i-1}\left[1 + \lambda g_{(j)}\right].$$

If λ = 0, then

$$\text{I-COA}_g\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = \sum_{i=1}^{n} f_{(i)} \cdot g_{(i)},$$

where $(\cdot)$: $\{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that $u_{(1)} \geq u_{(2)} \geq \cdots \geq u_{(n)}$, i.e., $\langle u_{(i)}, f_{(i)}\rangle$ is the 2-tuple with $u_{(i)}$ the ith largest value in the set $\{u_1, \ldots, u_n\}$, $A_{(i)} = \{x_{(1)}, \ldots, x_{(i)}\}$ $i \geq 1$ and $A_{(0)} = \phi$.

Proof. Let $(\cdot)$: $\{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation such that $u_{(1)} \geq u_{(2)} \geq \cdots \geq u_{(n)}$, i.e., $\langle u_{(i)}, f_{(i)}\rangle$ is the 2-tuple with $u_{(i)}$ the ith largest value in the set $\{u_1, \ldots, u_n\}$. Since $A_{(i)} = \{x_{(1)}, \ldots, x_{(i)}\}$ and $A_{(i-1)} = \{x_{(1)}, \ldots, x_{(i-1)}\}$, according to Equation 6, if $\lambda \neq 0$, then

$$g\left(A_{(i)}\right) - g\left(A_{(i-1)}\right) = g_{(i)} \cdot \left\{1 + \lambda \cdot \frac{1}{\lambda}\left(\prod_{j=1}^{i-1}\left[1 + \lambda g_{(j)}\right] - 1\right)\right\}$$

$$= g_{(i)} \cdot \left(\prod_{j=1}^{i-1}\left[1 + \lambda g_{(j)}\right]\right).$$

So

$$\text{I-COA}_g\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = \sum_{i=1}^{n} f_{(i)} \cdot g_{(i)} \prod_{j=1}^{i-1}\left[1 + \lambda g_{(j)}\right]$$

If $\lambda = 0$, then

$$g(A_{(i)}) - g(A_{(i-1)}) = g_{(i)}.$$

So

$$\text{I-COA}_g\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = \sum_{i=1}^{n} f_{(i)} \cdot g_{(i)}$$

This completes the proof of the proposition. ■

From Proposition 7 and Definition 2, we immediately have the following conclusion:

PROPOSITION 8. *For fuzzy measure μ, if m is an additive fuzzy measure, then*

$$\text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right)$$

$$= \text{IOWA}_w\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right),$$

where $w = (w_1, \ldots, w_n)$ is a set of weights or weighting vector associated with the IOWA operator with $w_i = m_{(i)}$. Furthermore, if $m_{(i)} = 1/n$, the I-COA operator is reduced to an arithmetic average operator.

Moreover, one can readily see that any IOWA, which has associated a set of weights or weighting vector $W = (w_1, \ldots, w_n)$ with it with $w_i \in [0, 1]$,

$\sum_{i=1}^{n} w_i = 1$, is also equivalent to an induced Choquet ordered averaging operator with a fuzzy measure m:

$$m(A) = \sum_{i \in A} w_i, \quad (A \subseteq X).$$

So we have the following conclusion:

PROPOSITION 9. *If the IOWA operator has an associated weighting vector* $w = (w_1, w_2, \ldots, w_n)^T$ *with* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$*, then there is an additive fuzzy measure m such that*

$$\text{IOWA}_w(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle)$$
$$= \text{I-COA}_m(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle).$$

If the weight associated with the IOWA operator is expressed by RIM quantifiers Q, similar to the relation between OWA and Choquet integral operators,[37] according to Proposition 9, we easily obtained the following conclusion:

COROLLARY 1. *For every RIM quantifiers Q,*

$$\text{IOWA}_Q(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle)$$
$$= \text{I-COA}_m(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle),$$

where m can be expressed by

$$m(A) = Q(|A|/n) \forall A \subseteq X.$$

PROPOSITION 10. *If the 2-tuple* $\langle u_i, f_i\rangle$ *can be expressed by* $\langle h(f_i), f_i\rangle$*, where h is an increasing function, then*

$$\text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = C_m(f_1, \ldots, f_n).$$

