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Exponential type functional equation and its Hyers–Ulam stability

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Abstract

We give the solution of the functional equation $f(x+y)+\lambda f(x)f(y)=\Phi(x,y)$ under some conditions. Also we show its Hyers–Ulam stability.

Keywords: Functional equation; Hyers–Ulam stability

1. Introduction and results

In [1], S. Butler posed the following problem: Show that for $d<-1$ there are exactly two solutions $f:\mathbb{R}\to\mathbb{R}$ of the functional equation

$$f(x+y)-f(x)f(y)=d\sin x\sin y \quad (x,y\in\mathbb{R}). \tag{1}$$

M.Th. Rassias excellently answered this problem [6]. Recently, S.-M. Jung has proved that Eq. (1) has the Hyers–Ulam stability [4]. Here we note that (1) is of the form:

$$f(x+y)+\lambda f(x)f(y)=\Phi(x,y) \quad (x,y\in\mathbb{R}), \tag{2}$$

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where λ is a nonzero complex number and Φ is a nonzero complex continuous function on $\mathbb{R}^2$ with $\Phi(x, y) = \Phi(y, x)$ $(x, y \in \mathbb{R})$. The symmetric condition $\Phi(x, y) = \Phi(y, x)$ is necessary for Eq. (2) to have a solution. In this paper, we consider the following two special cases:

$$\text{(I)} \quad \Phi(x, y) = \phi(x + y), \qquad \text{(II)} \quad \Phi(x, y) = \psi(x)\psi(y) \quad (x, y \in \mathbb{R}),$$

where ϕ and ψ are nonzero complex continuous functions on $\mathbb{R}$.

Our first purpose is to solve the functional equation (2) in case (I) or (II). The case (I) is settled as follows:

Theorem 1. *Let λ be a nonzero complex number and ϕ a nonzero complex continuous function on $\mathbb{R}$. If the functional equation*

$$f(x + y) + \lambda f(x)f(y) = \phi(x + y) \quad (x, y \in \mathbb{R}) \tag{3}$$

has a solution f, then ϕ is written as $\phi(x) = \rho e^{\alpha x}$ $(x \in \mathbb{R})$, where ρ and α are complex numbers and $\rho \neq 0$. In fact, the solution f of the equation

$$f(x + y) + \lambda f(x)f(y) = \rho e^{\alpha(x+y)} \quad (x, y \in \mathbb{R}) \tag{4}$$

is given by

$$f(x) = \frac{-1 \pm \sqrt{1 + 4\lambda\rho}}{2\lambda} e^{\alpha x} \quad (x \in \mathbb{R}). \tag{5}$$

The symbol $\sqrt{z}$ denotes one of the square roots of the complex number z.

In this paper, the ambiguity in the choice of the square roots of nonzero z is absorbed, because we will deal with two square roots at the same time.

A part of the case (II) is described like Theorem 1.

Theorem 2. *Let λ be a nonzero complex number and ψ a nonzero complex continuous function on $\mathbb{R}$ with $\psi(0) \neq 0$. If the functional equation*

$$f(x + y) + \lambda f(x)f(y) = \psi(x)\psi(y) \quad (x, y \in \mathbb{R}) \tag{6}$$

has a solution f, then ψ satisfies $\psi(x)\psi(y) = \rho e^{\alpha(x+y)}$ $(x, y \in \mathbb{R})$ for some complex numbers ρ, α with $\rho \neq 0$. For such ψ, (6) *becomes* (4).

We next change the assumption $\psi(0) \neq 0$ into $\psi(0) = 0$ in Theorem 2.

Theorem 3. *Let λ be a nonzero complex number and ψ be a nonzero complex function on $\mathbb{R}$ with $\psi(0) = 0$ and $\psi(c) \neq 0$ for some $c \in \mathbb{R}$. Then the functional equation* (6) *has a solution if and only if*

$$\begin{aligned} &\lambda^2\psi(c)^2\psi(x)\psi(y)A^2 \\ &\quad + \lambda\psi(c)\{\psi(x + y)\psi(c) - \psi(y + c)\psi(x) - \psi(x + c)\psi(y)\}A \\ &\quad + \psi(x + c)\psi(y + c) - \psi(x + y + c)\psi(c) - \lambda\psi(c)^2\psi(x)\psi(y) = 0 \end{aligned} \tag{7}$$

holds for every $x, y \in \mathbb{R}$, where A is a solution of the quadratic equation

$$\lambda^2\psi(c)^4z^2 - \lambda\psi(2c)\psi(c)^2z + \psi(2c)^2 - \psi(3c)\psi(c) - \lambda\psi(c)^4 = 0. \tag{8}$$

In this case, the solution f of (6) *is given by*

$$f(x) = A\psi(x) - \frac{1}{\lambda\psi(c)}\psi(x+c) \quad (x \in \mathbb{R}). \tag{9}$$

Corollary 4. *Let λ be a nonzero complex number and ψ be a nonzero complex function on $\mathbb{R}$ with $\psi(0) = 0$ and $\psi(c) \neq 0$ for some $c \in \mathbb{R}$. Suppose that*

$$\psi(c)\psi(x+y) = \psi(x)\psi(y+c) + \psi(x+c)\psi(y) \tag{10}$$

holds for every $x, y \in \mathbb{R}$. Then the functional equation (6) *has a solution and the solution f is given by*

$$f(x) = \pm\frac{1}{\lambda}\sqrt{\lambda + \frac{\psi(3c)}{\psi(c)^3}}\,\psi(x) - \frac{1}{\lambda\psi(c)}\psi(x+c) \quad (x \in \mathbb{R}). \tag{11}$$

The example of the function ψ satisfying (10) will be given in Section 5.

