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Space–time analyticity of weak solutions to linear parabolic systems with variable coefficients

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Abstract

Analytic smoothing properties of a general, strongly parabolic linear Cauchy problem of second order in $\mathbb{R}^N \times (0, T)$ with analytic coefficients (in space and time variables) are investigated. They are expressed in terms of holomorphic continuation of global (weak) L^2-type solutions to the system. Given $0 < T' < T \leqslant \infty$, it is proved that any L^2-type solution $u : \mathbb{R}^N \times (0, T) \to \mathbb{R}^M$ possesses a bounded holomorphic continuation $u(x + \mathrm{i}y, \sigma + \mathrm{i}\tau)$ into a complex domain in $\mathbb{C}^N \times \mathbb{C}$ defined by $(x, \sigma) \in \mathbb{R}^N \times (T', T)$, $|y| < A'$ and $|\tau| < B'$, where $A', B' > 0$ are constants depending upon T'. The proof uses the extension of a solution to an L^2-type solution in a domain in $\mathbb{C}^N \times \mathbb{C}$, such that this extension satisfies the Cauchy–Riemann equations. The holomorphic extension is thus obtained in a Hardy space H^2. Applications include *market completion* by European options in Finance.

Keywords: Space–time analyticity; Parabolic PDE; Holomorphic continuation; Hardy space; Market completeness; European option

1. Introduction

In this article we investigate analyticity (in space and time variables) of weak (L^2-type) solutions $\mathbf{u} = (u_1, \dots, u_M) : \mathbb{R}^N \times (0, T) \to \mathbb{R}^M$ (or $\mathbb{C}^M$) of the classical Cauchy problem for a strongly parabolic system of M linear partial differential equations of order $2m$ ($m \geqslant 1$ – an

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integer) with analytic coefficients, but with initial data $\mathbf{u}_0$ only in $\mathbf{L}^2(\mathbb{R}^N) = [L^2(\mathbb{R}^N)]^M$. This Cauchy problem has the following general form,

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{P}\left(x, t, \dfrac{1}{\mathrm{i}}\dfrac{\partial}{\partial x}\right)\mathbf{u} = \mathbf{f}(x, t) & \text{for } (x, t) \in \mathbb{R}^N \times (0, T); \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{1}$$

Here, $\partial/\partial x = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$ stands for the gradient and $\xi \mapsto \mathbf{P}(x, t, \xi)$ is a polynomial of order $2m$ in the variable $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ (or $\mathbb{C}^N$); its coefficients are $M \times M$ matrices (real or complex) which are assumed to be real analytic (jointly) in both variables $x \in \mathbb{R}^N$ and $t \in (0, T)$. As usual, $\mathbb{R}^N$ and $\mathbb{C}^N$, respectively, denote the N-dimensional real and complex Euclidean spaces, $\mathrm{i} = \sqrt{-1}$, and $M, N \in \mathbb{N}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$.

We impose certain standard *strong ellipticity* and *analyticity* hypotheses on the coefficients of the partial differential operator $\mathbf{P}(x, t, \frac{1}{\mathrm{i}}\frac{\partial}{\partial x})$ and on the function $\mathbf{f}(x, t)$ as well. Assuming only $\mathbf{u}_0 \in \mathbf{L}^2(\mathbb{R}^N)$, in this work we show that the (unique) weak solution $\mathbf{u} = \mathbf{u}(x, t)$ of problem (1) is real analytic in $(x, t) \in \mathbb{R}^N \times (0, T)$.

This claim is motivated by the standard formula for the solution of the Cauchy problem for the heat equation in $\mathbb{R}^N$ (with the Laplace operator Δ, i.e., $\mathbf{P}(x, t, \frac{1}{\mathrm{i}}\frac{\partial}{\partial x}) = -\Delta$, $\mathbf{f}(x, t) = 0$, and $M = 1$); see e.g. F. John [44, Chapt. 7, Sect. 1, Eq. (1.11), p. 209]. The heat equation case has been significantly generalized in P. Takáč et al. [70, Theorem 2.1, p. 429], where only the leading coefficients of the operator $\mathbf{P}(x, t, \frac{1}{\mathrm{i}}\frac{\partial}{\partial x})$ are assumed to be constant, but it is required that $\mathbf{u}_0 \in \mathbf{L}^\infty(\mathbb{R}^N) = [L^\infty(\mathbb{R}^N)]^M$. The main contribution of our present article is that we are able to *remove* the hypothesis that the leading coefficients must be constant. In contrast to [70, Proposition A.4, p. 446], this means that we *cannot* calculate the Green function for the Cauchy problem with the leading coefficients only,

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (-1)^m \displaystyle\sum_{|\alpha|=2m} \mathbf{P}^{(\alpha)}(x, t)\dfrac{\partial^{|\alpha|}\mathbf{u}}{\partial x^\alpha} = \mathbf{0} & \text{for } (x, t) \in \mathbb{R}^N \times (0, T); \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{2}$$

and then simply take advantage of the variation-of-constants formula [70, Eq. (3.22), p. 437] in order to obtain the solution of the original problem (1). Here, $\partial^{|\alpha|}\mathbf{u}/\partial x^\alpha = \frac{\partial^{|\alpha|}\mathbf{u}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$ denotes the (mixed) partial derivative of $\mathbf{u}$ with a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{Z}_+)^N$ of order $|\alpha| = \alpha_1 + \cdots + \alpha_N$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and each $\mathbf{P}^{(\alpha)}(x, t)$ is an $M \times M$ matrix.

Instead of using the Green function method (see [70]), we establish an a priori L^2-type estimate directly for holomorphic (i.e., complex analytic) extensions of solutions of the Cauchy problem (1) to a complex parabolic domain $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$ in $\mathbb{C}^N \times \mathbb{C}$ with initial data $\mathbf{u}_0$ from a Hardy space $\mathbf{H}^2(\mathfrak{X}^{(r)}) = [H^2(\mathfrak{X}^{(r)})]^M$ of holomorphic functions whose domain $\mathfrak{X}^{(r)} = \mathbb{R}^N + \mathrm{i}Q^{(r)}$ is a tube in $\mathbb{C}^N$ with base $Q^{(r)} = (-r, r)^N$, for some $0 < r < \infty$, see E.M. Stein and G. Weiss [69, Chapt. III]. We will see that this a priori L^2-type estimate over the domain $\Gamma_T^{(T_0)}(\kappa_0, \nu_0) \subset \mathbb{C}^N \times \mathbb{C}$ depends on the L^2-norm of the initial data $\mathbf{u}_0$ over $\mathbb{R}^N$ only. Consequently, we can combine an approximation procedure for the initial data from $\mathbf{L}^2(\mathbb{R}^N)$ by (holomorphic) functions from $\mathbf{H}^2(\mathfrak{X}^{(r)})$ with the uniqueness of the weak solution $\mathbf{u} = \mathbf{u}(x, t)$ of problem (1) in order to conclude that this weak solution is the (locally uniform) limit of a sequence of holomorphic functions in $\Gamma_T^{(T_0)}(\kappa_0,\nu_0)$ and thus itself holomorphic in $\Gamma_T^{(T_0)}(\kappa_0,\nu_0)$. Moreover, this limit satisfies the same a priori L^2-type estimate over $\Gamma_T^{(T_0)}(\kappa_0,\nu_0)$.

As in [70], our method is based on the simple fact that a function $u:\mathbb{R}^N\times(0,T)\to\mathbb{R}$ (or $\mathbb{C}$) is real analytic if and only if it has a holomorphic extension $\tilde{u}:\Omega\to\mathbb{C}$ to some complex domain Ω such that $\mathbb{R}^N\times(0,T)\subset\Omega\subset\mathbb{C}^N\times\mathbb{C}$, i.e., $u=\tilde{u}|_{\mathbb{R}^N\times(0,T)}$, the restriction of $\tilde{u}$ to $\mathbb{R}^N\times(0,T)$. If the domain Ω is fixed then the holomorphic extension $\tilde{u}$ of u to Ω is always unique, see e.g. F. JOHN [44, Chapt. 3, Sect. 3(c), pp. 70–72]. Thus, in order to show that the weak solution $\mathbf{u}=\mathbf{u}(x,t)$ of problem (1) is real analytic in $\mathbb{R}^N\times(0,T)$, it suffices to construct a holomorphic extension $\tilde{\mathbf{u}}$ of $\mathbf{u}$ to some complex domain Ω ($\mathbb{R}^N\times(0,T)\subset\Omega\subset\mathbb{C}^N\times\mathbb{C}$). Due to the uniqueness, we often drop the tilde ($\tilde{}$) in the notation for the (unique) holomorphic extension. Analogous ideas (holomorphic extension, uniqueness, and Bergman and Szegő spaces of holomorphic functions) were used earlier in N. HAYASHI [31–34].

In order to provide a quick, nontechnical hint to our approach, we now give a weaker version of our main result, Theorem 3.3 in Section 3, for a single equation in one space dimension ($M=N=1$),

$$\begin{cases} \dfrac{\partial u}{\partial t}=a(x,t)\dfrac{\partial^2 u}{\partial x^2}+b(x,t)\dfrac{\partial u}{\partial x}+c(x,t)u+f(x,t) & \text{for } (x,t)\in\mathbb{R}^1\times(0,T);\\ u(x,0)=u_0(x) & \text{for } x\in\mathbb{R}^1. \end{cases} \tag{3}$$

We begin with the complexifications of the spatial and temporal variables, $x\in\mathbb{R}$ and $t\in(0,T)$, respectively: Given $r,T\in(0,\infty)$ and $T'\in[0,T)$, we introduce the complex domains

$$\mathfrak{X}^{(r)}\stackrel{\text{def}}{=}\{z=x+\mathrm{i}y\in\mathbb{C}\colon |y|<r\}=\mathbb{R}+\mathrm{i}(-r,r),$$
$$\mathfrak{T}^{(r)}_{T',T}\stackrel{\text{def}}{=}\{t=\sigma+\mathrm{i}\tau\in\mathbb{C}\colon T'<\sigma<T \text{ and } |\tau|<r\}=(T',T)+\mathrm{i}(-r,r);$$

their closures in the complex plane $\mathbb{C}=\mathbb{R}\oplus\mathrm{i}\mathbb{R}$ are denoted by $\overline{\mathfrak{X}}^{(r)}$ and $\overline{\mathfrak{T}}^{(r)}_{T',T}$, respectively. The Banach space of all continuous ($L^2(\mathbb{R}^1)$-valued) functions $u:[0,T]\to L^2(\mathbb{R}^1)$ is denote by $C([0,T]\to L^2(\mathbb{R}^1))$; it is endowed with the natural supremum norm $\sup_{t\in[0,T]}\|u(\cdot,t)\|_{L^2(\mathbb{R}^1)}<\infty$.

Theorem 1.1 *($M=N=1$). Let $0<T<\infty$ and assume that there are some constants $A,B>0$ such that all coefficients a, b, and c, the partial derivative $\partial a/\partial x$, and the function f are bounded, continuously differentiable functions in the Cartesian product $\overline{\mathfrak{X}}^{(A)}\times\overline{\mathfrak{T}}^{(B)}_{0,T}$, with $\mathfrak{Re}\,a\geqslant\text{const}>0$, and all a, b, c, and f are holomorphic in $\mathfrak{X}^{(A)}\times\mathfrak{T}^{(B)}_{0,T}$. Furthermore, we assume that f satisfies*

$$\int_{-\infty}^{\infty}\left|f(x+\mathrm{i}y,t)\right|^2\,\mathrm{d}x\leqslant\text{const}<\infty\quad\text{for all } y\in[-A,A] \text{ and } t\in\overline{\mathfrak{T}}^{(B)}_{0,T}.$$

Then, given any $u_0\in L^2(\mathbb{R}^1)$, the Cauchy problem (3) possesses a unique weak solution $u\in C([0,T]\to L^2(\mathbb{R}^1))$. For each $T'\in(0,T)$, this solution can be (uniquely) extended to a *holomorphic function in* $\mathfrak{X}^{(A')} \times \mathfrak{T}^{(B')}_{T',T}$, *where* $A' \in (0, A)$ *and* $B' \in (0, B)$ *are some constants that depend on* A, B, *and* T', *but are* independent *from* u_0 *and* T. *Moreover, this* (*unique holomorphic*) *extension of* u, *denoted again by* u, *satisfies*

$$\int_{-\infty}^{\infty} \left|u(x+\mathrm{i}y,t)\right|^2 \mathrm{d}x \leqslant \mathrm{const} < \infty \quad \textit{for all } y \in \left(-A', A'\right) \textit{ and } t \in \mathfrak{T}^{(B')}_{T',T},$$

where the last constant may depend on T *exponentially.*

Remark 1.2. We will specify the dependence of $A' = A'(T')$ and $A' = A'(T')$ on $T' \in (0,T)$ in Theorem 3.3 (Section 3). In particular, we have $A'(T') \geqslant c_1(T')^{(1/2)+\varepsilon}$ and $B'(T') \geqslant c_2 T'$ for all $T' > 0$ small enough, where $\varepsilon \in (0,1)$ is an arbitrary constant and the constants $c_1, c_2 > 0$ are sufficiently small. If the leading coefficient $a(x,t)$ equals to a positive constant, we may take $\varepsilon = 0$; cf. P. Takáč et al. [70, Theorem 2.1, p. 429]. However, if $a(x,t)$ is not constant, our present methods do not allow us to take $\varepsilon = 0$, although we dare to conjecture that it should be possible.

Remark 1.3. It follows obviously from Theorem 1.1 that every *weak* solution $u \in C([0,T] \to L^2(\mathbb{R}^1))$ to the Cauchy problem (3) (defined e.g. in L.C. Evans [19, Chapt. 7, §1.1, p. 352], or J.-L. Lions [56, Chapt. IV, §1, p. 44], or [57, Chapt. III, Eq. (1.11), p. 102]) is *classical* in the sense that it is of class C^∞ over the open set $(0,T) \times \mathbb{R}^1$ and verifies Eq. (3) pointwise and the initial condition $u(\cdot,0) = u_0 \in L^2(\mathbb{R}^1)$ in the $L^2(\mathbb{R}^1)$-limit $\|u(\cdot,t) - u_0\|_{L^2(\mathbb{R}^1)} \to 0$ as $t \to 0+$. The main reason why we prefer to work with the notion of a *weak solution* as opposed to a *classical solution* of the Cauchy problem (3) is the fact that already a weak solution is *unique*. The uniqueness of a weak solution is an important technical argument in our proofs of Theorem 1.1 and Theorem 3.3 (Section 3).

In fact, we work sometimes also with the so-called *mild solutions* to the Cauchy problem (3) that make sense in $C([0,T] \to L^2(\mathbb{R}^1))$ and do not require any additional regularity knowledge; they are defined by the well-known variation-of-constants formula (A. Pazy [64, §5.7, p. 168]). Thus, they are even "weaker" than the weak solutions, but in our situation one can easily verify that every mild solution is also a weak solution to problem (3) and vice versa; see e.g. J.M. Ball [3] (or [64, Theorem on p. 259]).

The same remarks apply also to the more general Cauchy problem (1). □

This article is organized as follows. We introduce some basic notation (mostly complex domains) in Section 2. Our main analyticity result, Theorem 3.3, supplemented by an additional explanation in Proposition 3.4, is stated in Section 3. Their proofs are gradually built up in Sections 4 through 7: First, an important a priori L^2-type estimate is established in Lemma 3.4 (Section 4). Then, in Section 5, an equivalent characterization of the Hardy space $H^2(\mathfrak{X}^{(r)})$ over an N-dimensional strip $\mathfrak{X}^{(r)} = \mathbb{R}^N + \mathrm{i}(-r,r)^N$ in $\mathbb{C}^N$, $0 < r < \infty$, is provided by means of the Fourier–Laplace transform, its inverse, and Plancherel's theorem. The initial value Cauchy problem (1) with complex analytic initial data $\mathbf{u}_0 \in \mathbf{H}^2(\mathfrak{X}^{(r_0)})$ $(0 < r_0 < \infty)$ is solved in Proposition 6.1 (Section 6). The proofs of our main results are completed in Section 7. In Section 8 we present an application of Theorem 3.3 to the "martingale model" for *market completeness* in Mathematical Finance (M.H.A. Davis and J. Obłój [16, Sect. 3] and M. Romano and N. Touzi [65, Sect. 3]). This problem from Finance in fact "triggered" the mathematical research reported in the present article; cf. [16, Assumption (A4), p. 53]. An earlier, weaker analyticity result from P. TAKÁČ et al. [70, Theorem 2.1, p. 429] cannot be applied here for the reason mentioned at the beginning of this Introduction (*nonconstant* leading coefficients). Finally, Section 9 contains some historical remarks and comments concerning the analyticity of solutions to linear elliptic and parabolic systems and its applications to historically relevant classical problems.