Proof. Let $(\cdot)$: $\{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation such that $u_{(1)} \geq u_{(2)} \geq \cdots \geq u_{(n)}$. If $u_i = h(f_i)$, h is an increasing function, then the ith largest value of $(u_1, \ldots, u_n)$ is $u_{(i)}$ if and only if $f_{(i)}$ is the ith largest value of $(f_1, \ldots, f_n)$, according to Definitions 4 and 5, we have

$$\text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = C_m(f_1, \ldots, f_n). \qquad \blacksquare$$

If m is a non-additive fuzzy measure, for I-COA operator, assume that

$$w_i = m(A_i) - m(A_{i-1})$$

where A_i is any subset of X with $|A_i| = i$. Then according to Definition 2, it is obvious that

$$\text{I-COA}_m\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right)$$
$$= \text{IOWA}_w\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right).$$

According to the above analysis, we can obtain the following conclusion:

PROPOSITION 11. *Any commutative induced Choquet ordered averaging operator I-COA is an IOWA operator whose associated weights are* $w_i = m(A_i) - m(A_{i-1})$ $(i = 1, \ldots, n)$*, where* A_i *is any subset of X with* $|A_i| = i$.

The proof is similar to that of Theorem 13 in Ref. 27; here we do not duplicate it.

From Proposition 11, if we construct symmetric fuzzy measures with the help of the BUM function Q as follows:

$$m(A) = Q(|A|/n)\forall A \subseteq X.$$

Then the corresponding I-COA operator is reduced to the IOWA operator.

From the above analysis, it is easily known that the induced Choquet ordered averaging operator generalizes both the IOWA and Choquet integral operator. The IOWA operator is a special case of the induced Choquet ordered averaging operator.

3.3. The Induced Generalized Choquet Ordered Averaging Operator

Yager[38] proposed a generalized Choquet aggregation operator. Inspired by ideal of the generalized Choquet integral operator, we can present an induced generalized Choquet ordered averaging (I-GCOA) operator, which is defined as follows:

DEFINITION 6. *Let* f *be a positive real-valued function on X, and m be a fuzzy measure on X. An induced generalized Choquet ordered averaging (I-GCOA) operator of dimension n is a function I-GCOA:* $(R^+ \times R^+)^n \to R^+$, *which is defined to aggregate the set of second arguments of a list of n 2-tuples* $\{\langle u_1, f_1\rangle, \ldots, \langle u_n, f_n\rangle\}$ *according to the following expression:*

$$\text{I-GCOA}_{m,\beta}\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = \left(\sum_{i=1}^{n} f_{(i)}^{\beta}\left[m(A_{(i)}) - m(A_{(i-1)})\right]\right)^{1/\beta},$$

where $\beta \in R$, $(\cdot)$: $\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ *is a permutation such that* $u_{(1)} \geq u_{(2)} \geq \cdots \geq u_{(n)}$, *i.e.,* $\langle u_{(i)}, f_{(i)}\rangle$ *is the 2-tuple with* $u_{(i)}$ *the ith largest value in the set* $\{u_1, \ldots, u_n\}$, $A_{(i)} = \{x_{(1)}, \ldots, x_{(i)}\}$ $i \geq 1$ *and* $A_{(0)} = \phi$.

Let us now consider the form of the I-GCOA operator for some particular cases of β. As we have already noted when $\beta = 1$, we get the usual I-COA operator. When $\beta = 2$, we get

$$\text{I-GCOA}_{m,\beta}\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = \left(\sum_{i=1}^{n} f_{(i)}^{2}\left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right]\right)^{1/2}.$$

Let $w_{(i)} = m(A_{(i)}) - m(A_{(i-1)})$, then

$$\text{I-GCOA}_{m,\beta}\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) = \left(\sum_{i=1}^{n} f_{(i)}^{2} w_{(i)}\right)^{1/2},$$

which has the similar form to the order weighted square mean aggregation.