The second purpose of this paper is to investigate the Hyers–Ulam stability of Eq. (2). The Hyers–Ulam stability is the concept based on Ulam's problem [8] and Hyers' result [3]. We may find some results on the Hyers–Ulam stability of various equations in many papers (for example, [2,4,5]). First, we show the Hyers–Ulam stability of Eq. (4) that appeared in Theorems 1 and 2, under the assumption that α is purely imaginary.

Theorem 5. *Let λ and ρ be nonzero complex numbers and θ a real number. For the functional equation*

$$f(x+y) + \lambda f(x)f(y) = \rho e^{i\theta(x+y)} \quad (x, y \in \mathbb{R}), \tag{12}$$

there exists a constant $K = K(\lambda, \rho)$ with the following property: For any nonnegative number $\varepsilon < |\rho|$ and any complex function g on $\mathbb{R}$ satisfying

$$\left|g(x+y) + \lambda g(x)g(y) - \rho e^{i\theta(x+y)}\right| \leqslant \varepsilon \quad (x, y \in \mathbb{R}), \tag{13}$$

there is a solution f of (12) *such that*

$$\left|g(x) - f(x)\right| \leqslant K\left(\varepsilon + \sqrt{\varepsilon}\right) \quad (x \in \mathbb{R}). \tag{14}$$

Next, we consider the Hyers–Ulam stability of Eq. (6) in Theorem 2.

Theorem 6. *Let λ, ψ and c be as in Corollary* 4, *and suppose that ψ is bounded on $\mathbb{R}$. For the functional equation* (6), *there exists a constant $K = K(\lambda, \psi, c)$ with the following property: For any nonnegative number $\varepsilon < |\psi(c)|^2$ and any complex function g on $\mathbb{R}$ satisfying*

$$\left|g(x+y) + \lambda g(x)g(y) - \psi(x)\psi(y)\right| \leqslant \varepsilon \quad (x, y \in \mathbb{R}), \tag{15}$$

there is a solution f of (6) *such that* (14) *holds.*

Corollary 4 and Theorem 6 are generalizations of the results by M.Th. Rassias [6] and by S.-M. Jung [4]. We explain this fact in the last section.

2. Lemmas

In this section, we deal with the general functional equation (2). First we consider the exceptional case $\Phi(x, y) = 0$.

Lemma 1. *Let λ be a nonzero complex number. If a nonzero complex continuous function f on $\mathbb{R}$ satisfies*

$$f(x+y) + \lambda f(x)f(y) = 0 \quad (x, y \in \mathbb{R}), \tag{16}$$

then there exists a complex number α such that $f(x) = -\frac{1}{\lambda}e^{\alpha x}$ $(x \in \mathbb{R})$.

Proof. Define $g(x) = -\lambda f(x)$ for $x \in \mathbb{R}$. Then g is a nonzero complex continuous function on $\mathbb{R}$. Also, (16) implies

$$g(x+y) = g(x)g(y) \quad (x, y \in \mathbb{R}).$$

It is well known that such a function g is written as $g(x) = e^{\alpha x}$ $(x \in \mathbb{R})$ for some complex number α (see [7, Chapter 8, Exercise 6]). Hence $f(x) = -\frac{1}{\lambda}g(x) = -\frac{1}{\lambda}e^{\alpha x}$ for $x \in \mathbb{R}$. □

Let λ be a nonzero complex number and Φ a nonzero complex function on $\mathbb{R}^2$. Choose a point $(a, b) \in \mathbb{R}^2$ so that $\Phi(a, b) \neq 0$. For any complex function g on $\mathbb{R}$, define a function $g_{a,b}$ on $\mathbb{R}$ by

$$g_{a,b}(x) = \frac{1}{\lambda \Phi(a,b)}\big(\lambda g(a)\Phi(x, b) + \Phi(x, a+b) - \Phi(x+b, a)\big) \quad (x \in \mathbb{R}). \tag{17}$$

Lemma 2. *Let λ, Φ and (a, b) be as above. If $\varepsilon \geqslant 0$ and if a complex function g on $\mathbb{R}$ satisfies*

$$\big|g(x+y) + \lambda g(x)g(y) - \Phi(x, y)\big| \leqslant \varepsilon \quad (x, y \in \mathbb{R}), \tag{18}$$

then

$$\big|g(x) - g_{a,b}(x)\big| \leqslant \frac{2 + |\lambda|(|g(x)| + |g(a)|)}{|\lambda \Phi(a,b)|}\varepsilon \quad (x \in \mathbb{R}). \tag{19}$$

Proof. Putting $y = a + b$ in (18), we obtain

$$\big|g(x+a+b) + \lambda g(x)g(a+b) - \Phi(x, a+b)\big| \leqslant \varepsilon.$$

Replacing x by $x + b$ and y by a in (18), we obtain

$$\big|g(x+a+b) + \lambda g(x+b)g(a) - \Phi(x+b, a)\big| \leqslant \varepsilon.$$

Hence we compute

$$\begin{aligned}
&\big|\lambda \Phi(a,b)\big|\big|g(x) - g_{a,b}(x)\big| \\
&\quad = \big|\lambda \Phi(a,b)g(x) - \lambda g(a)\Phi(x,b) - \Phi(x, a+b) + \Phi(x+b, a)\big| \\
&\quad = \big|-\lambda g(x)\big(g(a+b) + \lambda g(a)g(b) - \Phi(a,b)\big) \\
&\qquad + \lambda g(a)\big(g(x+b) + \lambda g(x)g(b) - \Phi(x,b)\big) \\
&\qquad + \big(g(x+a+b) + \lambda g(x)g(a+b) - \Phi(x, a+b)\big) \\
&\qquad - \big(g(x+a+b) + \lambda g(x+b)g(a) - \Phi(x+b, a)\big)\big|
\end{aligned}$$