2. Notation

Typically, we denote by $x = (x_1, x_2, \dots, x_N)$ and $y = (y_1, y_2, \dots, y_N)$ points in $\mathbb{R}^N$ and by $z = (z_1, z_2, \dots, z_N)$ points in $\mathbb{C}^N$. We often write $\zeta = \xi + \mathrm{i}\eta$ for $\zeta \in \mathbb{C}$ and $\xi, \eta \in \mathbb{R}$, i.e., $\mathfrak{Re}\,\zeta = \xi$ and $\mathfrak{Im}\,\zeta = \eta$. Similarly, $z = x + \mathrm{i}y$ for $z \in \mathbb{C}^N$ and $x, y \in \mathbb{R}^N$, or equivalently $z_i = x_i + \mathrm{i}y_i$ ($i = 1, 2, \dots, N$) for $z_i \in \mathbb{C}$ and $x_i, y_i \in \mathbb{R}$, i.e., $\mathfrak{Re}\,z = x$ and $\mathfrak{Im}\,z = y$. Hence, we identify $\mathbb{C}^N = \mathbb{R}^N \oplus \mathrm{i}\mathbb{R}^N$ (or simply $\mathbb{C}^N = \mathbb{R}^N + \mathrm{i}\mathbb{R}^N$) as vector spaces over the field $\mathbb{R}$ and thus consider $\mathbb{R}^N$ to be a (vector) subspace of $\mathbb{C}^N$. We use a bar ($\bar{\ }$) to denote the complex conjugate $\bar{\zeta}$ of a number $\zeta \in \mathbb{C}$. The complex conjugate of a vector $z \in \mathbb{C}^N$ is denoted by $\bar{z} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_N)$. Similarly, the complex conjugate function of a complex-valued function $f(z)$ (for $f : \mathbb{C}^N \to \mathbb{C}$, for instance) is denoted by $\bar{f}(z) \equiv \overline{f(\bar{z})}$. Furthermore, we denote by $(z, w) = \sum_{i=1}^N z_i \bar{w}_i$ the standard Euclidean inner product of $z, w \in \mathbb{C}^N$ and by $|z| = (\sum_{i=1}^N |z_i|^2)^{1/2}$ the induced (Euclidean) norm of $z \in \mathbb{C}^N$. We will often use the sum (ℓ^1-) and the maximum (ℓ^∞-) norms of $z \in \mathbb{C}^N$, respectively:

$$|z|_1 = \sum_{i=1}^N |z_i| \quad \text{and} \quad |z|_\infty = \sup_{1 \leqslant i \leqslant N} |z_i|.$$

Finally, we write $z \cdot w = \sum_{i=1}^N z_i w_i$ for $z, w \in \mathbb{C}^N$, which is not to be confused with the inner product $(z, w) = \sum_{i=1}^N z_i \bar{w}_i$ if $w \notin \mathbb{R}^N$.

The vector space (over the field $\mathbb{R}$) of all real-valued (square) $M \times M$ matrices $\mathbf{A} = (a_{ij})_{i,j=1}^M$ is denoted by $\mathbb{R}^{M \times M}$ and its vector subspace of all symmetric matrices (i.e., $a_{ij} = a_{ji}$) by $\mathbb{R}^{M \times M}_{\mathrm{sym}}$. Similarly, the vector space (over the field $\mathbb{C}$) of all complex-valued $M \times M$ matrices $\mathbf{A} = (a_{ij})_{i,j=1}^M$ is denoted by $\mathbb{C}^{M \times M}$ and its subset of all Hermitian matrices (i.e., $a_{ij} = \bar{a}_{ji}$) by $\mathbb{C}^{M \times M}_{\mathrm{sym}}$, which is a vector subspace of $\mathbb{C}^{M \times M}$ over the field $\mathbb{R}$ only.

Given $r \in (0, \infty)$, we denote by $Q^{(r)} = (-r, r)^N = \{y \in \mathbb{R}^N \colon |y|_\infty < r\}$ the N-dimensional open cube in $\mathbb{R}^N$ with side lengths $2r$, and by $\overline{Q}^{(r)} = [-r, r]^N$ its closure.

In order to formulate our main hypotheses, given $r, T \in (0, \infty)$ and $T' \in [0, T)$, we introduce the following complex domains for the complexifications of the spatial and temporal variables, $x \in \mathbb{R}^N$ and $t \in (0, T)$, respectively:

$$\mathfrak{X}^{(r)} \stackrel{\text{def}}{=} \{z = x + \mathrm{i}y \in \mathbb{C}^N \colon |y|_\infty < r\} = \mathbb{R}^N + \mathrm{i}Q^{(r)}, \tag{4}$$

$$\mathfrak{T}^{(r)}_{T',T} \stackrel{\text{def}}{=} \{t = \sigma + \mathrm{i}\tau \in \mathbb{C} \colon T' < \sigma < T \text{ and } |\tau| < r\}. \tag{5}$$

The former, $\mathfrak{X}^{(r)}$, is a *tube* with *base* $Q^{(r)}$ (often called a *strip*, as in [10]) and the latter, $\mathfrak{T}^{(r)}_{T',T}$, is a rectangle. Notice that $\mathfrak{T}^{(r)}_{T',T}$ is an open neighborhood of the interval (T', T) in the complex plane $\mathbb{C}$.

Our techniques will use holomorphic semigroups in an open *sector*

$$\Delta_\vartheta \stackrel{\text{def}}{=} \{t = \varrho e^{i\theta} \in \mathbb{C}\colon\ \varrho > 0 \text{ and } \theta \in (-\vartheta, \vartheta)\} \tag{6}$$

with a given angle $\vartheta \in (0, \pi/2)$, but often locally in time in an open *triangle*

$$\begin{aligned}\Delta_\vartheta^{(T)} &\stackrel{\text{def}}{=} \Delta_\vartheta \cap \{t \in \mathbb{C}\colon\ 0 < \Re\mathfrak{e}\, t < T\} \\ &= \{t = \varrho e^{i\theta} \in \mathbb{C}\colon\ 0 < \varrho < T/\cos\theta \text{ and } |\theta| < \vartheta\}\end{aligned} \tag{7}$$

where $0 < T < \infty$. Their respective closures in $\mathbb{C}$ are denoted by $\overline{\Delta}_\vartheta$ and $\overline{\Delta}_\vartheta^{(T)}$; both contain the origin $0 \in \mathbb{C}$. Finally, for $0 < T' \leqslant T < \infty$ we abbreviate

$$\Delta_\vartheta^{T',T} \stackrel{\text{def}}{=} \Delta_\vartheta^{(T)} \cap \{t \in \mathbb{C}\colon\ |\Im\mathfrak{m}\, t| < T' \cdot \tan\vartheta\} = \bigcup_{0 \leqslant \xi \leqslant T-T'} \bigl(\xi + \Delta_\vartheta^{(T')}\bigr) \tag{8}$$

and denote by $\overline{\Delta}_\vartheta^{T',T}$ its closure in $\mathbb{C}$.

Throughout this article we work with complex-valued functions; hence, all Banach and Hilbert spaces of functions we consider are complex (over the field $\mathbb{C}$). We work with the standard inner product in $L^2(\mathbb{R}^N)$ defined by $(u, v)_{L^2} \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} u\bar{v}\, dx$ for $u, v \in L^2(\mathbb{R}^N)$. The induced norm is abbreviated by $\|u\|_{L^2} \equiv \|u\|_{L^2(\mathbb{R}^N)}$.

Given a domain Ω in $\mathbb{R}^p$ (or $\mathbb{C}^p = \mathbb{R}^p \oplus i\mathbb{R}^p$, $p \in \mathbb{N}$), we denote by $C^k(\Omega)$ ($k \in \mathbb{Z}_+$) the vector space of all k-times continuously differentiable functions $f : \Omega \to \mathbb{C}$ and by $C^k(\overline{\Omega})$ the vector space of all $f : \overline{\Omega} \to \mathbb{C}$ such that $f|_\Omega \in C^k(\Omega)$ and each partial derivative of f of order $\leqslant k$ can be extended to a continuous function on $\overline{\Omega}$. Of course, $f|_\Omega$ stands for the restriction of f to Ω and all partial derivatives are taken in the real variable sense. If Ω is bounded then $C^k(\overline{\Omega})$ is a Banach space endowed with a maximum-type norm. If Ω is not bounded in general then we denote by $C^0_{\text{unif}}(\overline{\Omega})$ the vector space of all uniformly continuous functions $f : \overline{\Omega} \to \mathbb{C}$. Finally, if $\Omega \subset \mathbb{C}^p$ is a complex domain, we denote by $A(\Omega)$ the Fréchet space of all holomorphic functions $f : \Omega \to \mathbb{C}$ endowed with the (complete metrizable) topology of uniform convergence on compact subsets of Ω.

Remark 2.1. We will often use the following classical fact; see e.g. F. JOHN [44, Theorem, p. 70] or S.G. KRANTZ [53, Definition II, p. 3]: Let $\Omega \subset \mathbb{C}^m$ be a complex domain ($m \geqslant 1$). A continuously differentiable function $h : \Omega \to \mathbb{C}$ is holomorphic *if and only if* it verifies the Cauchy–Riemann equations in Ω, i.e., $\partial h/\partial \bar{z}_i = 0$ in Ω; $i = 1, 2, \ldots, m$. □

Of course, we abbreviate the partial differential operators

$$\frac{\partial}{\partial z_i} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_i} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right). \tag{9}$$

3. Statement of the main result

Let us abbreviate the derivatives

$$D_x \stackrel{\text{def}}{=} \frac{1}{i}\frac{\partial}{\partial x}, \qquad D_x^\alpha \stackrel{\text{def}}{=} i^{-|\alpha|}\frac{\partial^{|\alpha|}}{\partial x^\alpha} \quad \text{for } \alpha \in (\mathbb{Z}_+)^N, \quad \text{and} \quad \partial_t^\ell \stackrel{\text{def}}{=} \frac{\partial^\ell}{\partial t^\ell} \quad \text{for } \ell \in \mathbb{Z}_+.$$

We assume that the operator

$$\mathbf{P}(x,t,D_x) = \sum_{|\alpha|,|\beta|\leqslant m} D_x^\alpha\big(\mathbf{P}^{\alpha\beta}(x,t)D_x^\beta\big) \equiv \sum_{|\alpha|,|\beta|\leqslant m} \mathrm{i}^{-|\alpha|-|\beta|}\frac{\partial^{|\alpha|}}{\partial x^\alpha}\left(\mathbf{P}^{\alpha\beta}(x,t)\frac{\partial^{|\beta|}}{\partial x^\beta}\right), \tag{10}$$

for $(x,t)\in\mathbb{R}^N\times(0,T)$, is a linear partial differential operator of order $2m$ in divergence form with the coefficients $\mathrm{i}^{-|\alpha|-|\beta|}\mathbf{P}^{\alpha\beta}(x,t)$ indexed by $\alpha,\beta\in(\mathbb{Z}_+)^N$ with $|\alpha|\leqslant m$ and $|\beta|\leqslant m$, where each $\mathbf{P}^{\alpha\beta}(x,t)=(P_{jk}^{\alpha\beta})_{j,k=1}^M$ is an $M\times M$ matrix with real (or complex) entries $P_{jk}^{\alpha\beta}=P_{jk}^{\alpha\beta}(x,t)$. The reader is referred to A. Friedman [25, Part 1, Sect. 12, pp. 32–37] or F. John [44, Chapt. 6, Sect. 2, pp. 190–195] for general facts about such operators.

We assume that the operator $\mathbf{P}$ and the function $\mathbf{f}$ satisfy the following hypotheses in the product domain $\Omega=\mathfrak{X}^{(r_0)}\times\Delta_{\vartheta_0}^{T_0,T}\subset\mathbb{C}^N\times\mathbb{C}$, with some $r_0\in(0,\infty)$, $0<T_0\leqslant T<\infty$, and $\vartheta_0\in(0,\pi/2)$; we denote by $\overline{\Omega}$ the closure of Ω:

Hypotheses.

(H1) For each pair $\alpha,\beta\in(\mathbb{Z}_+)^N$ with $|\alpha|\leqslant m$ and $|\beta|\leqslant m$, the entries $P_{jk}^{\alpha\beta}:\overline{\Omega}\to\mathbb{C}$ ($j,k=1,2,\dots,M$) of the coefficient $\mathbf{P}^{\alpha\beta}=(P_{jk}^{\alpha\beta})_{j,k=1}^M$ belong to $C^1(\overline{\Omega})\cap L^\infty(\Omega)\cap A(\Omega)$. Moreover, we assume that also all partial derivatives $\frac{\partial^{|\alpha'|}}{\partial x^{\alpha'}}P_{jk}^{\alpha\beta}(x,t)$ of order $|\alpha'|\leqslant|\alpha|$ ($\alpha'\in(\mathbb{Z}_+)^N$) are in $C^1(\overline{\Omega})$. The entries $P_{jk}^{\alpha\beta}$ of the leading coefficients ($|\alpha|=|\beta|=m$) are assumed to belong also to $C^0_{\mathrm{unif}}(\overline{\Omega})$ besides being in $C^1(\overline{\Omega})\cap L^\infty(\Omega)\cap A(\Omega)$.

(H2) The operator $\mathbf{P}$ is *strongly elliptic* in $\overline{\Omega}$, i.e., there exists a constant $c\in(0,\infty)$ such that the inequality

$$\Re\mathfrak{e}\left(\sum_{j,k=1}^M\sum_{|\alpha|=|\beta|=m}P_{jk}^{\alpha\beta}(z,t)\xi^{\alpha+\beta}\eta_k\bar{\eta}_j\right)\geqslant c|\xi|^{2m}|\boldsymbol{\eta}|^2 \tag{11}$$

holds for all $(z,t)\in\overline{\Omega}$ and for all $\xi=(\xi_1,\dots,\xi_N)\in\mathbb{R}^N$ and $\boldsymbol{\eta}=(\eta_1,\dots,\eta_M)\in\mathbb{C}^M$, where $\xi^{\alpha+\beta}=\xi_1^{\alpha_1+\beta_1}\dots\xi_N^{\alpha_N+\beta_N}$ and $\alpha=(\alpha_1,\dots,\alpha_N)\in(\mathbb{Z}_+)^N$, $\beta=(\beta_1,\dots,\beta_N)\in(\mathbb{Z}_+)^N$.

(H3) The components $f_j:\overline{\Omega}\to\mathbb{C}$ ($j=1,2,\dots,M$) of the function $\mathbf{f}=(f_1,\dots,f_M)$ belong to $C^1(\overline{\Omega})\cap A(\Omega)$ and $\mathbf{f}:\overline{\Omega}\to\mathbb{C}^M$ satisfies

$$\int_{\mathbb{R}^N}\big|\mathbf{f}(x+\mathrm{i}y,t)\big|^2\,\mathrm{d}x\leqslant K^2\quad\text{for all } y\in\overline{Q}^{(r_0)}\text{ and } t\in\overline{\Delta}_{\vartheta_0}^{T_0,T}, \tag{12}$$

where $K\in(0,\infty)$ is a constant.

A simple, but more restrictive alternative to formulate hypotheses (H1)–(H3) is to replace $\Omega = \mathfrak{X}^{(r_0)} \times \Delta_{\vartheta_0}^{T_0,T}$ by a larger, but simpler product domain $\Omega_0 = \mathfrak{X}^{(r_0)} \times \mathfrak{T}_{0,T}^{(\tau_0)}$ with $\tau_0 = T_0 \cdot \tan\vartheta_0$, thanks to $\Delta_{\vartheta_0}^{T_0,T} \subset \mathfrak{T}_{0,T}^{(\tau_0)}$.