Consider now the case when $\beta = -1$ here, we get

$$\text{I-GCOA}_{m,\beta}\left(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle\right) \times \left(\sum_{i=1}^{n} f_{(i)}^{-1}\left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right]\right)^{-1} = \frac{1}{\sum_{i=1}^{n} \frac{m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)}{f_{(i)}}}$$

$$= \frac{\prod_{i=1}^{n} f_{(i)}}{\sum_{i=1}^{n}\left[m\left(A_{(i)}\right) - m\left(A_{(i-1)}\right)\right]\left(\prod_{\substack{j=1\\ j\neq i}}^{n} f_{(j)}\right)}.$$

Let $w_{(i)} = m(A_{(i)}) - m(A_{(i-1)})$, then

$$\text{I-GCOA}_{m,\beta}(\langle u_1, f_1\rangle, \langle u_2, f_2\rangle, \ldots, \langle u_n, f_n\rangle) = \frac{\prod_{i=1}^{n} f_{(i)}}{\sum_{i=1}^{n} w_{(i)}\left(\prod_{\substack{j=1 \\ j\neq i}}^{n} f_{(j)}\right)},$$

which is closely related to the ordered Harmonic average.

4. AGGREGATION OF FUZZY PREFERENCE RELATIONS BY MEANS OF THE I-COA OPERATOR

Preference relation is a useful tool for expressing decision makers' preferences in GDM. Consider a GDM problem, Let $E = (e_1, e_2, \ldots, e_m)$ be a set of the experts involved in the decision process and $X = (x_1, x_2, \ldots, x_n)$ be a set of considered alternatives. In the process of decision making, an expert generally needs to provide his/her preferences for each pair of alternatives and then constructs a preference relation, which can be defined as follows:

DEFINITION 7. *A preference relation P on the set X is characterized by a function m_P: $X \times X \rightarrow D$, where D is the domain of representation of preference degrees.*

In some real decision-making problems, expert invited to make decision may not possess a precise or sufficient level of knowledge of the problem, or he/she is unable to discriminate explicitly the degree to which one alternative is better than others. In such cases, it is often used fuzzy preference relations[1,39] to express his/her preference by experts as follows:

DEFINITION 8. *A fuzzy preference relation $R = (r_{ij})_{n\times n}$ on the set X is a fuzzy set on the product set $X \times X$, which is characterized by a membership function*

$$r_{ij} = \mu_R(x_i, x_j) = X \times X \rightarrow [0, 1], \ \ for\ all\ i, j = 1, 2, \ldots, n$$

where r_{ij} denotes the preference degree of the alternative x_i over x_j. In particular, $r_{ij} = 0.5$ indicates indifference between x_i and x_j, $r_{ij} > 0.5$ indicates that x_i is preferred to x_j, and $r_{ij} < 0.5$ indicates that x_j is preferred to x_i. In this case, the preference relation R is usually assumed additive reciprocal, i.e.,

$$r_{ij} + r_{ji} = 1, \quad \forall i, \quad j = 1, 2, \ldots, n.$$

As we have mentioned, a fundamental aspect of the I-COA operators is the reordering of the arguments to be aggregated, based on the order-inducing variable. However, it is clear that a set of values can be reordered in a different way. In this section, we presented two special cases of I-COA operators for GDM with fuzzy preference relations.

4.1. The Importance Induced Choquet Ordered Averaging (I-I-COA) Operator

In multicriteria decision making, Choquet aggregation explicitly models the importance of not only individual criteria but their subsets, as well as various interactions between the criteria. But in real decision problems, the overall importance of a criterion $i \in N$ (where N denotes a criteria set) is not solely determined by itself i but also by all other criteria $T, i \in T$. Suppose that $m(i)$ denotes the importance degree of i, we may have $m(i) = 0$, suggesting that element is unimportant, but it may happen that for many subsets $T \subseteq N, m(T \cup i)$ is much greater than $m(T)$, suggesting that i is actually an important element in the decision.

In 1953, Shapley[40] proposed a definition of a coefficient of importance, based on a set of reasonable axioms. The importance index or Shapley value of criterion i with respect to g is defined by

$$\phi(m, i) = \sum_{T \subseteq N \setminus i} \frac{(n-t-1)!t!}{n!}[m(T \cup i) - m(T)], \tag{10}$$

where n and t denote the cardinality of set N and T, respectively.

In game theory, the Shapley value is used to express a power index. It can be interpreted as a weighted average value of the marginal contribution $m(T \cup i) - m(T)$ of element i alone in all combinations. The use of the Shapley value in multicriteria decision making was proposed in 1992 by Murofushi.[41] A basic property of the Shapley value is

$$\sum_{i=1}^{n} \phi(m, i) = 1.$$

Note also that, when m is additive, we clearly have

$$\phi(m, i) = m(i), i \in N. \tag{11}$$

If m is nonadditive then some criteria are dependent and Equation 11 generally does not hold anymore. This shows that it is sensible to search for a coefficient of overall importance for each criterion.