$$\begin{aligned}&\leqslant |\lambda|\big|g(x)\big|\big|g(a+b)+\lambda g(a)g(b)-\Phi(a,b)\big|\\&\quad+|\lambda|\big|g(a)\big|\big|g(x+b)+\lambda g(x)g(b)-\Phi(x,b)\big|\\&\quad+\big|g(x+a+b)+\lambda g(x)g(a+b)-\Phi(x,a+b)\big|\\&\quad+\big|g(x+a+b)+\lambda g(x+b)g(a)-\Phi(x+b,a)\big|\\&\leqslant |\lambda|\big|g(x)\big|\varepsilon+|\lambda|\big|g(a)\big|\varepsilon+\varepsilon+\varepsilon=\big(2+|\lambda|\big(\big|g(x)\big|+\big|g(a)\big|\big)\big)\varepsilon.\end{aligned}$$

Dividing both sides of the resulting inequality by $|\lambda\Phi(a,b)|$, we get (19). □

Lemma 3. *Let Φ, λ and (a,b) be as in Lemma 2. If the functional equation (2) has a solution f, then f is of the form*

$$f(x)=\frac{1}{\lambda\Phi(a,b)}\big(\lambda f(a)\Phi(x,b)+\Phi(x,a+b)-\Phi(x+b,a)\big)\quad (x\in\mathbb{R}).$$

Proof. In Lemma 2, put $\varepsilon=0$ and $g=f$. □

3. Proofs of Theorems 1–3 and Corollary 4

Proof of Theorem 1. Suppose that there exists a complex function f on $\mathbb{R}$ satisfying (3). Put $A=f(0)$. Substituting $y=0$ in (3), we obtain

$$\phi(x)=(1+\lambda A)f(x)\quad (x\in\mathbb{R}). \tag{20}$$

Since ϕ is nonzero and continuous, $1+\lambda A\neq 0$ and f is a nonzero continuous function on $\mathbb{R}$. Combining (20) with (3), we get $Af(x+y)-f(x)f(y)=0$ $(x,y\in\mathbb{R})$. Moreover, $A\neq 0$, because $A=0$ implies $f=0$. Thus we have the equation $f(x+y)-\frac{1}{A}f(x)f(y)=0$ $(x,y\in\mathbb{R})$. According to Lemma 1, there is a complex number α such that

$$f(x)=Ae^{\alpha x}\quad (x\in\mathbb{R}). \tag{21}$$

If we put

$$\rho=A(1+\lambda A), \tag{22}$$

then $\rho\neq 0$ and

$$\phi(x)=(1+\lambda A)Ae^{\alpha x}=\rho e^{\alpha x}\quad (x\in\mathbb{R})$$

by (20) and (21). Moreover, (22) implies $A=(-1\pm\sqrt{1+4\lambda\rho})/2\lambda$, and so (21) leads to (5). Conversely, a straightforward computation shows that the functions f defined by (5) satisfy (4). □

Proof of Theorem 2. Suppose that there exists a complex function f on $\mathbb{R}$ satisfying (6). Substituting $y=0$ in (6), we obtain

$$\psi(x)=\frac{1+\lambda f(0)}{\psi(0)}f(x)=Af(x)\quad (x\in\mathbb{R}),$$

where $A=(1+\lambda f(0))/\psi(0)$. Since ψ is nonzero, $A\neq 0$, and so $f(x)=\psi(x)/A$ $(x\in\mathbb{R})$. Hence (6) yields

$$\psi(x+y)+\frac{\lambda-A^2}{A}\psi(x)\psi(y)=0\quad (x,y\in\mathbb{R}).$$

Here $\lambda - A^2 \neq 0$, for if not, then $\psi = 0$. Applying Lemma 1, we find a complex number α such that $\psi(x) = (-A/(\lambda - A^2))e^{\alpha x}$ ($x \in \mathbb{R}$). If we put $\rho = A^2/(\lambda - A^2)^2$, then $\rho \neq 0$ and

$$\psi(x)\psi(y) = \frac{A^2}{(\lambda - A^2)^2} e^{\alpha x} e^{\alpha y} = \rho e^{\alpha(x+y)} \quad (x, y \in \mathbb{R}). \qquad \square$$

Proof of Theorem 3. Suppose that the functional equation (6) has a solution f. It follows from Lemma 3 with $\Phi(x, y) = \psi(x)\psi(y)$ ($x, y \in \mathbb{R}$) and $a = b = c$, that f is of the form

$$\begin{aligned} f(x) &= \frac{1}{\lambda\psi(c)^2}\big(\lambda f(c)\psi(x)\psi(c) + \psi(x)\psi(2c) - \psi(x+c)\psi(c)\big) \\ &= A\psi(x) - \frac{1}{\lambda\psi(c)}\psi(x+c) \quad (x \in \mathbb{R}), \end{aligned} \tag{23}$$

where $A = (\lambda f(c)\psi(c) + \psi(2c))/\lambda\psi(c)^2$. Since f is a solution of (6), if we substitute (23) into (6), then we obtain

$$\begin{aligned} &A\psi(x+y) - \frac{1}{\lambda\psi(c)}\psi(x+y+c) \\ &\quad + \lambda\left(A\psi(x) - \frac{1}{\lambda\psi(c)}\psi(x+c)\right)\left(A\psi(y) - \frac{1}{\lambda\psi(c)}\psi(y+c)\right) \\ &= \psi(x)\psi(y) \end{aligned} \tag{24}$$

for all $x, y \in \mathbb{R}$. In particular, if we put $x = y = c$ in (24), then we get

$$\lambda^2\psi(c)^4A^2 - \lambda\psi(2c)\psi(c)^2A + \psi(2c)^2 - \psi(3c)\psi(c) - \lambda\psi(c)^4 = 0.$$

This implies that A is a solution of the quadratic equation (8). By a simple calculation, we see from (24) that (7) holds for every $x, y \in \mathbb{R}$.