The strong ellipticity inequality (11) can be improved as follows:

Remark 3.1. We combine inequality (11) (in hypothesis (H2)) with the fact that all coefficient entries $P_{jk}^{\alpha\beta}$ are bounded in $\overline{\Omega}$ (for $\alpha, \beta \in (\mathbb{Z}_+)^N$ and $j, k = 1, 2, \ldots, M$, in hypothesis (H1)) in order to conclude that, in a smaller domain $\Omega' = \mathfrak{X}^{(r_0)} \times \Delta_{\vartheta_0'}^{T_0,T} \subset \Omega$, with some number $\vartheta_0' \in (0, \vartheta_0]$, inequality (11) holds in the following qualitatively stronger form, cf. S. Agmon [1, Theorem 7.12, inequality (7.21), p. 87]:

$$\mathfrak{Re}\left(e^{i\theta} \cdot \sum_{j,k=1}^{M} \sum_{|\alpha|=|\beta|=m} P_{jk}^{\alpha\beta}(z,t)\xi^{\alpha+\beta}\eta_k\bar{\eta}_j\right) \geqslant c'|\xi|^{2m}|\boldsymbol{\eta}|^2 \tag{11'}$$

for all $\theta \in [-\vartheta_0', \vartheta_0']$, for all $(z,t) \in \overline{\Omega}'$, and for all $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_M) \in \mathbb{C}^M$, where $c' \in (0, c]$ is a constant. This claim follows easily from Euler's formula $e^{i\theta} = \cos\theta + i \cdot \sin\theta$ and the fact that $\sin\theta/\cos\theta = \tan\theta \to 0$ as $\theta \to 0$. Consequently, without loss of generality, we may remove the prime (′) from both ϑ_0' and c' in (11′) and assume that

$$\mathfrak{Re}\left(e^{i\theta} \cdot \sum_{j,k=1}^{M} \sum_{|\alpha|=|\beta|=m} P_{jk}^{\alpha\beta}(z,t)\xi^{\alpha+\beta}\eta_k\bar{\eta}_j\right) \geqslant c|\xi|^{2m}|\boldsymbol{\eta}|^2 \tag{13}$$

for all $\theta \in [-\vartheta_0, \vartheta_0]$ and for all $(z,t) \in \overline{\Omega}'$, $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$, and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_M) \in \mathbb{C}^M$, where $c > 0$ is a constant. For the sake of convenience, in our proofs we prefer to use inequality (13) in place of (11). □

The *Gårding inequality* (in the whole space $\mathbb{R}^N$) below is an important consequence of inequality (13); see e.g. S. Agmon [1, Theorem 7.6, p. 78]:

Corollary 3.2 *(Gårding's inequality). Under both hypotheses* (H1) *and* (H2)*, there exist some constants c_1 and c_2, $c_1 > 0$ and $0 \leqslant c_2 < \infty$, such that*

$$\begin{aligned}&\mathfrak{Re}\left[e^{i\theta} \cdot \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^N} \overline{D_x^\alpha \mathbf{w}} \cdot \mathbf{P}^{\alpha\beta}(x+iy,t) D_x^\beta \mathbf{w}\, dx\right] \\ &\quad \geqslant c_1 \sum_{|\alpha|=m} \left\| D_x^\alpha \mathbf{w} \right\|_{L^2(\mathbb{R}^N)}^2 - c_2 \|\mathbf{w}\|_{L^2(\mathbb{R}^N)}^2 \end{aligned} \tag{14}$$

holds for all $\mathbf{w} \in W^{m,2}(\mathbb{R}^N)$ and for all $\theta \in [-\vartheta_0, \vartheta_0]$, $y \in \overline{Q}^{(r_0)}$, and $t \in \overline{\Delta}_{\vartheta_0}^{T_0,T}$.

Proof. The reader is referred to S. Agmon [1, Theorem 7.6, pp. 78–86] for a proof. We remark that the proof of Gårding's inequality (see [1, Lemma 7.9, p. 81]) requires the *uniform equicontinuity* of the leading coefficients $\mathbf{P}^{\alpha\beta}(x+iy,t)$ as functions of $x \in \mathbb{R}^N$ parametrized by $y \in \overline{Q}^{(r_0)}$ and $t \in \overline{\Delta}_{\vartheta_0}^{T_0,T}$, where $\mathbf{P}^{\alpha\beta}(x+\mathrm{i}y,t) = (P_{jk}^{\alpha\beta})_{j,k=1}^{M}$ for $|\alpha| = |\beta| = m$. This is guaranteed by our hypothesis that all $P_{jk}^{\alpha\beta}$ ($|\alpha| = |\beta| = m$) belong to $C^0_{\mathrm{unif}}(\overline{\Omega})$ as functions of $(z,t) = (x+\mathrm{i}y,t) \in \overline{\Omega}$. $\square$

In order to give a reasonable lower estimate on the domain of holomorphy (i.e., the domain of complex analyticity) of a weak solution u to the Cauchy problem (1), we introduce a few more subsets of $\mathbb{C}^N \times \mathbb{C}$ (cf. P. Takáč et al. [70, p. 428]):

Let $\varepsilon \in \mathbb{R}$ be arbitrary, but fixed, $0 < \varepsilon \leqslant 1 - (1/2m)$. We introduce the function $\chi(s) \stackrel{\mathrm{def}}{=} s^{(1/2m)+\varepsilon}$ for $s \in \mathbb{R}_+ \stackrel{\mathrm{def}}{=} [0,\infty)$; it is monotonically increasing and concave, $\chi(0) = 0$ and $\chi(1) = 1$, and

$$\int_0^1 \chi'(s)^{2m/(2m-1)}\,\mathrm{d}s = \left(\frac{1}{2m} + \varepsilon\right)^{2m/(2m-1)} \frac{2m-1}{2m\varepsilon} < \infty. \tag{15}$$

We abbreviate

$$J(\sigma) \stackrel{\mathrm{def}}{=} \begin{cases} \int_0^\sigma \chi'(s)^{2m/(2m-1)}\,\mathrm{d}s = \mathrm{const}\cdot\sigma^{2m\varepsilon/(2m-1)} & \text{for } 0 \leqslant \sigma \leqslant 1; \\ J(1) + \sigma - 1 & \text{for } 1 < \sigma < \infty, \end{cases} \tag{16}$$

note that $0 < \varepsilon \leqslant 1 - (1/2m) \Longleftrightarrow 0 < 2m\varepsilon/(2m-1) \leqslant 1$.

The two constants $\kappa_0, \nu_0 \in (0,\infty)$ used below will be specified later (in Theorem 3.3), and $0 \leqslant s \leqslant \infty$. We set

$$\Pi^{(s)}(\kappa_0) = \bigl\{z = x + \mathrm{i}y \in \mathbb{C}^N\colon\ |y|_\infty < \kappa_0\chi(s)\bigr\}, \tag{17}$$

$$\Sigma^{(s)}(\nu_0) = \bigl\{t = \sigma + \mathrm{i}\tau \in \mathbb{C}\colon\ \nu_0|\tau| < \sigma = s\bigr\}, \tag{18}$$

and introduce the complex parabolic domain

$$\Lambda^{(s)}(\kappa_0,\nu_0) = \bigcup\bigl\{\Pi^{(r)}(\kappa_0) \times \Sigma^{(r)}(\nu_0)\colon\ r \in (0,s)\bigr\} \subset \mathbb{C}^N \times \mathbb{C} \tag{19}$$

together with its time translation by r units, for $0 \leqslant r < \infty$,

$$\Lambda_r^{(s)}(\kappa_0,\nu_0) = \bigl\{(z,t) \in \mathbb{C}^N \times \mathbb{C}\colon\ (z,t-r) \in \Lambda^{(s)}(\kappa_0,\nu_0)\bigr\}. \tag{20}$$

We now define our most important set in $\mathbb{C}^N \times \mathbb{C}$, for $0 \leqslant s \leqslant T \leqslant \infty$, the complex parabolic domain

$$\Gamma_T^{(s)}(\kappa_0,\nu_0) = \begin{cases} \bigcup\{\Lambda_r^{(s)}(\kappa_0,\nu_0)\colon\ r \in [0,T-s]\} & \text{if } s < T; \\ \Lambda^{(s)}(\kappa_0,\nu_0) & \text{if } s = T. \end{cases} \tag{21}$$

Given any $r \in [0,T)$, we observe that the (real) time r section of $\Gamma_T^{(s)}(\kappa_0,\nu_0)$ is given by

$$\Theta^{(r,s)}(\kappa_0,\nu_0) \stackrel{\text{def}}{=} \big\{(z,t)\in \Gamma_T^{(s)}(\kappa_0,\nu_0)\colon\ \Re\mathfrak{e}\, t = r\big\}$$
$$= \Pi^{(r')}(\kappa_0)\times \Sigma^{(r')}(\nu_0), \quad \text{where } r' = \min\{r,s\}. \tag{22}$$

The x section of $\Gamma_T^{(s)}(\kappa_0,\nu_0)$ is independent from $x\in\mathbb{R}^N$; hence, we may identify $\Gamma_T^{(s)}(\kappa_0,\nu_0)\simeq \mathbb{R}^N\times\hat{\Gamma}_T^{(s)}(\kappa_0,\nu_0)$ where

$$\hat{\Gamma}_T^{(s)}(\kappa_0,\nu_0) \stackrel{\text{def}}{=} \big\{(y,t)=(y,\sigma+\mathrm{i}\tau)\in\mathbb{R}^N\times\mathbb{C}\colon\ 0<\sigma<T \text{ together with } |y|_\infty<\kappa_0\chi(r) \text{ and } \nu_0|\tau|<r \text{ where } r=\min\{\sigma,s\}\big\}. \tag{23}$$

Our main result is as follows; recall that $\mathbf{L}^2(\mathbb{R}^N)=[L^2(\mathbb{R}^N)]^M$.

Theorem 3.3. *Let $M,N\geqslant 1$, $0<T<\infty$, and assume that all hypotheses* (H1)–(H3) *are satisfied with some constants $0<r_0<\infty$, $0<T_0\leqslant T$, and $0<\vartheta_0<\pi/2$. Then, given any $\mathbf{u}_0\in\mathbf{L}^2(\mathbb{R}^N)$, the Cauchy problem* (1) *possesses a unique weak solution $\mathbf{u}\in C([0,T]\to\mathbf{L}^2(\mathbb{R}^N))$. This solution can be (uniquely) extended to a holomorphic function in the domain $\Gamma_T^{(T_0)}(\kappa_0,\nu_0)$, where $\kappa_0,\nu_0\in(0,\infty)$ are arbitrary numbers satisfying $r_0'\stackrel{\text{def}}{=}\kappa_0\chi(T_0)\leqslant r_0$ and $\nu_0\geqslant\cot\vartheta_0$. Moreover, there exists a constant $C_0\in\mathbb{R}_+$ depending on r_0, T_0, ϑ_0, κ_0, and ν_0, but* independent *from $\mathbf{u}_0$, T $(T\geqslant T_0)$, and K, such that the* (*unique holomorphic*) *extension of $\mathbf{u}$, denoted again by $\mathbf{u}$, satisfies*

$$\int_{\mathbb{R}^N}\big|\mathbf{u}(x+\mathrm{i}y,t)\big|^2\,\mathrm{d}x\leqslant \mathrm{e}^{C_0J(\Re\mathfrak{e}\,t/T_0)}\big(\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^N)}+K^2\cdot\Re\mathfrak{e}\,t\big) \tag{24}$$

for all $(y,t)\in\hat{\Gamma}_T^{(T_0)}(\kappa_0,\nu_0)$.

Recall that the constant K has been introduced in inequality (12) and the function J in Eq. (16). Also notice that the numbers κ_0 and ν_0 have been chosen in such a way that $\Gamma_T^{(T_0)}(\kappa_0,\nu_0)\subset\Omega=\mathfrak{X}^{(r_0)}\times\Delta_{\vartheta_0}^{T_0,T}$.

In fact, the desired estimate (24) follows easily from a more precise estimate below, (26), where we abbreviate

$$\varsigma_1(s)=\min\{s,1\}\quad\text{and}\quad\chi_1(s)=\chi\big(\varsigma_1(s)\big)\quad\text{for } s\in\mathbb{R}_+. \tag{25}$$

Proposition 3.4. *In the situation of Theorem* 3.3 *above, there exists a constant $c_0>0$* independent *from κ_0, ν_0, $\mathbf{u}_0$, T $(T\geqslant T_0)$, and K, such that we have also* (*with the same constant C_0 as in Theorem* 3.3)

$$\int_{\mathbb{R}^N}\big|\mathbf{u}\big(x+\mathrm{i}\chi_1(\sigma/T_0)y,\sigma+\mathrm{i}\varsigma_1(\sigma/T_0)\tau\big)\big|^2\,\mathrm{d}x$$
$$+\,c_0\sum_{|\alpha|\leqslant m}\int_0^\sigma\int_{\mathbb{R}^N}\big|D_x^\alpha\mathbf{u}\big(x+\mathrm{i}\chi_1(s/T_0)y,s+\mathrm{i}\varsigma_1(s/T_0)\tau\big)\big|^2\,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant \mathrm{e}^{C_0 J(\sigma/T_0)}\big(\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^N)} + K^2\sigma\big) \quad \textit{for all } \sigma \in (0,T) \tag{26}$$

and for all $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$ *satisfying* $|y|_\infty < r_0'$ *and* $\nu_0|\tau| < T_0$.

Remark 3.5. Notice that $\min\{\chi(s), \chi(T_0)\} = \chi_1(s/T_0)\chi(T_0)$ for all $s \geqslant 0$ and $0 < T_0 < \infty$. Furthermore, if $0 \leqslant s \leqslant T_0$ then $s + \mathrm{i}\varsigma_1(s/T_0)\tau = (1 + \mathrm{i}(\tau/T_0))s$. In contrast, for $s \geqslant T_0$ we have $\chi_1(s/T_0) = \varsigma_1(s/T_0) = 1$, so that $x + \mathrm{i}\chi_1(s/T_0)y = x + \mathrm{i}y$ and $s + \mathrm{i}\varsigma_1(s/T_0)\tau = s + \mathrm{i}\tau$ in (26). It is now easy to derive (24) from (26) with $t = \sigma + \mathrm{i}\varsigma_1(\sigma/T_0)\tau$, where $0 < \sigma < T$ and $\nu_0|\tau| < T_0$, i.e., $t = (1 + \mathrm{i}(\tau/T_0))\sigma$ if $0 \leqslant \sigma \leqslant T_0$, whereas $t = \sigma + \mathrm{i}\tau$ if $T_0 \leqslant \sigma \leqslant T$.

Remark 3.6. As we have already mentioned in Remark 1.2, if the leading coefficients $\mathbf{P}^{\alpha\beta}$ ($|\alpha| = |\beta| = m$) of the operator $\mathbf{P}$ are matrices with constant entries (independent from both x and t) then we may take $\chi(s) = s^{(1/2m)+\varepsilon}$ ($s \geqslant 0$) with $\varepsilon = 0$ in (17) above, by P. TAKÁČ et al. [70, Theorem 2.1, p. 429]. Thus, property (15) is not needed in this case; it does not hold because of $\chi'(s)^{2m/(2m-1)} = (1/2m)^{2m/(2m-1)}s^{-1}$ for $s > 0$.

4. An a priori estimate

In addition to hypotheses (H1)–(H3), with some constants $0 < r_0 < \infty$, $0 < T_0 \leqslant T < \infty$, and $0 < \vartheta_0 < \pi/2$, we assume throughout this section that $\kappa_0, \nu_0 \in (0,\infty)$ are given numbers satisfying $\kappa_0\chi(T_0) \leqslant r_0$, and $\nu_0 \geqslant \cot\vartheta_0$. Recall that these conditions on κ_0 and ν_0 guarantee $\Gamma_T^{(T_0)}(\kappa_0,\nu_0) \subset \Omega = \mathfrak{X}^{(r_0)} \times \Delta_{\vartheta_0}^{T_0,T}$.

Assuming (*a priori*) that the "existence" and "holomorphic regularity" parts of Theorem 3.3 are valid, we now establish the *a priori* estimates (24) and (26). In fact, the next lemma is the most important auxiliary result of our present work.