In our case, we propose to use the importance indexes associated with each one of the experts as the order-inducing values. Thus, the ordering of the preference values is first induced by the ordering of the experts from most to least important one. We call this importance index based I-COA operator as the importance I-COA (I-I-COA) operator and denote it as$\mathrm{I-COA}_m^I$.

Definition 9. *If a set of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{R^1, \ldots, R^m\}$, and each expert e_k has an importance index, $\phi(m, e_k) \in [0, 1]$, associated with him or her, then an I-I-COA operator of dimension n, $\mathrm{I}-\mathrm{COA}^I_m$, is an I-COA operator whose set of order inducing values is the set of importance indexes.*

Example 2. Suppose that we have a set of three experts $E = \{e_1, e_2, e_3\}$ and a set of three alternatives $X = \{x_1, x_2, x_3\}$, three experts provide the following fuzzy preference relations on the set of three alternatives:

$$R^1 = \begin{pmatrix} 0.5 & 0.75 & 0.87 \\ 0.25 & 0.5 & 0.66 \\ 0.13 & 0.34 & 0.5 \end{pmatrix}; R^2 = \begin{pmatrix} 0.5 & 0.66 & 0.94 \\ 0.34 & 0.5 & 0.87 \\ 0.06 & 0.13 & 0.5 \end{pmatrix}; R^3 = \begin{pmatrix} 0.5 & 0.66 & 0.75 \\ 0.34 & 0.5 & 0.66 \\ 0.25 & 0.34 & 0.5 \end{pmatrix}.$$

Suppose that $g(e_1) = 0.30$, $g(e_2) = 0.40$, and $g(e_3) = 0.50$. Then λ of expert can be determined: $\lambda = -0.45$. According to Equation 7, we can calculate that

$$g(e_1, e_2) = 0.646, \quad g(e_1, e_3) = 0.7325, \quad g(e_2, e_3) = 0.18, \quad g(e_1, e_2, e_3) = 1.$$

According to Equation 10, we have

$$\phi(g, e_1) = 0.243, \quad \phi(g, e_2) = 0.332, \quad \phi(g, e_3) = 0.425.$$

According to Definition 5 or Proposition 7c, the collective fuzzy preference relation is

$$R = \text{I-COA}^I_g(\langle \phi(g, e_3), R^1 \rangle, \langle \phi(g, e_2), R^2 \rangle, \langle \phi(g, e_3), R^3 \rangle) = \begin{pmatrix} 0.5 & 0.68 & 0.83 \\ 0.32 & 0.5 & 0.73 \\ 0.17 & 0.27 & 0.5 \end{pmatrix},$$

whose elements can be considered as the preference of one alternative over another for most of the more important experts.

Remark 4. From the above example, in general, we have the following conclusion: If a group of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of reciprocal fuzzy preference relations, $\{P^1, \ldots, P^m\}$, $r^k_{ij} + r^k_{ji} = 1, \forall i, jk$, and if $\{u_1, \ldots, u_m\}$ is a set of order-inducing (importance index) values associated with the set of experts, then the collective preference relation, $R = (r_{ij})$, obtained by using an I-COA operator is also reciprocal.

Let $(\cdot)$: $\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ a permutation such that $u_{(1)} \geq u_{(2)} \geq \cdots \geq u_{(n)}$. Then

$$r_{ij} = \mathrm{I} - \mathrm{COA}_m^I \left(\langle u_1, r_{ij}^1\rangle, \ldots, \langle u_n, r_{ij}^n\rangle\right) = \sum_{k=1}^{m} r_{ij}^{(k)} \left(m\left(A_{(k)}\right) - m\left(A_{(k-1)}\right)\right),$$

where $A_{(k)} = \{e_{(1)}, \ldots, e_{(k)}\}$, $u_k = \phi(m, e_k)$.