Conversely, suppose that (7) holds for every $x, y \in \mathbb{R}$. We will show that the functional equation (6) has a solution. In fact, if we put $f(x) = A\psi(x) - \psi(x+c)/\lambda\psi(c)$ for $x \in \mathbb{R}$, then we have

$$\begin{aligned} &f(x+y) + \lambda f(x)f(y) \\ &= A\psi(x+y) - \frac{1}{\lambda\psi(c)}\psi(x+y+c) \\ &\quad + \lambda\left(A\psi(x) - \frac{1}{\lambda\psi(c)}\psi(x+c)\right)\left(A\psi(y) - \frac{1}{\lambda\psi(c)}\psi(y+c)\right) \\ &= \frac{1}{\lambda\psi(c)^2}\big[\lambda^2\psi(c)^2\psi(x)\psi(y)A^2 \\ &\quad + \lambda\psi(c)\big\{\psi(x+y)\psi(c) - \psi(y+c)\psi(x) - \psi(x+c)\psi(y)\big\}A \\ &\quad + \psi(x+c)\psi(y+c) - \psi(x+y+c)\psi(c)\big] \end{aligned} \tag{25}$$

for every $x, y \in \mathbb{R}$. Since (7) is assumed to hold, it follows from (25) that

$$f(x+y) + \lambda f(x)f(y) = \frac{1}{\lambda\psi(c)^2}\lambda\psi(c)^2\psi(x)\psi(y) = \psi(x)\psi(y)$$

for every $x, y \in \mathbb{R}$. This proves that the functional equation (6) has a solution.

From the argument above, we see that the solution of the functional equation (6) is given by (9), where A is a solution of the quadratic equation (8). This completes the proof. $\square$

Proof of Corollary 4. Note that if we put $x = y = c$ in (10), then we obtain

$$\psi(2c) = 0. \tag{26}$$

We will prove that (7) holds for every $x, y \in \mathbb{R}$, where A is a solution of the quadratic equation (8): By (26) with (8), A satisfies

$$\lambda^2\psi(c)^4A^2 - \psi(3c)\psi(c) - \lambda\psi(c)^4 = 0. \tag{27}$$

Fix $x, y \in \mathbb{R}$ arbitrarily. By (27) we have

$$\lambda^2\psi(c)^2\psi(x)\psi(y)A^2 - \lambda\psi(c)^2\psi(x)\psi(y) = \frac{\psi(3c)}{\psi(c)}\psi(x)\psi(y).$$

Therefore, to prove (7) it is enough to show that

$$\frac{\psi(3c)}{\psi(c)}\psi(x)\psi(y) = \psi(x+y+c)\psi(c) - \psi(x+c)\psi(y+c). \tag{28}$$

To do this, put $y = 2c$ in (10). With (26), we have

$$\psi(c)\psi(x+2c) - \psi(x)\psi(3c) = 0.$$

This implies that

$$\frac{\psi(3c)}{\psi(c)}\psi(x)\psi(y) = \psi(x+2c)\psi(y). \tag{29}$$

If we replace x by $x + c$ in (10), then we get

$$\psi(c)\psi(x+y+c) - \psi(x+c)\psi(y+c) - \psi(x+2c)\psi(y) = 0,$$

and hence

$$\psi(x+2c)\psi(y) = \psi(c)\psi(x+y+c) - \psi(x+c)\psi(y+c). \tag{30}$$

By (29) and (30), we conclude that (28) holds. By Theorem 3, the functional equation (6) has a solution and the solution f is of the form

$$f(x) = A\psi(x) - \frac{1}{\lambda\psi(c)}\psi(x+c) \quad (x \in \mathbb{R}).$$

Since A satisfies (27), we see that

$$A = \pm\frac{1}{\lambda}\sqrt{\lambda + \frac{\psi(3c)}{\psi(c)^3}},$$

and the proof is complete. □

4. Proofs of Theorems 5 and 6

Proof of Theorem 5. Choose ε so that $0 \leqslant \varepsilon < |\rho|$, and let g be a complex function on $\mathbb{R}$ satisfying (13). We first observe that g is bounded on $\mathbb{R}$. Putting $y = 0$ in (13), we obtain

$$\left|\left(1 + \lambda g(0)\right)g(x) - \rho e^{i\theta x}\right| \leqslant \varepsilon \quad (x \in \mathbb{R}).$$

Since $\varepsilon < |\rho|$, it is impossible that $1 + \lambda g(0) = 0$. Hence $1 + \lambda g(0) \neq 0$. Thus $|g(x)| \leqslant (|\rho| + \varepsilon)/|1 + \lambda g(0)|$ for all $x \in \mathbb{R}$. This means that g is bounded on $\mathbb{R}$. Put $m = \sup\{|g(x)|: x \in \mathbb{R}\}$. Since (13) implies

$$\left|\lambda g(x)g(y)\right| \leqslant \left|g(x+y)\right| + |\rho| + \varepsilon < \left|g(x+y)\right| + 2|\rho| \quad (x, y \in \mathbb{R}),$$

it follows that $|\lambda| m^2 \leqslant m + 2|\rho|$, and so

$$m \leqslant \frac{1+\sqrt{1+8|\lambda\rho|}}{2|\lambda|}.$$

Put $\Phi(x,y) = \rho e^{i\theta(x+y)}$ $(x, y \in \mathbb{R})$. Since $\Phi(0,0) = \rho$, (17) with $a = b = 0$ becomes

$$g_{0,0}(x) = \frac{1}{\lambda\rho}\bigl(\lambda g(0)\rho e^{i\theta x} + \rho e^{i\theta x} - \rho e^{i\theta x}\bigr) = g(0)e^{i\theta x} \quad (x \in \mathbb{R}),$$

and Lemma 2 says that (13) implies that

$$\begin{aligned} \bigl|g(x) - g_{0,0}(x)\bigr| &\leqslant \frac{2+|\lambda|(|g(x)|+|g(0)|)}{|\lambda\rho|}\varepsilon \\ &\leqslant \frac{2+2|\lambda|m}{|\lambda\rho|}\varepsilon \leqslant \frac{3+\sqrt{1+8|\lambda\rho|}}{|\lambda\rho|}\varepsilon \quad (x \in \mathbb{R}). \end{aligned} \tag{31}$$