Lemma 4.1. *Assume that* $\mathbf{u} : \mathbb{R}^N \times (0,T) \to \mathbb{C}^M$ *is a weak solution to the Cauchy problem* (1) *with given initial data* $\mathbf{u}_0 \in \mathbf{L}^2(\mathbb{R}^N)$, *such that* $\mathbf{u} \in C([0,T] \to \mathbf{L}^2(\mathbb{R}^N))$ *and* $\mathbf{u}$ *possesses a holomorphic extension to the complex domain* $\Gamma_T^{(T_0)}(\kappa_0,\nu_0)$, *denoted by* $\mathbf{u}$ *again, that has the following two properties*:

(i) *This extension satisfies a* local *version of the estimate in* (24), *that is, every pair* $(y_0,t_0) \in \hat{\Gamma}_T^{(T_0)}(\kappa_0,\nu_0)$ *has an open neighborhood G in* $\hat{\Gamma}_T^{(T_0)}(\kappa_0,\nu_0)$ *such that*

$$\sup_{(y,t)\in G} \int_{\mathbb{R}^N} \big|\mathbf{u}(x+\mathrm{i}y,t)\big|^2 \,\mathrm{d}x < \infty. \tag{27}$$

(ii) *For each fixed* $\tau_1 \in \mathbb{R}$ *with* $\nu_0|\tau_1| < 1$ *we have, as* $\sigma \to 0+$,

$$\sup_{|y|_\infty < \kappa_0\chi(\sigma)} \int_{\mathbb{R}^N} \big|\mathbf{u}\big(x+\mathrm{i}y,(1+\mathrm{i}\tau_1)\sigma\big) - \mathbf{u}_0(x)\big|^2 \,\mathrm{d}x \longrightarrow 0. \tag{28}$$

Then the extension $\mathbf{u}$ *satisfies the partial differential equation* (1) *in* $\Gamma_T^{(T_0)}(\kappa_0,\nu_0)$ *in the classical sense at every point* $(x,y,t) \simeq (x+\mathrm{i}y,t) \in \Gamma_T^{(T_0)}(\kappa_0,\nu_0)$, *with the partial derivatives being* *taken with respect to x and t as indicated in* (1). *Moreover, there exists a constant $C_0 \in \mathbb{R}_+$,* independent *from $\mathbf{u}_0$, $\mathbf{u}$, T $(T \geqslant T_0)$, and K, such that (the holomorphic extension of) $\mathbf{u}$ satisfies the estimate in* (24) *for all $(y,t) \in \hat{\Gamma}_T^{(T_0)}(\kappa_0, \nu_0)$. Similarly, also the estimate in* (26) *is valid.*

Consequently, both, the weak solution to the Cauchy problem (1) *and its holomorphic extension to $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$ must be unique* (*which justifies our notation for the extension*).

We will see that the constant C_0 *does depend* on T_0 which is the lower bound for T $(0 < T_0 \leqslant T < \infty)$.

Remark 4.2. (a) In Lemma 4.1 above, it suffices to assume that hypotheses (H1)–(H3) are satisfied in the smaller domain $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$ rather than in $\Omega = \mathfrak{X}^{(r_0)} \times \Delta_{\vartheta_0}^{T_0,T}$. This claim will be obvious from our proof of the lemma given below.

(b) We *may* also *replace* the hypothesis that the coefficient entries $P_{jk}^{\alpha\beta}$ (in hypothesis (H1), for $|\alpha|, |\beta| \leqslant m$ and $j,k = 1,2,\dots,M$) and the components f_j (in hypothesis (H3), for $j = 1,2,\dots,M$) are *holomorphic* in the complex domain $\Omega = \mathfrak{X}^{(r_0)} \times \Delta_{\vartheta_0}^{T_0,T}$, by the hypothesis that the holomorphic extension of $\mathbf{u}$ *satisfies* the partial differential equation (1) in $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$ in the classical sense. Also this claim follows easily from (the first paragraph of) our proof. □

Proof of Lemma 4.1. As we have already mentioned in the Introduction (Section 1), the holomorphic extension of a real analytic function from a real domain (open and connected in $\mathbb{R}^p$, $p \in \mathbb{N}$) to a complex domain (in $\mathbb{C}^p$) is always unique (F. John [44, Chapt. 3, Sect. 3(c), pp. 70–72]). Consequently, since $\mathbf{u}$ is holomorphic in $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$ and satisfies Eq. (1) in $\mathbb{R}^N \times (0,T)$ in the classical sense, it must satisfy the same equation throughout $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$. Here, we have used the hypothesis that all entries $P_{jk}^{\alpha\beta}$ of all coefficients $\mathbf{P}^{\alpha\beta}$ and all components f_j of $\mathbf{f}$ are holomorphic in the complex domain $\Omega = \mathfrak{X}^{(r_0)} \times \Delta_{\vartheta_0}^{T_0,T} \supset \Gamma_T^{(T_0)}(\kappa_0, \nu_0)$.

Function $\mathbf{u}$ being holomorphic in $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$, it can be expressed by the Cauchy integral formula for polydiscs (see e.g. S.G. Krantz [53, Theorem 1.2.2 (p. 24)], or F. John [44, Chapt. 3, Sect. 3(c), Eq. (3.22c), p. 71]). From this formula every partial derivative $\frac{\partial^{|\alpha|+\ell}\mathbf{u}}{\partial x^\alpha \partial t^\ell} = \frac{\partial^{|\alpha|+\ell}\mathbf{u}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N} \partial t^\ell}$ of $\mathbf{u}$ at any point (z,t) near $(z_0,t_0) \in \Gamma_T^{(T_0)}(\kappa_0, \nu_0)$ (of any order $|\alpha| = \alpha_1 + \dots + \alpha_N$ in $x_1, \dots, x_N$ and $\ell \in \mathbb{Z}_+$ in t) is obtained in the form of an integral of $\mathbf{u}$ over the Cartesian product of $(N+1)$ circles of the same radius $\varrho > 0$ centered at the components of (z_0,t_0), see [44, Chapt. 3, Sect. 3(c), Eq. (3.22f), p. 71]. We take $\varrho > 0$ small enough, such that the closed polydisc (defined by these circles)

$$\overline{D}^{(\varrho)} = \bigl\{(z,t) \in \mathbb{C}^N \times \mathbb{C}\colon\ |z - z_0|_\infty \leqslant \varrho \text{ and } |t - t_0| \leqslant \varrho\bigr\}$$

is contained in $\mathbb{R}^N \times G$, i.e., $(z,t) = (x + \mathrm{i}y, t) \in \overline{D}^{(\varrho)} \Longrightarrow (y,t) \in G$, where G is the open neighborhood of (y_0,t_0) in $\hat{\Gamma}_T^{(T_0)}(\kappa_0, \nu_0)$ as specified in Lemma 4.1, property (i). As above, $(z_0,t_0) = (x_0 + \mathrm{i}y_0, t_0)$. We apply (27) to this representation of $\frac{\partial^{|\alpha|+\ell}\mathbf{u}}{\partial x^\alpha \partial t^\ell}(z,t)$ at every point $(z,t) \in \overline{D}^{(\varrho/2)}$ to conclude that also this partial derivative satisfies the Cauchy derivative estimate (cf. [44, Chapt. 3, Sect. 3(c), Eq. (3.23), p. 71])

$$\sup_{(y,t)\in G'} \int_{\mathbb{R}^N} \left| \frac{\partial^{|\alpha|+\ell}\mathbf{u}}{\partial x^\alpha \partial t^\ell}(x+\mathrm{i}y,t) \right|^2 \mathrm{d}x < \infty, \tag{29}$$

where G' is another open neighborhood of (y_0, t_0) in $\hat{\Gamma}_T^{(T_0)}(\kappa_0, \nu_0)$ with closure $\overline{G}' \subset G$.

Recall that $\chi(s) = s^{(1/2m)+\varepsilon}$ for $s \in \mathbb{R}_+$, where $\varepsilon \in \mathbb{R}$ is a constant, $0 < \varepsilon \leqslant 1 - (1/2m)$, together with $\varsigma_1(s) = \min\{s, 1\}$ and $\chi_1(s) = \chi(\varsigma_1(s))$ for $s \in \mathbb{R}_+$, by (25). In addition, let $\tau \in \mathbb{R}$ be arbitrary with $\nu_0|\tau| < T_0$. This choice of the function χ and the number τ guarantees

$$\begin{gathered}\bigl(\chi_1(\sigma/T_0)y, \sigma + \mathrm{i}\varsigma_1(\sigma/T_0)\tau\bigr) \in \hat{\Gamma}_T^{(T_0)}(\kappa_0, \nu_0) \\ \text{for all } (y,\sigma) \in \mathbb{R}^N \times (0,T) \text{ with } |y|_\infty < r_0', \end{gathered}\tag{30}$$

where $r_0' = \kappa_0 \chi(T_0) \leqslant r_0$ by our hypotheses, and

$$\hat{\Gamma}_T^{(T_0)}(\kappa_0,\nu_0) = \bigl\{(y,t) = (y, \sigma+\mathrm{i}\tau) \in \mathbb{R}^N \times \mathbb{C}\colon\ 0 < \sigma < T,\ |y|_\infty < r_0' \\ \chi_1(\sigma/T_0),\ \text{and } \nu_0|\tau| < T_0\varsigma_1(\sigma/T_0)\bigr\},$$

by Eq. (23). Equivalently to (30), substituting

$$\begin{gathered}(z,t) = \bigl(x + \mathrm{i}\chi_1(\sigma/T_0)y, \sigma + \mathrm{i}\varsigma_1(\sigma/T_0)\tau\bigr) \\ \text{for } (y,\sigma) \in \mathbb{R}^N \times (0,T) \text{ with } |y|_\infty < r_0',\end{gathered}$$

we have $(z,t) \in \Gamma_T^{(T_0)}(\kappa_0, \nu_0)$.

Consequently, fixing an arbitrary $y \in \mathbb{R}^N$ with $|y|_\infty < r_0'$, we may define the function

$$\mathbf{v}(x,s) \equiv \mathbf{v}(x,s;y) \stackrel{\text{def}}{=} \mathbf{u}(z,t) \quad \text{of } (x,s) \in \mathbb{R}^N \times (0,T), \tag{31}$$

where we have introduced the substitution

$$(z,t) \stackrel{\text{def}}{=} \bigl(x + \mathrm{i}\chi_1(s/T_0)y, s + \mathrm{i}\varsigma_1(s/T_0)\tau\bigr) \quad \text{for } (x,s) \in \mathbb{R}^N \times (0,T). \tag{32}$$

Notice that $\mathbf{v}(x,s) = \mathbf{u}(x+\mathrm{i}y, s+\mathrm{i}\tau)$ whenever $T_0 \leqslant s \leqslant T$. Since $\mathbf{u}$ is holomorphic in $\Gamma_T^{(T_0)}(\kappa_0,\nu_0)$, with a help from the Cauchy–Riemann equations we have

$$\partial_s \mathbf{v}(x,s) = \bigl(1 + \mathrm{i}\tau_1(s)\bigr)\partial_t \mathbf{u}(z,t) - T_0^{-1}\chi_1'(s/T_0) D_x \mathbf{u}(z,t) y \tag{33}$$

for every $s \in (0,T) \setminus \{0\}$, where

$$\begin{cases} \tau_1(s) = \tau/T_0 \quad \text{and} \quad \chi_1'(s) = \chi'(s) & \text{for } 0 \leqslant s \leqslant T_0; \\ \tau_1(s) = \chi_1'(s) = 0 & \text{for } T_0 < s \leqslant T, \end{cases} \tag{34}$$

and $D_x\mathbf{u}(z,t)y = \sum_{i=1}^N D_{x_i}\mathbf{u}(z,t)y_i \in \mathbb{C}^M$. Observe that the corresponding partial derivatives of the functions $\mathbf{u}$ and $\mathbf{v}$ with respect to the space variables coincide, for $(x,s) \in \mathbb{R}^N \times (0,T)$:

$$D_x^\alpha \mathbf{v}(x,s) = D_x^\alpha \mathbf{u}\bigl(x + \mathrm{i}\chi_1(s/T_0)y, s + \mathrm{i}\varsigma_1(s/T_0)\tau\bigr) \equiv D_x^\alpha \mathbf{u}(z,t).$$

Now we substitute Eq. (1) with the operator (10) into (33); the last equation combined with the initial condition $\mathbf{u}(\cdot,0)=\mathbf{u}_0$ renders the following Cauchy problem for $\mathbf{v}$:

$$\begin{cases} \partial_s \mathbf{v}(x,s) + \big(1+\mathrm{i}\tau_1(s)\big)\mathbf{P}(z,t,D_x)\mathbf{v}(x,s) & \\ \quad = -T_0^{-1}\chi_1'(s/T_0)D_x\mathbf{v}(x,s)y + \big(1+\mathrm{i}\tau_1(s)\big)\mathbf{f}(z,t) & \text{for } (x,t)\in\mathbb{R}^N\times(0,T); \\ \mathbf{v}(x,0)=\mathbf{u}_0(x) & \text{for } x\in\mathbb{R}^N. \end{cases} \tag{35}$$

By (29), all summands in Eq. (35) above are in $L^2(\mathbb{R}^N)$ with respect to the space variable $x\in\mathbb{R}^N$; their $L^2(\mathbb{R}^N)$-norms are uniformly bounded for $s\in[a,b]$, whenever $0<a<b<T$. We split the operator $\mathbf{P}$ into its principal part $\mathbf{Q}$ and the lower-order part $\mathbf{R}$, i.e.,

$$\begin{aligned} \mathbf{P}(z,t,D_x) &= \mathbf{Q}(x,s,D_x)+\mathbf{R}(x,s,D_x) \\ &\stackrel{\text{def}}{=} \sum_{|\alpha|=|\beta|=m} D_x^\alpha\big(\mathbf{P}^{\alpha\beta}(x,t)D_x^\beta\big) + \sum_{\substack{|\alpha|,|\beta|\leqslant m\\ |\alpha|+|\beta|\leqslant 2m-1}} D_x^\alpha\big(\mathbf{P}^{\alpha\beta}(x,t)D_x^\beta\big), \end{aligned} \tag{36}$$

and substitute

$$\mathbf{g}(x,s)\stackrel{\text{def}}{=}\mathbf{f}\big(x+\mathrm{i}\chi_1(s/T_0)y,\, s+\mathrm{i}\varsigma_1(s/T_0)\tau\big)\equiv\mathbf{f}(z,t). \tag{37}$$

With a help from Eq. (35) we calculate

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^N}\big|\mathbf{v}(x,s)\big|^2\,\mathrm{d}x &= \frac{1}{2}\int_{\mathbb{R}^N}(\partial_s\mathbf{v}\cdot\bar{\mathbf{v}}+\mathbf{v}\cdot\partial_s\bar{\mathbf{v}})\,\mathrm{d}x \\ &= -\,\Re\mathfrak{e}\bigg[\big(1+\mathrm{i}\tau_1(s)\big)\int_{\mathbb{R}^N}\bar{\mathbf{v}}\cdot\mathbf{P}(z,t,D_x)\mathbf{v}\,\mathrm{d}x\bigg] \\ &\quad - T_0^{-1}\chi_1'(s/T_0)\cdot\Re\mathfrak{e}\int_{\mathbb{R}^N}\overline{\mathbf{v}(x,s)}\cdot D_x\mathbf{v}(x,s)y\,\mathrm{d}x \\ &\quad + \Re\mathfrak{e}\bigg[\big(1+\mathrm{i}\tau_1(s)\big)\int_{\mathbb{R}^N}\mathbf{f}(z,t)\cdot\overline{\mathbf{v}(x,s)}\,\mathrm{d}x\bigg] \end{aligned}$$

which, upon substituting (36) and (37), becomes

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^N}\big|\mathbf{v}(x,s)\big|^2\,\mathrm{d}x \\ &\quad + \Re\mathfrak{e}\bigg[\big(1+\mathrm{i}\tau_1(s)\big)\sum_{|\alpha|=|\beta|=m}\int_{\mathbb{R}^N}\overline{D_x^\alpha\mathbf{v}}\cdot\mathbf{P}^{\alpha\beta}(z,t)D_x^\beta\mathbf{v}\,\mathrm{d}x\bigg] \\ &\quad + \Re\mathfrak{e}\bigg[\big(1+\mathrm{i}\tau_1(s)\big)\sum_{\substack{|\alpha|,|\beta|\leqslant m\\ |\alpha|+|\beta|\leqslant 2m-1}}\int_{\mathbb{R}^N}\overline{D_x^\alpha\mathbf{v}}\cdot\mathbf{P}^{\alpha\beta}(z,t)D_x^\beta\mathbf{v}\,\mathrm{d}x\bigg] \end{aligned}$$

$$
\begin{aligned}
&= -T_0^{-1}\chi_1'(s/T_0)\cdot \Re\mathfrak{e}\int_{\mathbb{R}^N} \overline{\mathbf{v}(x,s)}\cdot D_x\mathbf{v}(x,s)\,y\,\mathrm{d}x \\
&\quad + \Re\mathfrak{e}\left[\big(1+\mathrm{i}\tau_1(s)\big)\int_{\mathbb{R}^N} \mathbf{g}(x,s)\cdot\overline{\mathbf{v}(x,s)}\,\mathrm{d}x\right] \qquad (38)
\end{aligned}
$$

for $s \in (0,T)\setminus\{0\}$. We estimate the individual integrals as follows.