On the other hand,

$$\begin{aligned} r_{ji} = \mathrm{I} - \mathrm{COA}_m^I \left(\langle u_1, r_{ji}^1\rangle, \ldots, \langle u_n, r_{ji}^n\rangle\right) &= \sum_{k=1}^{m} r_{ji}^{(k)} \left(m\left(A_{(k)}\right) - m\left(A_{(k-1)}\right)\right) \\ &= \sum_{k=1}^{m} \left(1 - r_{ij}^{(k)}\right) \left(m\left(A_{(k)}\right) - m\left(A_{(k-1)}\right)\right) \\ &= \sum_{k=1}^{m} \left(m\left(A_{(k)}\right) - m\left(A_{(k-1)}\right)\right) \\ &\quad - \sum_{k=1}^{m} r_{ij}^{(k)} \left(m\left(A_{(k)}\right) - m\left(A_{(k-1)}\right)\right) \\ &= 1 - r_{ij} \end{aligned}$$

Thus R is reciprocal.

4.2. The Preference Induced Choquet Ordered Averaging (P-I-COA) Operator

If $R^k = (r_{ij}^k)$ is a fuzzy preference relation on the set of alternatives $\{x_1, \ldots, x_n\}$ then the total sum of the elements of each row i, $\bar{r}_i^k = \sum_j r_{ij}^k$, can be interpreted as the total preference of the alternative x_i. The resulting value obtained by dividing an element of that row, r_{ir}^k, by $\bar{r}_i^k$, $\bar{r}_{ir}^k = r_{ir}^k / \bar{r}_i^k$, can be interpreted as the relative preference contribution of that particular element to the total preference of the alternative x_i. $\bar{R}^k = (\bar{r}_{ij}^k)$ is the relative preference matrix. These relative preference values can be used as the order-inducing values of an I-COA operator to aggregate a set of fuzzy preference relations. We call this a preference I-COA (P-I-COA) operator and denote it as $\mathrm{I} - \mathrm{COA}_m^P$.

DEFINITION 10. *If a set of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{R^1, \ldots, R^m\}$, then a P-I-COA operator of dimension n,* $\mathrm{I\text{-}COA}_m^P$, *is an I-COA* *operator whose set of order-inducing values is the set of relative preference values associated with each one of the arguments to aggregate.*

Example 3. Using the same data as in Example 2, the corresponding relative preference matrices, $\bar{R}^k$, are

$$\bar{R}^1 = \begin{pmatrix} 0.24 & 0.35 & 0.41 \\ 0.18 & 0.35 & 0.47 \\ 0.13 & 0.35 & 0.52 \end{pmatrix}; \bar{R}^2 = \begin{pmatrix} 0.24 & 0.31 & 0.45 \\ 0.20 & 0.29 & 0.51 \\ 0.09 & 0.19 & 0.72 \end{pmatrix}; \bar{R}^3 = \begin{pmatrix} 0.26 & 0.35 & 0.39 \\ 0.23 & 0.33 & 0.44 \\ 0.23 & 0.31 & 0.46 \end{pmatrix}.$$

Suppose that $g(e_1) = 0.30$, $g(e_2) = 0.40$, and $g(e_3) = 0.50$. Then λ of experts can be determined: $\lambda = -0.45$. According to Equation 7, we can calculate that

$$g(e_1, e_2) = 0.646, \quad g(e_1, e_3) = 0.7325, \quad g(e_2, e_3) = 0.18, \quad g(e_1, e_2, e_3) = 1.$$

According to Definition 5 or Proposition 7, the collective fuzzy preference relation is

$$R = \text{I-COA}_g^P \left(\langle r_{ij}^{-1}, R^1 \rangle, \langle r_{ij}^{-2}, R^2 \rangle, \langle r_{ij}^{-3}, R^3 \rangle \right) = \begin{pmatrix} 0.5 & 0.69 & 0.86 \\ 0.32 & 0.5 & 0.74 \\ 0.17 & 0.28 & 0.5 \end{pmatrix}.$$

Remark 5. The collective preference relation obtained by the application of the P-I-COA operator does not have the reciprocity property. This is due to the fact that this P-I-COA operator behaves as a P-IOWA operator, which normally does not maintain the reciprocity property.[17]

4.3. A Process Based on I-COA Operators to GDM Problems with Fuzzy Preference Relations

Suppose that we have a group of experts, $E = \{e_1, \ldots, e_m\}$, which provide his/her preference about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{R^1, \ldots, R^m\}$, $R^k = (r_{ij}^k)$, and $r_{ij}^k \in [0, 1]$, which are additive reciprocal, i.e., $r_{ij}^k + r_{ji}^k = 1, \forall i, j, k$.