Next, we substitute $x = y = 0$ in (13) to see that $|g(0) + \lambda g(0)^2 - \rho| \leqslant \varepsilon$. This inequality can be written in the form

$$|\lambda|\bigl|g(0) - A_1\bigr|\bigl|g(0) - A_2\bigr| \leqslant \varepsilon,$$

where

$$A_1 = \frac{-1+\sqrt{1+4\lambda\rho}}{2\lambda} \quad \text{and} \quad A_2 = \frac{-1-\sqrt{1+4\lambda\rho}}{2\lambda}.$$

Hence we have either

$$\bigl|g(0) - A_1\bigr| \leqslant \sqrt{\frac{\varepsilon}{|\lambda|}} \quad \text{or} \quad \bigl|g(0) - A_2\bigr| \leqslant \sqrt{\frac{\varepsilon}{|\lambda|}}.$$

Write A for A_i with $|g(0) - A_i| \leqslant \sqrt{\varepsilon/|\lambda|}$, and put $f(x) = Ae^{i\theta x}$ for $x \in \mathbb{R}$. By Theorem 1, we know that f is a solution of (12). Moreover, we have

$$\bigl|g_{0,0}(x) - f(x)\bigr| = \bigl|\bigl(g(0) - A\bigr)e^{i\theta x}\bigr| = \bigl|g(0) - A\bigr| \leqslant \sqrt{\frac{\varepsilon}{|\lambda|}} \quad (x \in \mathbb{R}). \tag{32}$$

Put

$$K = \max\left\{\frac{3+\sqrt{1+8|\lambda\rho|}}{|\lambda\rho|}, \frac{1}{\sqrt{|\lambda|}}\right\}.$$

Then (31) and (32) show that

$$\bigl|g(x) - f(x)\bigr| \leqslant \bigl|g(x) - g_{0,0}(x)\bigr| + \bigl|g_{0,0}(x) - f(x)\bigr| \leqslant K\bigl(\varepsilon + \sqrt{\varepsilon}\bigr) \quad (x \in \mathbb{R}). \qquad \square$$

Remark 1. In Theorem 5, we only proved the Hyers–Ulam stability for $\varepsilon < |\rho|$. In general, the Hyers–Ulam stability does not hold for $\varepsilon \geqslant |\rho|$. To see this, pick $\varepsilon \geqslant |\rho|$ arbitrarily. Put $g(x) = -e^x/\lambda$ for $x \in \mathbb{R}$. Then

$$g(x+y) + \lambda g(x)g(y) = 0$$

holds for every $x, y \in \mathbb{R}$. In particular, (13) is satisfied since $|\rho| \leqslant \varepsilon$. On the other hand, it follows from Theorem 1 that the solution f of the functional equation (12) is given by

$$f(x) = Ae^{i\theta x} \quad (x \in \mathbb{R}),$$

where $A = (-1 \pm \sqrt{1+4\lambda\rho})/2\lambda$. Since $|f(x)| = |A|$ on $\mathbb{R}$, we have

$$\sup_{x\in\mathbb{R}}\bigl|g(x) - f(x)\bigr| \geqslant \frac{1}{|\lambda|}\sup_{x\in\mathbb{R}} e^x - |A| = \infty.$$

This implies that if $\varepsilon \geqslant |\rho|$, then there exists an approximate solution g of (12) such that g is not near to any solution f of (12).

Remark 2. In Theorem 5, we only considered the functional equation (4) with $\alpha = i\theta$. If we consider the case where the real part $\operatorname{Re}\alpha \neq 0$, then the following is true: If $\varepsilon, M \geqslant 0$ and if

$$f(x+y) + \lambda f(x)f(y) = \rho e^{\alpha(x+y)}, \qquad \bigl|g(x+y) + \lambda g(x)g(y) - \rho e^{\alpha(x+y)}\bigr| \leqslant \varepsilon$$

and $|g(x) - f(x)| \leqslant M$ for all $x, y \in \mathbb{R}$, then $g = f$.

In fact, put $h = g - f$, and hence $|h(x)| \leqslant M$ for all $x \in \mathbb{R}$. By hypothesis,

$$\begin{aligned}
&\bigl|h(x+y) + f(x+y) + \lambda\bigl(h(x) + f(x)\bigr)\bigl(h(y) + f(y)\bigr) - \rho e^{\alpha(x+y)}\bigr| \\
&\quad = \bigl|g(x+y) + \lambda g(x)g(y) - \rho e^{\alpha(x+y)}\bigr| \leqslant \varepsilon
\end{aligned}$$

for all $x, y \in \mathbb{R}$. Since f is a solution of the functional equation (4), we have

$$\bigl|h(x+y) + \lambda\bigl(h(x)h(y) + h(x)f(y) + h(y)f(x)\bigr)\bigr| \leqslant \varepsilon \quad (x, y \in \mathbb{R}).$$