First, we estimate the second term on the left-hand side of (38) from below. To this end, let us set $\theta_1 = \arctan\tau_1$; hence $\nu_0|\tan\theta_1| = \nu_0|\tau_1| < 1 \leqslant \nu_0\cdot\tan\vartheta_0$ and thus we may take advantage of inequality (13) with $\theta = \theta_1$. Consequently, we can apply *Gårding's inequality* (14) from Corollary 3.2 in order to get

$$
\begin{aligned}
&\Re\mathfrak{e}\left[\big(1+\mathrm{i}\tau_1(s)\big)\sum_{|\alpha|=|\beta|=m}\int_{\mathbb{R}^N} \overline{D_x^\alpha\mathbf{w}}\cdot\mathbf{P}^{\alpha\beta}(z,t)D_x^\beta\mathbf{w}\,\mathrm{d}x\right] \\
&\quad \geqslant c_1\sum_{|\alpha|=m}\big\|D_x^\alpha\mathbf{w}\big\|^2_{L^2(\mathbb{R}^N)} - c_2\|\mathbf{w}\|^2_{L^2(\mathbb{R}^N)} \qquad (39)
\end{aligned}
$$

for all $\mathbf{w}\in W^{m,2}(\mathbb{R}^N)$ and for all $s\in(0,T)$. Here, c_1 and c_2 are some constants, $c_1 > 0$ and $0 \leqslant c_2 < \infty$, that can be chosen to be independent from $\mathbf{w}$, s, and $\tau\in\mathbb{R}$ satisfying $\nu_0|\tau| < T_0$. Thanks to the Gagliardo–Nirenberg inequalities (see, e.g., A. FRIEDMAN [25, Part 1, Sect. 9, Theorem 9.3 on p. 24]), an equivalent norm in the Sobolev space $\mathbf{W}^{m,2}(\mathbb{R}^N) = [W^{m,2}(\mathbb{R}^N)]^M$ is given by

$$
\|\mathbf{w}\|_{W^{m,2}(\mathbb{R}^N)} \stackrel{\text{def}}{=} \big([\mathbf{w}]^2_{m,2} + \|\mathbf{w}\|^2_{L^2(\mathbb{R}^N)}\big)^{1/2}
$$

with the seminorm

$$
[\mathbf{w}]_{m,2} \equiv [\mathbf{w}]_{W^{m,2}(\mathbb{R}^N)} \stackrel{\text{def}}{=} \left(\sum_{|\alpha|=m}\big\|D_x^\alpha\mathbf{w}\big\|^2_{L^2(\mathbb{R}^N)}\right)^{1/2}.
$$

Each summand (integral) in the third term on the left-hand side of Eq. (38) is estimated by the Cauchy, Gagliardo–Nirenberg, and Young inequalities, for all $\alpha,\beta\in(\mathbb{Z}_+)^N$ such that $|\alpha|\leqslant m$, $|\beta|\leqslant m$, and $|\alpha|+|\beta|\leqslant 2m-1$:

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} \overline{D_x^\alpha\mathbf{w}}\cdot\mathbf{P}^{\alpha\beta}(z,t)D_x^\beta\mathbf{w}\,\mathrm{d}x\right| \leqslant \big\|\mathbf{P}^{\alpha\beta}\big\|_{L^\infty(\Omega)}\big\|D_x^\alpha\mathbf{w}\big\|_{L^2}\big\|D_x^\beta\mathbf{w}\big\|_{L^2} \\
&\quad \leqslant c_{\alpha\beta}[\mathbf{w}]_{m,2}^{\frac{|\alpha|+|\beta|}{m}}\|\mathbf{w}\|_{L^2}^{2-\frac{|\alpha|+|\beta|}{m}} \leqslant \epsilon[\mathbf{w}]^2_{m,2} + c'_{\alpha\beta}\epsilon^{-\frac{|\alpha|+|\beta|}{2m-|\alpha|-|\beta|}}\|\mathbf{w}\|^2_{L^2} \\
&\quad \leqslant \epsilon[\mathbf{w}]^2_{m,2} + c'_{\alpha\beta}\big(\epsilon^{-(2m-1)}+1\big)\|\mathbf{w}\|^2_{L^2} \qquad (40)
\end{aligned}
$$

for all $\mathbf{w}\in W^{m,2}(\mathbb{R}^N)$, $s\in(0,T)$, and for all $\epsilon>0$. Similarly to c_1 and c_2, also $c_{\alpha\beta}$ and $c'_{\alpha\beta}$ are some positive constants independent from $\mathbf{w}$, s, and τ satisfying $\nu_0|\tau| < T_0$. The same reasoning as above yields (for $|\alpha| = 0$ and $|\beta| = 1$)

$$
\begin{aligned}
&T_0^{-1}\left|\chi_1'(s/T_0)\cdot\Re\mathfrak{e}\int_{\mathbb{R}^N}\overline{\mathbf{w}(x,s)}\cdot D_x\mathbf{w}(x,s)y\,\mathrm{d}x\right| \\
&\quad\leqslant \hat{c}T_0^{-1}\chi_1'(s/T_0)[\mathbf{w}]_{m,2}^{1/m}\|\mathbf{w}\|_{L^2}^{2-\frac{1}{m}} \\
&\quad\leqslant \epsilon[\mathbf{w}]_{m,2}^2+\hat{c}'\epsilon^{-1/(2m-1)}\bigl(T_0^{-1}\chi_1'(s/T_0)\bigr)^{2m/(2m-1)}\|\mathbf{w}\|_{L^2}^2,
\end{aligned}
\tag{41}
$$

where $\hat{c}$ and $\hat{c}'$ are some positive constants, and

$$
\begin{aligned}
\left|\int_{\mathbb{R}^N}\mathbf{g}(x,s)\cdot\overline{\mathbf{w}(x)}\,\mathrm{d}x\right| &\leqslant \biggl(\int_{\mathbb{R}^N}\bigl|\mathbf{g}(x,s)\bigr|^2\,\mathrm{d}x\biggr)^{1/2}\|\mathbf{w}\|_{L^2} \\
&\leqslant \frac{1}{2}\bigl(1+\tau_1(s)^2\bigr)^{-1/2}K^2+\frac{1}{2}\bigl(1+\tau_1(s)^2\bigr)^{1/2}\|\mathbf{w}\|_{L^2}^2,
\end{aligned}
\tag{42}
$$

where K is the constant introduced in inequality (12).

Next, we apply inequalities (39) through (42) to Eq. (38) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^N}\bigl|\mathbf{v}(x,s)\bigr|^2\,\mathrm{d}x+c_1\bigl[\mathbf{v}(\cdot,s)\bigr]_{m,2}^2-c_2\bigl\|\mathbf{v}(\cdot,s)\bigr\|_{L^2}^2 \\
&\quad\leqslant \tilde{c}_m\epsilon\bigl[\mathbf{v}(\cdot,s)\bigr]_{m,2}^2+\tilde{c}_m'\bigl(\epsilon^{-(2m-1)}+1\bigr)\bigl\|\mathbf{v}(\cdot,s)\bigr\|_{L^2}^2 \\
&\qquad+\hat{c}'\epsilon^{-1/(2m-1)}\bigl(T_0^{-1}\chi_1'(s/T_0)\bigr)^{2m/(2m-1)}\bigl\|\mathbf{v}(\cdot,s)\bigr\|_{L^2}^2+\frac{1}{2}K^2
\end{aligned}
\tag{43}
$$

for all $s\in(0,T)\setminus\{0\}$ and for all $\epsilon>0$, where also the constants $\tilde{c}_m$, $\tilde{c}_m'$, and $\hat{c}'$ (all >0) are independent from $\mathbf{v}$, s, T, and τ with $\nu_0|\tau|<T_0$. Taking $\epsilon=c_1/(2\tilde{c}_m)$ and setting

$$
V_0(s)\stackrel{\text{def}}{=}\bigl\|\mathbf{v}(\cdot,s)\bigr\|_{L^2}^2\quad\text{and}\quad V_1(s)\stackrel{\text{def}}{=}V_0(s)+c_1\int_0^s\bigl[\mathbf{v}\bigl(\cdot,s'\bigr)\bigr]_{m,2}^2\,\mathrm{d}s'
$$

for $s\in[0,T)$, we deduce from (43) that

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}V_j(s) &\leqslant \bigl[c_2+\tilde{c}_m'\bigl((2\tilde{c}_m/c_1)^{2m-1}+1\bigr)\bigr]V_k(s) \\
&\qquad+\hat{c}'(2\tilde{c}_m/c_1)^{1/(2m-1)}\bigl(T_0^{-1}\chi_1'(s/T_0)\bigr)^{2m/(2m-1)}V_k(s)+\frac{1}{2}K^2 \\
&=\bigl[c_3+c_4\bigl(T_0^{-1}\chi_1'(s/T_0)\bigr)^{2m/(2m-1)}\bigr]V_k(s)+\frac{1}{2}K^2
\end{aligned}
\tag{44}
$$

for $j=1$ and $k=0$, and hence for any $j,k\in\{0,1\}$, where c_3 and c_4 are some positive constants, and

$$\int_0^\sigma \chi_1'(s/T_0)^{2m/(2m-1)}\,\mathrm{d}s = T_0 \int_0^{\sigma/T_0} \chi_1'(s')^{2m/(2m-1)}\,\mathrm{d}s'$$
$$= T_0 \cdot J\bigl(\min\{\sigma/T_0, 1\}\bigr) \quad \text{for } \sigma \in \mathbb{R}_+,$$

by (16). We apply Gronwall's inequality to (44) with $j = k \in \{0, 1\}$ to conclude that

$$V_j(\sigma) \leqslant \mathrm{e}^{E(\sigma)-E(\sigma_0)} V_j(\sigma_0) + K^2 \int_{\sigma_0}^{\sigma} \mathrm{e}^{E(\sigma)-E(s)}\,\mathrm{d}s$$
$$\leqslant \mathrm{e}^{E(\sigma)-E(\sigma_0)} \bigl(V_j(\sigma_0) + K^2(\sigma - \sigma_0)\bigr) \tag{45}$$

whenever $0 < \sigma_0 \leqslant \sigma \leqslant T$, where

$$E(\sigma) \stackrel{\text{def}}{=} 2\int_0^\sigma \bigl[c_3 + c_4\bigl(T_0^{-1}\chi_1'(s/T_0)\bigr)^{2m/(2m-1)}\bigr]\,\mathrm{d}s$$
$$= 2c_3\sigma + 2c_4 T_0^{-1/(2m-1)} \cdot J\bigl(\min\{\sigma/T_0, 1\}\bigr)$$

satisfies

$$E(\sigma) \leqslant C_0 J(\sigma/T_0) \quad \text{for every } \sigma \in [0, T]. \tag{46}$$

Here, $C_0 \in \mathbb{R}_+$ is a constant independent from $\sigma \in [0, T]$, $y \in \mathbb{R}^N$ ($|y|_\infty < r_0'$), $\tau \in \mathbb{R}$ ($\nu_0|\tau| < T_0$), $\mathbf{u}_0$, $\mathbf{u}$, T ($T \geqslant T_0$), and K.

Finally, (28) guarantees $V_0(s) \to V_0(0)$ as $s \to 0+$. Consequently, (45) (for $j = 0$, as $\sigma_0 \to 0+$) and (46) imply

$$V_0(\sigma) \leqslant \mathrm{e}^{C_0 J(\sigma/T_0)}\bigl(V_0(0) + K^2\sigma\bigr) = \mathrm{e}^{C_0 J(\sigma/T_0)}\bigl(\|\mathbf{u}_0\|_{L^2}^2 + K^2\sigma\bigr) \tag{47}$$

for every $\sigma \in [0, T)$. The last inequality yields (26) immediately with $c_0 = 0$ only, and (24) as well. Furthermore, inequality (47) applied to the right-hand side of (44) with $j = 1$ and $k = 0$ guarantees, upon integration with respect to $s \in [0, \sigma]$, for $\sigma \in [0, T)$,

$$V_1(\sigma) - V_1(0) \leqslant E(\sigma)\mathrm{e}^{C_0 J(\sigma/T_0)}\bigl(\|\mathbf{u}_0\|_{L^2}^2 + K^2\sigma\bigr) + K^2\sigma.$$

This forces $\limsup_{\sigma\to 0+} V_1(\sigma) \leqslant V_1(0) = V_0(0) = \|\mathbf{u}_0\|_{L^2}^2$. Since $\lim_{\sigma\to 0+} V_0(\sigma) = V_0(0)$ evidently implies $\liminf_{\sigma\to 0+} V_1(\sigma) \geqslant V_1(0)$, we must have also $\lim_{\sigma\to 0+} V_1(\sigma) = V_1(0)$. The same procedure as for $V_0(\sigma)$ applied to (45) now for $V_1(\sigma)$ yields

$$V_1(\sigma) \leqslant \mathrm{e}^{C_0 J(\sigma/T_0)}\bigl(\|\mathbf{u}_0\|_{L^2}^2 + K^2\sigma\bigr) \quad \text{for every } \sigma \in [0, T).$$

This inequality renders (26) with a constant $c_0 > 0$.

The proof of the lemma is complete. □

5. Hardy spaces and Fourier transforms

In order to be able to take advantage of the a priori estimates (24) and (26) established in Lemma 4.1, we will approximate arbitrary initial data $\mathbf{u}_0 \in \mathbf{L}^2(\mathbb{R}^N) = [L^2(\mathbb{R}^N)]^M$ by (holomorphic) functions from the Hardy space $\mathbf{H}^2(\mathfrak{X}^{(r_0)}) = [H^2(\mathfrak{X}^{(r_0)})]^M$, where $r_0 \in (0,\infty)$ is the constant from hypotheses (H1)–(H3). Recall that $\mathfrak{X}^{(r)} = \mathbb{R}^N + \mathrm{i}Q^{(r)}$ with $Q^{(r)} = (-r,r)^N$, for $0 < r < \infty$. We denote by $H^2(\mathfrak{X}^{(r)})$ the Hardy space of type H^2 over the strip $\mathfrak{X}^{(r)}$, i.e., $H^2(\mathfrak{X}^{(r)})$ is the vector space which consists of all holomorphic (i.e., complex analytic) functions $u : \mathfrak{X}^{(r)} \to \mathbb{C}$ that have finite norm

$$\begin{aligned} \|u\|_{H^2} \equiv \|u\|_{H^2(\mathfrak{X}^{(r)})} &\stackrel{\text{def}}{=} \sup_{|y|_\infty < r} \left\| u(\cdot + \mathrm{i}y) \right\|_{L^2(\mathbb{R}^N)} \\ &= \sup_{|y|_\infty < r} \left(\int_{\mathbb{R}^N} \left| u(x+\mathrm{i}y) \right|^2 \mathrm{d}x \right)^{1/2} < \infty \quad \left(y \in \mathbb{R}^N \right). \end{aligned} \tag{48}$$

It is not difficult to verify that $H^2(\mathfrak{X}^{(r)})$ is a Banach space. We refer to E.M. Stein and G. Weiss [69, Chapt. III] for basic theory of Hardy spaces; the most relevant results about $H^2(\mathfrak{X}^{(r)})$ can be found in [69, Chapt. III, §2, pp. 91–101, and §6.12, pp. 127–128]. It is important to note that the base $Q^{(r)}$ of the tube $\mathfrak{X}^{(r)}$ is an open polyhedron with 2^N vertices of type $y_0 = (\pm r, \pm r, \dots, \pm r) \in [-r,r]^N \subset \mathbb{R}^N$.