In the following, we will utilize the above two induced Choquet ordered averaging operators to propose an approach to GDM with fuzzy preference relations, which involves the following steps:

Step 1. Confirm the fuzzy density $g_i = g(e_i)$ of each expert. According to Equation 7, parameter λ of expert can be determined. According to Equation 10, the importance index of expert e_k, $\phi(g, e_k)$ can be obtained.

Step 2. Using the induced Choquet integral operator

$$r_{ij} = \text{I-COA}_g^I\left(\left\langle \phi(g,e_1), r_{ij}^1\right\rangle, \left\langle \phi(g,e_2), r_{ij}^2\right\rangle, \ldots, \left\langle \phi(g,e_m), r_{ij}^m\right\rangle\right)$$

$$= \sum_{k=1}^{m} r_{ij}^{(k)}\left(g\left(A_{(k)}\right) - g\left(A_{(k-1)}\right)\right),$$

where $\langle \phi(g,e_{(k)}), r_{ij}^{(k)}\rangle$ is the 2-tuple with $\phi(g,e_{(k)})$ the kth largest value in the set $\{\phi(g,e_1),\ldots,\phi(g,e_m)\}$, $A_{(k)} = \{e_{(1)},\ldots,e_{(k)}\}$, aggregate every expert preference $r_{ij}^k(k = 1,2,\ldots,m)$ to a collective fuzzy preference relation $R = (r_{ij})$.

Step 3. Confirm the fuzzy density $g_j = g(x_j)$ of each alternative. According to Equation 7, parameter λ of alternative can be determined.

Step 4. According to the collective fuzzy preference relation R, we can obtain its relative preference matrix $\bar{R} = (\bar{r}_{ij})$. Then using the induced Choquet ordered averaging operator

$$r_i = \text{I-COA}_g^P\left(\langle \bar{r}_{i1}, r_{i1}\rangle, \langle \bar{r}_{i2}, r_{i2}\rangle, \ldots, \langle \bar{r}_{in}, r_{in}\rangle\right)$$

$$= \sum_{j=1}^{n} r_{i(j)}\left(g(B_{(j)}) - g\left(B_{(j-1)}\right)\right)$$

$$= \sum_{j=1}^{n} r_{i(j)} g\left(x_{(j)}\right) \prod_{r=1}^{j-1}\left[1 + \lambda g\left(x_{(r)}\right)\right],$$

where $\langle \bar{r}_{i(j)}, r_{i(j)}\rangle$ is the 2-tuple with $\bar{r}_{i(j)}$ the jth largest value in the set $\{\bar{r}_{i1},\ldots,\bar{r}_{in}\}$, $B_{(j)} = \{x_{(1)},\ldots,x_{(j)}\}$, aggregate fuzzy preference of alternative x_i over the other alternatives into a collective overall preference value of alternative x_i.

Step 5. Finally, according to the collective overall preference value r_i, we choose the best alternative(s). The largest is r_i, the best is the alternative x_i.

Step 6. End.

5. CONCLUSIONS

In general, there are always interactive phenomena among preferences of experts in GDM problems. It is not suitable to aggregate preference information by conventional additive linear operators. To overcome the limitation, in this paper, we introduced the I-COA operator and investigated its properties. We have shown that the IOWA and Choquet integral operators are subclasses of the I-COA operator.

Furthermore, an induced generalized Choquet ordered averaging operator is introduced. We have studied the use of the I-COA operator in the aggregation of fuzzy preference relations in GDM problems. We have defined two I-COA operators: the I-I-COA operator, which induces the ordering of the argument values based on the importance indexes of the information sources and the P-I-COA operator, which induces the ordering of the arguments based on the relative preference associated with each one of them. Finally, a process for GDM problems with fuzzy preference relations based on I-COA operators was presented. The main advantage of this proposal is that we take into consideration not only the reliability of the sources of information (experts) according to their importance indexes but also the interactive phenomena among preferences of experts are considered in the aggregation process, which can avoid losing and distorting the given preference information, and makes the final results in accordance with the real decision problems.

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