By the triangle inequality,

$$\begin{aligned}
\bigl|\lambda h(y)f(x)\bigr| &\leqslant \varepsilon + \bigl|h(x+y)\bigr| + |\lambda|\bigl|h(x)\bigr|\bigl(\bigl|h(y)\bigr| + \bigl|f(y)\bigr|\bigr) \\
&\leqslant \varepsilon + M + |\lambda| M\bigl(M + \bigl|f(y)\bigr|\bigr)
\end{aligned}$$

for all $x, y \in \mathbb{R}$ since $|h| \leqslant M$ on $\mathbb{R}$. Now assume, on the contrary, that there exists $x_0 \in \mathbb{R}$ such that $h(x_0) \neq 0$. Then we get

$$\bigl|\lambda h(x_0)f(x)\bigr| \leqslant \varepsilon + M + |\lambda| M\bigl(M + \bigl|f(x_0)\bigr|\bigr),$$

and hence

$$\bigl|f(x)\bigr| \leqslant \frac{\varepsilon + M + |\lambda| M(M + |f(x_0)|)}{|\lambda h(x_0)|}$$

for all $x \in \mathbb{R}$. Therefore, f is a bounded function on $\mathbb{R}$. On the other hand, since f is a solution of Eq. (4), f is of the form $f(x) = Ae^{\alpha x}$ for $x \in \mathbb{R}$ by Theorem 1, where $A = (-1 \pm \sqrt{1+4\lambda\rho})/2\lambda$. Because $\operatorname{Re}\alpha \neq 0$, f is an unbounded function on $\mathbb{R}$, which is a contradiction. We thus conclude that $h = 0$ on $\mathbb{R}$, that is, $g = f$.

Proof of Theorem 6. Choose ε so that $0 \leqslant \varepsilon < |\psi(c)|^2$, and let g be a complex function satisfying (15). Putting $\Phi(x, y) = \psi(x)\psi(y)$ $(x, y \in \mathbb{R})$ and $a = b = c$ in (17), and using (26), we see that

$$\begin{aligned}
g_{c,c}(x) &= \frac{1}{\lambda\psi(c)^2}\bigl(\lambda g(c)\psi(x)\psi(c) + \psi(x)\psi(2c) - \psi(x+c)\psi(c)\bigr) \\
&= \frac{g(c)}{\psi(c)}\psi(x) - \frac{1}{\lambda\psi(c)}\psi(x+c) \quad (x \in \mathbb{R}).
\end{aligned} \tag{33}$$

Since ψ is bounded on $\mathbb{R}$, so is $g_{c,c}$. Also, Lemma 2 says that (15) implies

$$\bigl|g(x) - g_{c,c}(x)\bigr| \leqslant \frac{2 + |\lambda|(|g(x)| + |g(c)|)}{|\lambda\psi(c)^2|}\varepsilon \quad (x \in \mathbb{R}). \tag{34}$$

Hence

$$\left(1-\frac{\varepsilon}{|\psi(c)|^2}\right)\left|g(x)\right| \leqslant \left|g_{c,c}(x)\right| + \frac{2+|\lambda g(c)|}{|\lambda\psi(c)^2|}\varepsilon \quad (x \in \mathbb{R}).$$

Since $\varepsilon < |\psi(c)|^2$ and since $g_{c,c}$ is bounded on $\mathbb{R}$, it follows that g is also bounded on $\mathbb{R}$. Put $m = \sup\{|g(x)|\colon x \in \mathbb{R}\}$ and $M_\psi = \sup\{|\psi(x)|\colon x \in \mathbb{R}\}$ (recall that ψ is bounded). By (15), we have $|\lambda g(x)g(y)| \leqslant |g(x+y)| + |\psi(x)\psi(y)| + \varepsilon$ $(x, y \in \mathbb{R})$, and so $|\lambda|m^2 \leqslant m + M_\psi^2 + |\psi(c)|^2$. Hence m is less than or equal to

$$M_g = \frac{1+\sqrt{1+4(M_\psi^2+|\psi(c)|^2)|\lambda|}}{2|\lambda|}.$$

Clearly, the constant M_g depends only on λ, ψ and c. Since $|g(x)| \leqslant M_g$ for all $x \in \mathbb{R}$, it follows from (34) that

$$\left|g(x) - g_{c,c}(x)\right| \leqslant \frac{2+2|\lambda|M_g}{|\lambda\psi(c)^2|}\varepsilon \quad (x \in \mathbb{R}). \tag{35}$$

Next, we substitute $x = 2c$ in (33) and use (26) to see that

$$g_{c,c}(2c) = \frac{g(c)}{\psi(c)}\psi(2c) - \frac{1}{\lambda\psi(c)}\psi(3c) = -\frac{\psi(3c)}{\lambda\psi(c)}. \tag{36}$$

The same substitution turns (35) into

$$\left|g(2c) - g_{c,c}(2c)\right| \leqslant \frac{2+2|\lambda|M_g}{|\lambda\psi(c)^2|}\varepsilon,$$

while the substitution $x = y = c$ in (15) provides $|g(2c) + \lambda g(c)^2 - \psi(c)^2| \leqslant \varepsilon$. Hence

$$\begin{aligned}\left|\lambda g(c)^2 - \psi(c)^2 + g_{c,c}(2c)\right| &\leqslant \left|\lambda g(c)^2 - \psi(c)^2 + g(2c)\right| + \left|g_{c,c}(2c) - g(2c)\right| \\ &\leqslant \left(1 + \frac{2+2|\lambda|M_g}{|\lambda\psi(c)^2|}\right)\varepsilon.\end{aligned}$$