An important characterization of $H^2(\mathfrak{X}^{(r)})$ is obtained by means of the Fourier transformation and Plancherel's theorem. Let $F, f \in L^2(\mathbb{R}^N)$ be arbitrary functions, such that f is the Fourier transform of F, i.e.,

$$f(\xi) = \int_{\mathbb{R}^N} \mathrm{e}^{-2\pi\mathrm{i}x\cdot\xi} F(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^N, \tag{49}$$

$$F(x) = \int_{\mathbb{R}^N} \mathrm{e}^{2\pi\mathrm{i}x\cdot\xi} f(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}^N. \tag{50}$$

The following proposition is a direct consequence of results from [69, Chapt. III, Theorem 2.3 (p. 93), Corollary 2.9 (p. 97), and §6.12, pp. 127–128]. Recall that $|\xi|_1 = \sum_{i=1}^N |\xi_i|$ for $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N$; in particular, we have the product measure $\mathrm{e}^{4\pi r|\xi|_1}\,\mathrm{d}\xi = \prod_{i=1}^N \mathrm{e}^{4\pi r|\xi_i|}\,\mathrm{d}\xi_i$ on $\mathbb{R}^N$.

Proposition 5.1. *The function F is the restriction to $\mathbb{R}^N$ of a function from $H^2(\mathfrak{X}^{(r)})$, which we denote by F again,* if and only if *f satisfies*

$$\int_{\mathbb{R}^N} \left| f(\xi) \right|^2 \mathrm{e}^{4\pi r|\xi|_1}\,\mathrm{d}\xi < \infty. \tag{51}$$

If this is the case, then $F \in H^2(\mathfrak{X}^{(r)})$ is given by the inverse Fourier–Laplace transform of f, i.e.,

$$F(x+\mathrm{i}y)=\int\limits_{\mathbb{R}^N} \mathrm{e}^{2\pi\mathrm{i}(x+\mathrm{i}y)\cdot\xi} f(\xi)\,\mathrm{d}\xi, \quad x\in\mathbb{R}^N,\ y\in Q^{(r)}. \tag{52}$$

Moreover, the mapping $y\mapsto F(\cdot+\mathrm{i}y): Q^{(r)}\to L^2(\mathbb{R}^N)$ *extends to a continuous mapping from the closure* $\overline{Q}^{(r)}$ *of* $Q^{(r)}$ *(in* $\mathbb{R}^N$*) to* $L^2(\mathbb{R}^N)$*, the latter given by formula* (52) *again. Finally, the mapping* $F\mapsto f$ *is an isomorphism* (*both, algebraically and topologically*) *of the Hardy space* $H^2(\mathfrak{X}^{(r)})$ *onto the weighted Lebesgue space* $L^2(\mathbb{R}^N;\mathrm{e}^{4\pi r|\xi|_1}\,\mathrm{d}\xi)$ *with the norm given by* (51).

We abbreviate $L_r^2(\mathbb{R}^N)\overset{\text{def}}{=}L^2(\mathbb{R}^N;\mathrm{e}^{4\pi r|\xi|_1}\,\mathrm{d}\xi)$ and endow this weighted Lebesgue space with the standard norm (induced by the canonical inner product)

$$\|f\|_{L_r^2(\mathbb{R}^N)}\overset{\text{def}}{=}\Bigg(\int\limits_{\mathbb{R}^N}\big|f(\xi)\big|^2\mathrm{e}^{4\pi r|\xi|_1}\,\mathrm{d}\xi\Bigg)^{1/2}, \tag{53}$$

in accordance to (51), with the weight $\mathrm{e}^{4\pi r|\xi|_1}$ for $\xi\in\mathbb{R}^N$. However, if results from [69, Chapt. III] are applied directly to obtain Proposition 5.1, the space $L_r^2(\mathbb{R}^N)$ has to be endowed with the supremum-like norm

$$\|f\|_{L_r^2(\mathbb{R}^N)}^{(\sup)}\overset{\text{def}}{=}\sup_{y\in Q^{(r)}}\Bigg(\int\limits_{\mathbb{R}^N}\big|f(\xi)\big|^2\mathrm{e}^{4\pi y\cdot\xi}\,\mathrm{d}\xi\Bigg)^{1/2}. \tag{54}$$

We now show that these two norms on $L_r^2(\mathbb{R}^N)$ are indeed equivalent, by the following inequalities, valid for every $y=(y_1,y_2,\dots,y_N)\in Q^{(r)}$:

$$\begin{aligned}
\int\limits_{\mathbb{R}^N}\big|f(\xi)\big|^2\mathrm{e}^{4\pi y\cdot\xi}\,\mathrm{d}\xi &\leqslant \max_{y_0=(\pm r,\pm r,\dots,\pm r)\in\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\big|f(\xi)\big|^2\mathrm{e}^{4\pi y_0\cdot\xi}\,\mathrm{d}\xi \\
&\leqslant \|f\|_{L_r^2(\mathbb{R}^N)}^2=\int\limits_{\mathbb{R}^N}\big|f(\xi)\big|^2\mathrm{e}^{4\pi r|\xi|_1}\,\mathrm{d}\xi=\int\limits_{\mathbb{R}^N}\big|f(\xi)\big|^2\prod_{i=1}^N\mathrm{e}^{4\pi r|\xi_i|}\,\mathrm{d}\xi_i \\
&\leqslant \int\limits_{\mathbb{R}^N}\big|f(\xi)\big|^2\prod_{i=1}^N\big(\mathrm{e}^{4\pi r\xi_i}+\mathrm{e}^{-4\pi r\xi_i}\big)\,\mathrm{d}\xi_i \\
&\leqslant 2^N\cdot\max_{y_0=(\pm r,\pm r,\dots,\pm r)\in\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\big|f(\xi)\big|^2\mathrm{e}^{4\pi y_0\cdot\xi}\,\mathrm{d}\xi \\
&\leqslant 2^N\cdot\sup_{x\in Q^{(r)}}\int\limits_{\mathbb{R}^N}\big|f(\xi)\big|^2\mathrm{e}^{4\pi x\cdot\xi}\,\mathrm{d}\xi=2^N\big(\|f\|_{L_r^2(\mathbb{R}^N)}^{(\sup)}\big)^2.
\end{aligned}$$

Here, we have made use of the convexity and monotonicity of the exponential function on $\mathbb{R}$ (in the first and second inequalities above, respectively) combined with $\mathrm{e}^{4\pi y\cdot\xi}\,\mathrm{d}\xi=$

$\prod_{i=1}^{N} \mathrm{e}^{4\pi y_i \xi_i}\, \mathrm{d}\xi_i$. Observe that $y \in Q^{(r)}$ may be written as a convex combination of the vertices $y_0^\ell = (\pm r, \pm r, \dots, \pm r) \in \mathbb{R}^N$ ($\ell = 1, 2, \dots, 2^N$), i.e., $y = \sum_{\ell=1}^{2^N} \gamma_\ell y_0^\ell$ with some coefficients $\gamma_\ell \geqslant 0$ satisfying $\sum_{\ell=1}^{2^N} \gamma_\ell = 1$. The exponential function being convex, we thus have $\mathrm{e}^{4\pi y\cdot\xi} \leqslant \sum_{\ell=1}^{2^N} \gamma_\ell \mathrm{e}^{4\pi y_0^\ell \cdot \xi}$. In particular, we have shown the desired norm equivalence

$$\|f\|_{L_r^2(\mathbb{R}^N)}^{(\sup)} \leqslant \|f\|_{L_r^2(\mathbb{R}^N)} \leqslant 2^{N/2} \cdot \|f\|_{L_r^2(\mathbb{R}^N)}^{(\sup)}$$

together with the following representation,

$$L_r^2(\mathbb{R}^N) = \bigcap_{y_0 = (\pm r, \pm r, \dots, \pm r) \in \mathbb{R}^N} L^2\big(\mathbb{R}^N; \mathrm{e}^{4\pi y_0 \cdot \xi}\, \mathrm{d}\xi\big) = \bigcap_{y \in Q^{(r)}} L^2\big(\mathbb{R}^N; \mathrm{e}^{4\pi y\cdot\xi}\, \mathrm{d}\xi\big).$$

The following lemma plays an important rôle in the proof of Proposition 5.1. In particular, it renders the Fourier transform $f^{(y)}$ of the function $F(\cdot + \mathrm{i}y)$,

$$f^{(y)}(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi} F(x + \mathrm{i}y)\, \mathrm{d}x, \quad \xi \in \mathbb{R}^N. \tag{55}$$

Lemma 5.2. *In the situation of Proposition* 5.1, *F is the restriction to $\mathbb{R}^N$ of a function from* $H^2(\mathfrak{X}^{(r)})$ if and only if *condition* (51) *holds. If this is the case and $y \in Q^{(r)}$ is arbitrary, then both functions $x \mapsto F(x + \mathrm{i}y)$ and $\xi \mapsto \mathrm{e}^{-2\pi y\cdot\xi} f(\xi)$ are in $L^2(\mathbb{R}^N)$ and the latter is the Fourier transform of the former, i.e.,* $\mathrm{e}^{-2\pi y\cdot\xi} f(\xi) = f^{(y)}(\xi)$ *for $\xi \in \mathbb{R}^N$.*

Proof. The proof of this lemma is given in [69, Chapt. III], pp. 99–101. Its principal idea is to justify the use of Cauchy's theorem in order to shift from the integration plane $\mathrm{i}y + \mathbb{R}^N = \{x + \mathrm{i}y \in \mathbb{C}\colon x \in \mathbb{R}^N\}$ in the first integral below to $\mathbb{R}^N$ in the second one:

$$\begin{aligned} f^{(y)}(\xi) &= \int_{\mathbb{R}^N} \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi} F(x + \mathrm{i}y)\, \mathrm{d}x = \int_{\mathbb{R}^N} \mathrm{e}^{-2\pi \mathrm{i}(x - \mathrm{i}y)\cdot\xi} F(x)\, \mathrm{d}x \\ &= \mathrm{e}^{-2\pi y\cdot\xi} \int_{\mathbb{R}^N} \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi} F(x)\, \mathrm{d}x = \mathrm{e}^{-2\pi y\cdot\xi} f(\xi), \quad \xi \in \mathbb{R}^N. \end{aligned}$$

Naturally, a regularization procedure is applied. The shift is performed in each coordinate y_i ($i = 1, 2, \dots, N$) separately, so that it suffices to verify the second equality above for $y = (y_1, 0, \dots, 0)$ only, by applying Cauchy's theorem with respect to the variable $z_1 = x_1 + \mathrm{i}y_1 \in \mathbb{C}$. □

An alternative proof of Lemma 5.2 can be found in L. HÖRMANDER [38, Theorem 7.4.2, p. 192]; his proof uses the Cauchy–Riemann equations instead of Cauchy's theorem.

Remark 5.3. An easy direct inspection of the proofs of Proposition 5.1 and Lemma 5.2 shows that both these results remain valid also for the Banach spaces of vector-valued functions $\mathbf{L}^2(\mathbb{R}^N) = [L^2(\mathbb{R}^N)]^M$ and $\mathbf{H}^2(\mathfrak{X}^{(r)}) = [H^2(\mathfrak{X}^{(r)})]^M$ with obvious minor modifications, where $r \in (0, \infty)$ is a constant. □

Given $R \in (0,\infty)$, we denote by $E_R \equiv E_R(\mathbb{C}^N)$ the set of all entire holomorphic functions $F : \mathbb{C}^N \to \mathbb{C}$ (i.e., complex analytic functions in the entire space $\mathbb{C}^N$) with the following property: For each $m \in \mathbb{Z}_+$ there is a constant $C_m \geqslant 0$ such that

$$\left|F(z)\right| \leqslant C_m\left(1+|z|\right)^{-m} \exp\left(2\pi R|\Im\mathfrak{m}\, z|_1\right) \quad \text{for all } z \in \mathbb{C}^N.$$

By the Paley–Wiener–Schwartz theorem, as stated in L. Hörmander [38, Theorem 7.3.1, p. 181], such a function F is characterized by its Fourier transform f defined by (49): $f : \mathbb{R}^N \to \mathbb{C}$ is a C^∞ function with compact support contained in the closed cube $\overline{Q}^{(R)} = [-R, R]^N = \{y \in \mathbb{R}^N\colon |y|_\infty \leqslant R\}$. Allowing any $R \in (0,\infty)$ we observe that the set of such functions f forms a dense vector subspace of the weighted Lebesgue space $L^2_r(\mathbb{R}^N) = L^2(\mathbb{R}^N; \mathrm{e}^{4\pi r|\xi|_1}\,\mathrm{d}\xi)$. Hence, Proposition 5.1 has the following important consequence:

Corollary 5.4.

(a) *The set of* (*restrictions to* $\mathfrak{X}^{(r)}$ *of*) *all functions* $F \in \bigcup_{0<R<\infty} E_R$ *forms a dense vector subspace of the Hardy space* $H^2(\mathfrak{X}^{(r)})$.
(b) $H^2(\mathfrak{X}^{(r)})$ *is a dense vector subspace of the Lebesgue space* $L^2(\mathbb{R}^N)$.

Remark 5.5. From the arguments used in the proof of Corollary 5.4 above, combined with standard properties of the Fourier transformation (differentiation and pointwise multiplication by a polynomial), we deduce that $F \in E_R \Longrightarrow P(D_z)F(z),\ P(z)F(z) \in E_R$, holds for every complex polynomial $P(z)$ of $z \in \mathbb{C}^N$ and for every $R \in (0,\infty)$, where $D_z \stackrel{\text{def}}{=} \frac{1}{\mathrm{i}}\frac{\partial}{\partial z} \equiv -\mathrm{i}(\partial/\partial z_1, \ldots, \partial/\partial z_N)$, cf. (9). $\square$

Applying Proposition 5.1 once again we have also

Corollary 5.6. $F \in \mathfrak{X}^{(r)} \Longrightarrow P(D_z)F(z) \in \mathfrak{X}^{(s)}$ *for every complex polynomial* $P : \mathbb{C}^N \to \mathbb{C}$, *whenever* $0 < s < r < \infty$.

The conclusions of Remark 5.5 and Corollary 5.6 can also be derived directly, without recourse to Proposition 5.1, from the Bochner–Martinelli integral representation of holomorphic functions or from the Cauchy integral formula for polydiscs (see e.g. S.G. Krantz [53, Corollary 1.1.7 (p. 18) or Theorem 1.2.2 (p. 24)], respectively, or F. John [44, Chapt. 3, Sect. 3(c), Eqs. (3.22c) and (3.23), p. 71]).

6. The Cauchy problem in a Hardy space H^2

In order to be able to take advantage of the a priori estimates (24) and (26) established in Lemma 4.1, we will approximate arbitrary initial data $\mathbf{u}_0 \in \mathbf{L}^2(\mathbb{R}^N)$ by (holomorphic) functions from the Hardy space $\mathbf{H}^2(\mathfrak{X}^{(r_0)})$, where $r_0 \in (0,\infty)$ is the constant from hypotheses (H1)–(H3). Recall that $\mathfrak{X}^{(r)} = \mathbb{R}^N + \mathrm{i}Q^{(r)}$ with $Q^{(r)} = (-r, r)^N$, for $0 < r < \infty$, and $\Omega = \mathfrak{X}^{(r_0)} \times \Delta_{\vartheta_0}^{T_0,T} \subset \mathbb{C}^N \times \mathbb{C}$, with some $0 < T_0 \leqslant T < \infty$.

Proposition 6.1. *Let* $M, N \geqslant 1$, $0 < T < \infty$, *and assume that all hypotheses* (H1)–(H3) *are satisfied with some constants* $0 < r_0 < \infty$, $0 < T_0 \leqslant T$, *and* $0 < \vartheta_0 < \pi/2$. *Then, given any*

$\mathbf{u}_0 \in \mathbf{H}^2(\mathfrak{X}^{(r_0)})$, *the Cauchy problem* (1) *possesses a unique classical solution* $\mathbf{u} : \Omega \to \mathbb{C}^M$ *which is holomorphic in* Ω, *has a continuous extension to* $\mathfrak{X}^{(r_0)} \times [0, T]$ *satisfying* $\mathbf{u}(\cdot, 0) = \mathbf{u}_0$ *in* $\mathfrak{X}^{(r_0)}$, *and* $\mathbf{u}$ *has all properties assumed in Lemma* 4.1, *including* (i) *and* (ii).

Recall again that the numbers κ_0 and ν_0 have been chosen in such a way that $\Gamma_T^{(T_0)}(\kappa_0, \nu_0) \subset \Omega$.

Proof of Proposition 6.1. We will construct the desired solution $\mathbf{u} : \Omega \to \mathbb{C}^M$ to the Cauchy problem (1) by obtaining first its translations $\mathbf{u}^{(\zeta)}(x,t) = \mathbf{u}(x + \zeta, t)$ from the corresponding translated Cauchy problem (1) below, (56), as unknown functions of $(x,t) \in \mathbb{R}^N \times (0, T)$, for each fixed $\zeta = \xi + \mathrm{i}\eta \in \mathfrak{X}^{(r_0)}$:

Given a function $\mathbf{u}_0 \in \mathbf{H}^2(\mathfrak{X}^{(r_0)})$ and a point $\zeta = \xi + \mathrm{i}\eta \in \mathfrak{X}^{(r_0)}$, we observe that the translation function $\mathbf{u}_0^{(\zeta)}(x) \stackrel{\text{def}}{=} \mathbf{u}_0(x + \xi + \mathrm{i}\eta) = \mathbf{u}_0(x + \zeta)$ of $x \in \mathbb{R}^N$ is in $\mathbf{L}^2(\mathbb{R}^N)$ with the norm

$$\|\mathbf{u}_0^{(\zeta)}\|_{L^2(\mathbb{R}^N)} \leqslant \|\mathbf{u}_0\|_{H^2(\mathfrak{X}^{(r_0)})} < \infty.$$

Let us consider the corresponding translated Cauchy problem (1), i.e.,

$$\begin{cases} \dfrac{\partial \mathbf{u}^{(\zeta)}}{\partial t} + \mathbf{P}^{(\zeta)}\left(x, t, \dfrac{1}{\mathrm{i}}\dfrac{\partial}{\partial x}\right)\mathbf{u}^{(\zeta)} = \mathbf{f}^{(\zeta)}(x,t) & \text{for } (x,t) \in \mathbb{R}^N \times (0,T); \\ \mathbf{u}^{(\zeta)}(x, 0) = \mathbf{u}_0^{(\zeta)}(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{56}$$

Here, in accordance with our notation for $\mathbf{u}$ and $\mathbf{u}_0^{(\zeta)}$, we have denoted

$$\mathbf{P}^{(\zeta)}\left(x, t, \frac{1}{\mathrm{i}}\frac{\partial}{\partial x}\right) \stackrel{\text{def}}{=} \mathbf{P}\left(\zeta + x, t, \frac{1}{\mathrm{i}}\frac{\partial}{\partial x}\right),$$

$$\mathbf{f}^{(\zeta)}(x,t) \stackrel{\text{def}}{=} \mathbf{f}(\xi + x, t) \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,T).$$

By a pair of standard theorems for abstract parabolic systems due to J.-L. LIONS [56, Chapt. IV, Théorème 1.1 (§1, p. 46) and Théorème 2.1 (§2, p. 52)] (for alternative proofs, see also e.g. L.C. EVANS [19, Chapt. 7, §1.2(c), Theorems 3 and 4, pp. 356–358], J.-L. LIONS [57, Chapt. III, Theorem 1.2, p. 102], or H. TANABE [71, Chapt. 5, §5.5, Theorem 5.5.1, p. 150]), the Cauchy problem (56) possesses a unique weak solution $\mathbf{u}^{(\zeta)} : \mathbb{R}^N \times (0,T) \to \mathbb{C}^M$ such that

(i) $\mathbf{u}^{(\zeta)} \in C([0,T] \to \mathbf{L}^2(\mathbb{R}^N))$, i.e., $\mathbf{u}^{(\zeta)} : [0,T] \to \mathbf{L}^2(\mathbb{R}^N)$ is a continuous function,
(ii) $\mathbf{u}^{(\zeta)} \in L^2((0,T) \to \mathbf{W}^{m,2}(\mathbb{R}^N))$,
(iii) $\mathbf{u}^{(\zeta)} \in W^{1,2}((0,T) \to \mathbf{W}^{-m,2}(\mathbb{R}^N))$, i.e., both $\mathbf{u}^{(\zeta)}$ and its distributional time-derivative $\frac{\partial}{\partial t}\mathbf{u}^{(\zeta)}$ belong to $L^2((0,T) \to \mathbf{W}^{-m,2}(\mathbb{R}^N))$, and
(iv)

$$\begin{cases} \dfrac{\partial \mathbf{u}^{(\zeta)}}{\partial t} + \mathbf{P}^{(\zeta)}\left(x, t, \dfrac{1}{\mathrm{i}}\dfrac{\partial}{\partial x}\right)\mathbf{u}^{(\zeta)} = \mathbf{f}^{(\zeta)} & \text{in } L^2\big([0,T] \to \mathbf{W}^{-m,2}(\mathbb{R}^N)\big); \\ \mathbf{u}^{(\zeta)}(\cdot, 0) = \mathbf{u}_0^{(\zeta)} & \text{in } \mathbf{L}^2(\Omega). \end{cases} \tag{57}$$

We remark that $W^{-m,2}(\mathbb{R}^N)$ is the dual space of the Sobolev space $W^{m,2}(\mathbb{R}^N)$ and, consequently, $\mathbf{W}^{-m,2}(\mathbb{R}^N) = [W^{-m,2}(\mathbb{R}^N)]^M$ is the dual space of $\mathbf{W}^{m,2}(\mathbb{R}^N) = [W^{m,2}(\mathbb{R}^N)]^M$; similarly, $L^2((0,T) \to \mathbf{W}^{-m,2}(\mathbb{R}^N))$ is the dual space of $L^2((0,T) \to \mathbf{W}^{m,2}(\mathbb{R}^N))$.

Taking advantage of our differentiability hypotheses (H1) and (H3), respectively, on the coefficients of the partial differential operator $\mathbf{P}(x,t,\frac{1}{\mathrm{i}}\frac{\partial}{\partial x})$ and on the function $\mathbf{f}(x,t)$, we observe that if the initial data $\mathbf{u}_0 \in \mathbf{L}^2(\mathbb{R}^N)$ are C^∞-smooth (in the real-variable sense) then also the (unique) solution $\mathbf{u}^{(\zeta)}(x,t) = \mathbf{u}(x+\zeta,t)$ to the translated Cauchy problem (56) is C^∞-smooth in $\mathbb{R}^N \times [0,T]$, by Theorem 19 and Corollary (to Theorem 19) in A. FRIEDMAN [24, Chapt. 10], on p. 321 and p. 322, respectively. If the initial data $\mathbf{u}_0 \in \mathbf{L}^2(\mathbb{R}^N)$ are only continuously differentiable (in the real-variable sense) in an open polydisc neighborhood

$$D^{(\varrho)} = \left\{ z \in \mathbb{C}^N \colon |z-\zeta|_\infty < \varrho \right\}$$

of the point $\zeta \in \mathfrak{X}^{(r_0)}$, for some $\varrho > 0$ such that the closure $\overline{D}^{(\varrho)} \subset \mathfrak{X}^{(r_0)}$, then also the function

$$\zeta \mapsto \mathbf{u}^{(\zeta)}(x,t) = \mathbf{u}(x+\zeta,t) : D^{(\varrho)} \to \mathbf{L}^2\big(\mathbb{R}^N\big) \tag{58}$$

is strongly differentiable at every point $\zeta \in D^{(\varrho)}$ (again only in the real-variable sense, $\zeta = \xi + \mathrm{i}\eta \in D^{(\varrho)}$; see [24, Chapt. 10, p. 273] for a definition). This regularity result can be proved exactly as Theorem 19 and Corollary (to Theorem 19) in A. FRIEDMAN [24, Chapt. 10, pp. 321–322]. Moreover, in analogy with (24) we have the following standard estimates, with some constant $C_T \in (0,\infty)$:

$$\int\limits_{\mathbb{R}^N} \left| \frac{\partial}{\partial x_i} \mathbf{u}^{(\zeta)}(x,t) \right|^2 \mathrm{d}x \leqslant C_T \left\| \frac{\partial}{\partial x_i} \mathbf{u}_0^{(\zeta)} \right\|^2_{L^2(\mathbb{R}^N)}, \tag{59}$$

$$\int\limits_{\mathbb{R}^N} \left| \frac{\partial}{\partial y_i} \mathbf{u}^{(\zeta)}(x,t) \right|^2 \mathrm{d}x \leqslant C_T \left\| \frac{\partial}{\partial y_i} \mathbf{u}_0^{(\zeta)} \right\|^2_{L^2(\mathbb{R}^N)}, \tag{60}$$

for all $t \in [0,T]$ and $i = 1,2,\dots,N$, where $\frac{\partial}{\partial x_i}\mathbf{u}^{(\zeta)}(x,t)$ and $\frac{\partial}{\partial y_i}\mathbf{u}^{(\zeta)}(x,t)$, respectively, denote the strong partial derivatives of the function (58) with respect to x_i and y_i at $\zeta = \xi + \mathrm{i}\eta \in D^{(\varrho)}$, and similarly for $\mathbf{u}_0^{(\zeta)}$.

Consequently, for every $i = 1,2,\dots,N$ we are allowed to apply the operator $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + \mathrm{i}\frac{\partial}{\partial y_i})$ (cf. (9)) to the Cauchy problem (56) with ζ replaced by $z = (z_1, z_2, \dots, z_N) \in D^{(\varrho)}$ in order to conclude that the function $\mathbf{u}_i^{(z)} : \mathbb{R}^N \times (0,T) \to \mathbb{C}^M$ defined by $\mathbf{u}_i^{(z)}(x,t) \stackrel{\text{def}}{=} \frac{\partial}{\partial \bar{z}_i}\mathbf{u}^{(z)}(x,t)$ is a weak solution to the Cauchy problem (1) with the zero initial data $\mathbf{u}_i^{(z)}(x,0) \stackrel{\text{def}}{=} \frac{\partial}{\partial \bar{z}_i}\mathbf{u}^{(z)}(x,0) = \frac{\partial}{\partial \bar{z}_i}\mathbf{u}_0^{(z)}(x) = 0$ for $x \in \mathbb{R}^N$, thanks to $\mathbf{u}_0 \in \mathbf{H}^2(\mathfrak{X}^{(r_0)})$ being holomorphic in $\mathfrak{X}^{(r_0)}$. The weak solution $\mathbf{u}_i^{(z)} : \mathbb{R}^N \times (0,T) \to \mathbb{C}$ to this Cauchy problem being unique, we conclude that $\mathbf{u}_i^{(z)}(x,t) \equiv 0$ in $\mathbb{R}^N \times (0,T)$, for each $i = 1,2,\dots,N$, cf. (59) and (60). Hence, for every $t \in [0,T]$, the function $\mathbf{u}(\cdot,t)$ is holomorphic in $\mathfrak{X}^{(r_0)}$. The fact that also the function $\mathbf{u}(z,\cdot)$ is holomorphic in $\Delta_{\vartheta_0}^{T_0,T}$, for every $z \in \mathfrak{X}^{(r_0)}$, follows from Theorem 5.7.2 in H. TANABE [71, §5.7, p. 161], combined with [71, Theorem 5.7.6, §5.7.4, p. 179]. Finally, $\mathbf{u}$ being of class $C^\infty(\Omega)$ in $\Omega = \mathfrak{X}^{(r_0)} \times \Delta_{\vartheta_0}^{T_0,T} \subset \mathbb{C}^N \times \mathbb{C}$, we conclude that $\mathbf{u}$ is holomorphic in Ω. □

7. Proof of the main result

Now we are ready to prove our main result, Theorem 3.3. Proposition 3.4 is then a direct consequence of Theorem 3.3 and Lemma 4.1; we leave the details to the reader.

Proof of Theorem 3.3. The existence and uniqueness of a weak solution $\mathbf{u} \in C([0,T] \to \mathbf{L}^2(\mathbb{R}^N))$ to the Cauchy problem (1) is obtained by the same arguments as in the proof of Proposition 6.1 above (with $\zeta = 0 \in \mathbb{C}$): one may use KOMATSU's result [50, Theorem 3] or LIONS's theorems for abstract parabolic systems [56, Chapt. IV, Théorème 1.1 (§1, p. 46) and Théorème 2.1 (§2, p. 52)] (or [57, Chapt. III, Theorem 1.2, p. 102]).

It remains to show that this solution, $\mathbf{u} : \mathbb{R}^N \times (0,T) \to \mathbb{C}^M$, can be (uniquely) extended to a holomorphic function in the complex domain $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$.

Uniqueness. Assume that $\mathbf{u}$ possesses a holomorphic extension to the domain $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$, denoted again by $\mathbf{u}$. Then, by Lemma 4.1, the holomorphic extension of $\mathbf{u}$ satisfies the estimate in (24) for all $(y,t) \in \hat{\Gamma}_T^{(T_0)}(\kappa_0, \nu_0)$. Similarly, also the estimate in (26) is valid. If $\tilde{\mathbf{u}} : \Gamma_T^{(T_0)}(\kappa_0, \nu_0) \to \mathbb{C}^M$ is another holomorphic extension of $\mathbf{u}$, then the difference $\mathbf{w} = \tilde{\mathbf{u}} - \mathbf{u}$ satisfies Eq. (1) throughout $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$ in the classical sense with the initial data $\mathbf{u}_0 = \mathbf{w}(\cdot, 0) = \mathbf{0} \in \mathbb{C}^M$ in $\mathbb{R}^N$ and the function $\mathbf{f} \equiv \mathbf{0} \in \mathbb{C}^M$ throughout $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$. Hence, we may apply the a priori estimate in (24) to the difference $\mathbf{w}$ in place of $\mathbf{u}$, with $\|\mathbf{u}_0\|_{L^2(\mathbb{R}^N)} = 0$ and $K = 0$, in order to conclude that $\int_{\mathbb{R}^N} |\mathbf{w}(x + \mathrm{i}y, t)|^2 \,\mathrm{d}x = 0$ for all $(y,t) \in \hat{\Gamma}_T^{(T_0)}(\kappa_0, \nu_0)$. The function $\mathbf{w}$ being holomorphic in $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$, we have just proved that $\mathbf{w} \equiv \mathbf{0} \in \mathbb{C}^M$ throughout $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$, as desired.