With (36), the left side can be written as follows:

$$\begin{aligned}\left|\lambda g(c)^2 - \psi(c)^2 + g_{c,c}(2c)\right| &= |\lambda|\left|g(c)^2 - \frac{\psi(c)^2}{\lambda} - \frac{\psi(3c)}{\lambda^2\psi(c)}\right| \\ &= |\lambda|\left|g(c)^2 - \frac{\psi(c)^2}{\lambda^2}\left(\lambda + \frac{\psi(3c)}{\psi(c)^3}\right)\right| \\ &= |\lambda|\left|g(c) - A_1\psi(c)w\right|\left|g(c) - A_2\psi(c)\right|,\end{aligned}$$

where

$$A_1 = \frac{1}{\lambda}\sqrt{\lambda + \frac{\psi(3c)}{\psi(c)^3}} \quad \text{and} \quad A_2 = -\frac{1}{\lambda}\sqrt{\lambda + \frac{\psi(3c)}{\psi(c)^3}}.$$

Hence either $|g(c) - A_1\psi(c)|$ or $|g(c) - A_2\psi(c)|$ is bounded by

$$\sqrt{\frac{1}{|\lambda|}\left(1 + \frac{2+2|\lambda|M_g}{|\lambda\psi(c)^2|}\right)}\sqrt{\varepsilon}.$$

If $|g(c) - A_i\psi(c)|$ is truly bounded by the above quantity, then we write A instead of A_i and put

$$f(x) = A\psi(x) - \frac{1}{\lambda\psi(c)}\psi(x+c) \quad (x \in \mathbb{R}). \tag{37}$$

By Corollary 4, f is a solution of (6). By (33), (37) and our choice of $A = A_i$, we obtain

$$\begin{aligned} \left|g_{c,c}(x) - f(x)\right| &\leqslant \left|\frac{g(c)}{\psi(c)}\psi(x) - A\psi(x)\right| = \frac{1}{|\psi(c)|}\left|g(c) - A\psi(c)\right|\left|\psi(x)\right| \\ &\leqslant \frac{1}{|\psi(c)|}\sqrt{\frac{1}{|\lambda|}\left(1 + \frac{2+2|\lambda|M_g}{|\lambda\psi(c)^2|}\right)M_\psi\sqrt{\varepsilon}} \quad (x \in \mathbb{R}). \end{aligned} \tag{38}$$

Now put

$$K = \max\left\{\frac{2+2|\lambda|M_g}{|\lambda\psi(c)^2|}, \frac{1}{|\psi(c)|}\sqrt{\frac{1}{|\lambda|}\left(1 + \frac{2+2|\lambda|M_g}{|\lambda\psi(c)^2|}\right)M_\psi}\right\},$$

which depends only on λ, ψ and c. Then (35) and (38) yield (14). □

5. An example

Let us give some applications of Theorems 3 and 6.

Corollary 7. *Let λ and β be nonzero complex numbers. Then the functional equation*

$$f(x+y) + \lambda f(x)f(y) = \beta xy \quad (x, y \in \mathbb{R}) \tag{39}$$

has a solution and the solution f is given by

$$f(x) = \pm\sqrt{\frac{\beta}{\lambda}}x - \frac{1}{\lambda} \quad (x \in \mathbb{R}). \tag{40}$$

Proof. Put $\psi(x) = \sqrt{\beta}x$ for $x \in \mathbb{R}$. Take $c = 1$ in (8):

$$\lambda^2\beta^2z^2 - 2\lambda\beta\sqrt{\beta}z + \beta - \lambda\beta^2 = 0.$$

Let A be a solution of the quadratic equation above, that is,

$$A = \frac{\sqrt{\beta} \pm \beta\sqrt{\lambda}}{\lambda\beta}.$$

It is easy to see that $A\psi(x) - \psi(x+1)/\lambda\psi(1)$ is a solution of the functional equation (39). By Theorem 3, the solution of (39) is of the form (40). □

Remark 3. Let ψ be as in Corollary 7. It is obvious that ψ does not hold (10) for any $c, x, y \in \mathbb{R} \setminus \{0\}$. On the other hand, the functional equation (39) has a solution by Corollary 7. Therefore, (10) is not necessary but sufficient for Eq. (6) to have a solution.

Corollary 8. *Let λ, ρ, α, β be complex numbers with $\lambda \neq 0$, $\rho \neq 0$ and $\alpha \neq \beta$. Then the functional equation*

$$f(x+y) + \lambda f(x)f(y) = \rho\left(e^{\alpha x} - e^{\beta x}\right)\left(e^{\alpha y} - e^{\beta y}\right) \quad (x, y \in \mathbb{R}) \tag{41}$$

has a solution and the solution is given by

$$f(x) = \pm\frac{\sqrt{1+4\lambda\rho}}{2\lambda}\left(e^{\alpha x} - e^{\beta x}\right) - \frac{1}{2\lambda}\left(e^{\alpha x} + e^{\beta x}\right) \quad (x \in \mathbb{R}).$$

Proof. Define a complex function ψ on $\mathbb{R}$ by

$$\psi(x) = \sqrt{\rho}\left(e^{\alpha x} - e^{\beta x}\right) \quad (x \in \mathbb{R}). \tag{42}$$

Clearly, $\psi(0) = 0$. Take $c = \pi i/(\beta - \alpha)$. Then

$$e^{\beta c} = \exp\frac{\beta\pi i}{\beta - \alpha} = \exp\left(\pi i + \frac{\alpha\pi i}{\beta - \alpha}\right) = \exp\pi i \exp\frac{\alpha\pi i}{\beta - \alpha} = -e^{\alpha c}. \tag{43}$$

Hence

$$\psi(c) = \sqrt{\rho}\left(e^{\alpha c} - e^{\beta c}\right) = 2\sqrt{\rho}e^{\alpha c} \neq 0. \tag{44}$$

Also, ψ satisfies the condition (10), because we use (43) and (44) to compute

$$\begin{aligned}
&\psi(x)\psi(y+c) + \psi(x+c)\psi(y) \\
&\quad = \sqrt{\rho}\left(e^{\alpha x} - e^{\beta x}\right)\sqrt{\rho}\left(e^{\alpha(y+c)} - e^{\beta(y+c)}\right) + \sqrt{\rho}\left(e^{\alpha(x+c)} - e^{\beta(x+c)}\right)\sqrt{\rho}\left(e^{\alpha y} - e^{\beta y}\right) \\
&\quad = \rho\big(e^{\alpha c}e^{\alpha(x+y)} - e^{\beta c}e^{\alpha x+\beta y} - e^{\alpha c}e^{\beta x+\alpha y} + e^{\beta c}e^{\beta(x+y)} \\
&\qquad + e^{\alpha c}e^{\alpha(x+y)} - e^{\alpha c}e^{\alpha x+\beta y} - e^{\beta c}e^{\beta x+\alpha y} + e^{\beta c}e^{\beta(x+y)}\big) \\
&\quad = \rho e^{\alpha c}\big(e^{\alpha(x+y)} + e^{\alpha x+\beta y} - e^{\beta x+\alpha y} - e^{\beta(x+y)} \\
&\qquad + e^{\alpha(x+y)} - e^{\alpha x+\beta y} + e^{\beta x+\alpha y} - e^{\beta(x+y)}\big) \\
&\quad = 2\rho e^{\alpha c}\left(e^{\alpha(x+y)} - e^{\beta(x+y)}\right) = 2\sqrt{\rho}e^{\alpha c}\sqrt{\rho}\left(e^{\alpha(x+y)} - e^{\beta(x+y)}\right) \\
&\quad = \psi(c)\psi(x+y).
\end{aligned}$$

It follows from Corollary 4 that the functional equation (41) has a solution and the solution f is of the form (11). Moreover, since $\psi(3c) = \sqrt{\rho}(e^{3\alpha c} - e^{3\beta c}) = 2\sqrt{\rho}e^{3\alpha c}$ by (43), (11) becomes

$$\begin{aligned}
f(x) &= \pm\frac{1}{\lambda}\sqrt{\lambda + \frac{2\sqrt{\rho}e^{3\alpha c}}{(2\sqrt{\rho}e^{\alpha c})^3}}\sqrt{\rho}\left(e^{\alpha x} - e^{\beta x}\right) - \frac{1}{2\lambda\sqrt{\rho}e^{\alpha c}}\sqrt{\rho}\left(e^{\alpha(x+c)} - e^{\beta(x+c)}\right) \\
&= \pm\frac{1}{\lambda}\sqrt{\lambda + \frac{1}{4\rho}}\sqrt{\rho}\left(e^{\alpha x} - e^{\beta x}\right) - \frac{1}{2\lambda e^{\alpha c}}\left(e^{\alpha c}e^{\alpha x} - e^{\beta c}e^{\beta x}\right) \\
&= \pm\frac{\sqrt{1+4\lambda\rho}}{2\lambda}\left(e^{\alpha x} - e^{\beta x}\right) - \frac{1}{2\lambda}\left(e^{\alpha x} + e^{\beta x}\right).
\end{aligned}$$

The proof is complete. □

Taking $\alpha = i$, $\beta = -i$ and $\rho = -d/4$ in Corollary 8, we have the following:

Corollary 9. *Let λ and d be nonzero complex numbers. Then the functional equation*

$$f(x+y) + \lambda f(x)f(y) = d\sin x\sin y \quad (x, y \in \mathbb{R})$$

has a solution and the solution is given by

$$f(x) = \pm\frac{\sqrt{\lambda d - 1}}{\lambda}\sin x - \frac{1}{\lambda}\cos x \quad (x \in \mathbb{R}).$$

The case where $\lambda = -1$ and $d < -1$ in Corollary 9 is the result by Rassias:

Corollary 10. (Rassias [6]) *Let d be a real number with $d < -1$. Then the functional equation* (1) *has exactly two solutions $f(x) = \pm\sqrt{-d-1}\sin x + \cos x$.*

Next we consider the Hyers–Ulam stability of Eq. (41). If α and β are purely imaginary, then the function ψ defined by (42) is bounded on $\mathbb{R}$. Hence we can apply Theorem 6 and obtain the following corollary:

Corollary 11. *Let λ, ρ be nonzero complex numbers and a, b distinct real numbers. For the functional equation*

$$f(x+y) + \lambda f(x)f(y) = \rho\left(e^{iax} - e^{ibx}\right)\left(e^{iay} - e^{iby}\right) \quad (x, y \in \mathbb{R}), \tag{45}$$

there exists a constant $K = K(\lambda, \rho, a, b)$ with the following property: *For any nonnegative number $\varepsilon < 4|\rho|$ and any complex function g on $\mathbb{R}$ satisfying*

$$\left|g(x+y) + \lambda g(x)g(y) - \rho\left(e^{iax} - e^{ibx}\right)\left(e^{iay} - e^{iby}\right)\right| \leqslant \varepsilon \quad (x, y \in \mathbb{R}),$$

there is a solution f of (45) *such that* (14) *holds.*

Taking $\lambda = -1$, $a = 1$, $b = -1$ and $\rho = -d/4$ in Corollary 11, we obtain the result by S.-M. Jung:

Corollary 12. (S.-M. Jung [4]) *Let d be a real number with $d < -1$. For the functional equation* (1)*, there exists a constant $K = K(d)$ with the following property*: *For any nonnegative number $\varepsilon < |d|$ and any complex function g on $\mathbb{R}$ satisfying*

$$\left|g(x+y) - g(x)g(y) - d\sin x \sin y\right| \leqslant \varepsilon \quad (x, y \in \mathbb{R}),$$

there is a solution f of (1) *such that* (14) *holds.*

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