Existence. Given $\mathbf{u}_0 \in \mathbf{L}^2(\mathbb{R}^N)$, we make use of Corollary 5.4(b) to conclude that there is a sequence $\{\mathbf{u}_{0,n}\}_{n=1}^{\infty} \subset \mathbf{H}^2(\mathfrak{X}^{(r_0)})$ such that $\|\mathbf{u}_{0,n} - \mathbf{u}_0\|_{L^2(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Next, for each $n = 1, 2, 3, \ldots$, we apply Proposition 6.1 to conclude that the Cauchy problem (1) with the initial data $\mathbf{u}_{0,n}$ in place of $\mathbf{u}_0$ possesses a unique classical solution $\mathbf{u}_n : \Omega \to \mathbb{C}^M$ which is holomorphic in Ω, has a continuous extension to $\mathfrak{X}^{(r_0)} \times [0,T]$ satisfying $\mathbf{u}_n(\cdot, 0) = \mathbf{u}_{0,n}$ in $\mathfrak{X}^{(r_0)}$, and $\mathbf{u}_n$ has all properties assumed in Lemma 4.1 for $\mathbf{u}$, including (i) and (ii). For each pair $m, n \in \mathbb{N}$, the difference $\mathbf{w}_{m,n} = \mathbf{u}_m - \mathbf{u}_n$ satisfies Eq. (1) throughout $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$ in the classical sense with the function $\mathbf{f} \equiv \mathbf{0} \in \mathbb{C}^M$ throughout $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$. We apply the a priori estimate in (24) to the difference $\mathbf{w}_{m,n}$ in place of $\mathbf{u}$, with $K = 0$, to conclude that

$$\int_{\mathbb{R}^N} \left|\mathbf{u}_m(x + \mathrm{i}y, t) - \mathbf{u}_n(x + \mathrm{i}y, t)\right|^2 \mathrm{d}x \leqslant \mathrm{e}^{C_0 J(\Re t / T_0)} \|\mathbf{u}_{m,0} - \mathbf{u}_{n,0}\|_{L^2(\mathbb{R}^N)}^2 \tag{61}$$

for all $(y,t) \in \hat{\Gamma}_T^{(T_0)}(\kappa_0, \nu_0)$. Recalling $\|\mathbf{u}_{0,n} - \mathbf{u}_0\|_{L^2(\mathbb{R}^N)} \to 0$ as $n \to \infty$, we infer from inequality (61) that $\{\mathbf{u}_n(\cdot + \mathrm{i}y, t)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$, uniformly for $(y,t) \in \hat{\Gamma}_T^{(T_0)}(\kappa_0, \nu_0)$. We denote its limit by

$$\mathbf{u}(\cdot + \mathrm{i}y, t) \stackrel{\text{def}}{=} \lim_{n \to \infty} \mathbf{u}_n(\cdot + \mathrm{i}y, t) \quad \text{in } L^2(\mathbb{R}^N). \tag{62}$$

Function $\mathbf{u}_n$ being holomorphic in $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$, it can be expressed by the Cauchy integral formula for polydiscs (S.G. KRANTZ [53, Theorem 1.2.2 (p. 24)], or F. JOHN [44, Chapt. 3, Sect. 3(c), Eq. (3.22c), p. 71]), see the beginning of our proof of Lemma 4.1. From this formula we deduce by standard limiting arguments using inequality (61) that also the limit function $\mathbf{u}$ is expressed by the same Cauchy integral formula for polydiscs. It follows that also $\mathbf{u}$ is holomorphic in $\Gamma_T^{(T_0)}(\kappa_0, \nu_0)$, as desired. Obviously, Lemma 4.1 guarantees that $\mathbf{u}$ satisfies inequality (24).

The proof of Theorem 3.3 is complete. □

8. An application to Mathematical Finance

In this section we present a simple application of our main analyticity result, Theorem 3.3 (Section 3), to the problem of ***market completeness*** in Mathematical Finance. The reader is referred to M.H.A. Davis and J. Obłój [16], S.L. Heston [36], J.C. Hull [41], J. Hull and A. White [42], A.L. Lewis [55], E.M. Stein and J.C. Stein [68], and J.B. Wiggins [72] for additional important work on this subject. We closely follow the approach in [16, Sect. 3] labeled ***"martingale model"*** for market completeness. This method generalizes an earlier work by M. Romano and N. Touzi [65, Sect. 3] on market completion by European options in a *stochastic volatility* model. An alternative approach to *market completeness*, called *"market model"*, is presented in P.J. Schönbucher [66] and M. Schweizer and J. Wissel [67]. Another alternative approach to market completeness, which yields *"endogenous completeness"* of a diffusion driven equilibrium market, has been investigated recently in J. Hugonnier, S. Malamud, and E. Trubowitz [40] and D. Kramkov and S. Predoiu [52]. Also this approach imposes analyticity hypotheses on the *drift* and the *volatility coefficients* in order to obtain a complete market, the former [40] in both, space and time variables, the latter [52] in the time variable only.

Next, we apply our main analyticity result, Theorem 3.3, to the stochastic volatility model from J.-P. Fouque, G. Papanicolaou, and K.R. Sircar [22, p. 47]. The question of *applicability* of this analyticity result to other stochastic volatility models in [36,42,55,68,72,40,52, 13,43] is discussed at the end of this section (Remark 8.7).

We consider (a slight generalization of) the stochastic volatility model from [22] given under the *risk neutral measure* via Eqs. (2.18)–(2.19) in [22, p. 47]. The model is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, where $\mathbb{P}$ is the risk neutral probability measure, and the filtration satisfies the usual conditions. Since an equivalent martingale measure $\mathbb{P}^*$ is *not* unique, the market is ***incomplete***. We will show that a call option can be used to complete the market as proposed in [16]. If $X_t = \ln S_t$ denotes the (natural) logarithm of the *stock price* S_t at time $t \geqslant 0$, then the pair $(X_t, V_t)_{t\geqslant 0}$ satisfies the following system of *stochastic* differential equations,

$$\begin{cases} \mathrm{d}X_t = \left(r - \dfrac{1}{2} f(V_t)^2\right) \mathrm{d}t + f(V_t)\, \mathrm{d}W_t, \\ \mathrm{d}V_t = \left[\alpha(m - V_t) - \beta\left(\rho_t \dfrac{\mu - r}{f(V_t)} + \gamma_t \sqrt{1-\rho_t^2}\right)\right] \mathrm{d}t + \beta\big(\rho_t\, \mathrm{d}W_t + \sqrt{1-\rho_t^2}\, \mathrm{d}Z_t\big), \end{cases} \tag{63}$$

where $(W_t)_{t\geqslant 0}$ and $(Z_t)_{t\geqslant 0}$ are two independent Brownian motions, r, α, m, β, and μ are positive constants, and all functions f, $\gamma_t = \gamma(X_t, V_t)$, and $\rho_t = \rho(X_t, V_t)$,

$$f : \mathbb{R} \to (f_-, f_+), \quad 0 < f_- < f_+ < \infty,$$
$$\gamma : \mathbb{R}^2 \to (\gamma_-, \gamma_+), \quad -\infty < \gamma_- < \gamma_+ < \infty,$$
$$\rho : \mathbb{R}^2 \to (-\rho_0, \rho_0), \quad 0 < \rho_0 < 1,$$

are assumed to be real analytic. The constant r represents the *instantaneous interest rate*, α (m, respectively) is the *rate of mean reversion* (the *long-run mean level*) of the *stochastic volatility* V_t, β is the *volatility of volatility* (which determines *volatility risk*), $f(V_t)$ is the *instantaneous volatility level*, γ represents the *market price of volatility*, ρ_t is the *correlation process* between the *stock price* S_t and its *volatility* V_t, and the expression $(\mu - r)/f(V_t)$ represents the *excess return-to-risk ratio* (i.e., (excess return) / risk). Mathematically, we may and will allow function f to depend on both variables X_t and V_t, i.e., $f_t = f(X_t, V_t)$ in analogy with γ and ρ.

As proposed in [16, Sect. 4], let us consider a European option written in this market with *payoff* $\hat{h}(S_T) \geqslant 0$ at maturity $T > 0$. We assume that $h = \hat{h} \circ \exp \in L^2(\mathbb{R})$, so that $\hat{h}(\exp(x)) = h(x)$, $x \in \mathbb{R}$, and $\hat{h}(S_T) = h(X_T)$. As usual, for $x \in \mathbb{R}$ we abbreviate $x^+ \stackrel{\text{def}}{=} \max\{x, 0\}$ and $x^- \stackrel{\text{def}}{=} \max\{-x, 0\}$.

Remark 8.1. In order to be able to treat a more realistic choice of the *payoff function* $\hat{h} : \mathbb{R}_+ \to \mathbb{R}$, such as $\hat{h}$ is only bounded (for a put option given via $\hat{h}(s) = (K - s)^+$) or growing linearly $0 \leqslant \hat{h}(s) \leqslant c_1 s + c_2$ (for a call option given via $\hat{h}(s) = (s - K)^+$), where $K > 0$ is the strike price, $c_1, c_2 \geqslant 0$ are some constants, and $s \geqslant 0$, we would have to make the following changes: Replace the unknown function $p(x, v, t)$ below (the arbitrage-free option price), where $x = \ln s \in \mathbb{R}$, by the product $(k + x^2)^{-1}(\cosh v)^{-1} p(x, v, t)$ for a put option and by $\mathrm{e}^{-kx}(\cosh v)^{-1} p(x, v, t)$ for a call option, respectively, where the constant $k > 0$ has to be taken large enough. We refer to F. Baustian [5, Chapt. 4, pp. 91–93] or C. Erdmann [18, Chapt. 5, pp. 179–182] for this replacement procedure which is a matter of a simple, routine computation. For the convenience of an interested reader, we present another replacement (substitution) procedure below that works for both, European put and call options simultaneously:

We denote $\Pi(x, v) \stackrel{\text{def}}{=} \cosh x \cosh v$ for $(x, v) \in \mathbb{R}^2$. Given a constant $k > 0$ that has to be taken large enough also here, for the unknown function $p(x, v, t)$ we substitute $q(x, v, t) = p(x, v, t)/\Pi(kx, v)$ and derive the corresponding parabolic partial differential equation for the new unknown function $q(x, v, T - t)$ in place of $p(x, v, T - t)$, for all $(x, v) \in \mathbb{R}^2$ and $t \in \mathbb{R}_+$. In place of the time variable t we use $T - t$ ($\leqslant T$) so that from now on $t \in \mathbb{R}_+$ stands for the *time-to-maturity*. In case $h(x) = (\mathrm{e}^x - K)^{\pm}$ for $x \in \mathbb{R}$, one may choose any $k > 1$; this choice guarantees that the function $x \mapsto h(x)/\cosh kx$ belongs to $L^2(\mathbb{R})$. First, we calculate the partial derivatives of p with respect to x,

$$\frac{\partial p}{\partial x} = \left(\frac{\partial q}{\partial x} + kq(x, v, t)\tanh kx\right)\Pi(kx, v),$$

$$\frac{\partial^2 p}{\partial x^2} = \left(\frac{\partial^2 q}{\partial x^2} + 2k \cdot \frac{\partial q}{\partial x} \cdot \tanh kx + k^2 q(x, v, t)\right)\Pi(kx, v),$$

then the partial derivatives of p with respect to v and t,

$$\frac{\partial p}{\partial v} = \left(\frac{\partial q}{\partial v} + q(x, v, t)\tanh v\right)\Pi(kx, v),$$

$$\frac{\partial p}{\partial t} = \frac{\partial q}{\partial t} \cdot \Pi(kx, v), \qquad p(x, v, t) = q(x, v, t)\Pi(kx, v),$$

$$\frac{\partial^2 p}{\partial v^2} = \left(\frac{\partial^2 q}{\partial v^2} + 2 \cdot \frac{\partial q}{\partial v} \cdot \tanh v + q(x, v, t)\right)\Pi(kx, v),$$

$$\frac{\partial^2 p}{\partial x \partial v} = \left(\frac{\partial^2 q}{\partial x \partial v} + \frac{\partial q}{\partial x} \cdot \tanh v + \frac{\partial q}{\partial v} \cdot k \tanh kx \right. \\ \left. + q(x, v, t) \cdot k \tanh kx \tanh v \right) \Pi(kx, v).$$

It is obvious that, apart from the multiplicative factor $\Pi(kx, v)$, all partial derivatives of q on the right-hand sides of these formulas have *bounded*, real-analytic coefficients, thanks to $|\tanh x| < 1$ for all $x \in \mathbb{R}$. Their analyticity and boundedness extend easily to the complex strip

$$\mathbb{R} + \mathrm{i}(-r, r) = \{ z = x + \mathrm{i}y \in \mathbb{C}\colon\ x \in \mathbb{R},\ y \in (-r, r) \}$$

in $\mathbb{C}$, whenever $r > 0$ is small enough, such that $kr < \pi/2$.

From the formulas above it follows immediately that if $p(x, v, T - t)$ verifies the (initial value) Cauchy problem (1) then also $u(x, v, t) = q(x, v, T - t)$ verifies an analogous Cauchy problem, with the *same leading coefficients* (for the second-order partial derivatives) and all the remaining coefficients (for the first- and zeroth-order partial derivatives) satisfying the *same hypotheses* as those in Eq. (1) for $p(x, v, T - t)$; cf. Eqs. (65) and (66) below. Thanks to the factor $\Pi(kx, v) = \cosh kx \cosh v$, we observe that the function $(x, v) \mapsto q(x, v, T) = h(x)/\Pi(kx, v)$ belongs to $L^2(\mathbb{R}^2)$ at maturity T. We conclude that it suffices to consider the Cauchy problem (1) for q in place of p; i.e, for q in the Banach space $C([0, T] \to L^2(\mathbb{R}^2))$ upon the time variable substitution $t \mapsto T - t$. A more specific form of the Cauchy problem for q, rendered by system (63), is provided in (66) below. □

The *arbitrage-free price* A_t^h of the European option at time $t \in (0, T)$ is thus given by the expectation formula (with respect to the risk neutral probability measure $\mathbb{P}$)

$$A_t^h = \mathrm{e}^{-r(T-t)} \mathbb{E}_{\mathbb{P}}\big[\hat{h}(S_T) \,\big|\, \mathcal{F}_t\big] = \mathrm{e}^{-r(T-t)} \mathbb{E}_{\mathbb{P}}\big[h(X_T) \,\big|\, \mathcal{F}_t\big]. \tag{64}$$

From the Markov property of (X_t, V_t) and the Feynman–Kac formula (cf. A. Friedman [26, Chapt. 6] or B. Øksendal [63, Chapt. 8]) we deduce $A_t^h = p(X_t, V_t, t)$ where p solves the (terminal value) Cauchy problem

$$\begin{cases} \dfrac{\partial p}{\partial t} + \mathcal{G}_t p - rp = 0, & (x, v, t) \in \mathbb{R}^2 \times (0, T), \\ p(x, v, T) = h(x), & (x, v) \in \mathbb{R}^2, \end{cases} \tag{65}$$

with $\mathcal{G}_t$ being the (time-independent) infinitesimal generator of the time-homogeneous Markov process (X_t, V_t); cf. A. Friedman [26, Chapt. 2, p. 30] and the rigorous definition in [26, Eq. (3.5), p. 31]. Hence, the function $u : (x, v, t) \mapsto p(x, v, T - t)$ verifies a Cauchy problem of type (1) written in the following (general, nondivergence) form,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{N} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{j=1}^{N} b_j(x, t) \frac{\partial u}{\partial x_j} & \\ \quad - c(x, t)u = f(x, t) & \text{for } (x, t) \in \mathbb{R}^N \times (0, T); \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{66}$$

with the initial data $u_0(x,v) = p(x,v,T) = h(x,v)$ (at $t = 0$) and the coefficients

$$a(x,v,t) = \frac{1}{2}\begin{pmatrix} f(v)^2 & \beta f(v)\rho(x,v) \\ \beta f(v)\rho(x,v) & \beta^2 \end{pmatrix} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}},$$

$$b(x,v,t) = \begin{pmatrix} r - \frac{1}{2}f(v)^2 \\ \alpha(m-v) - \beta(\rho(x,v)\frac{\mu-r}{f(v)} + \gamma(x,v)\sqrt{1-\rho(x,v)^2}) \end{pmatrix} \in \mathbb{R}^2,$$

$$c(x,v,t) = -r \in \mathbb{R},$$

where the variable $x \in \mathbb{R}^N$ in (66) has been replaced by $(x,v) \in \mathbb{R}^2$.

Recalling the divergence form (10) of the operator $\mathbf{P}(x,t,\frac{1}{\mathrm{i}}\frac{\partial}{\partial x})$ from the Cauchy problem (1), we could rewrite the new Cauchy problem (66) in the divergence form as well, but this would not simplify our analysis here at all.

Since the coefficient $b = \binom{b_1}{b_2}$ in the drift term is unbounded as $|v| \to \infty$, because $b_2(x,v,t)$ contains the expression $\alpha(m-v)$, a suitable change of variables will turn out to be more convenient. A further change of variables also eliminates the coefficient c. Although we could change variables directly at the level of the Cauchy problem, it is more natural to change variables by altering the underlying stochastic processes (X_t, V_t) in order to obtain a possible financial interpretation of the new variables. To this end, we recall $X_t = \ln S_t$ and let

$$\tilde{S}_t = \mathrm{e}^{r(T-t)}S_t, \qquad \tilde{X}_t = \ln\tilde{S}_t = X_t + r(T-t), \quad \text{and}$$
$$\tilde{V}_t = m + \mathrm{e}^{-\alpha(T-t)}(V_t - m). \tag{67}$$

Then $\tilde{S}_t$ is the *forward price process* which is a $\mathbb{P}$-martingale. Itô's formula readily implies that the pair $(\tilde{X}_t, \tilde{V}_t)_{t\geqslant 0}$ verifies (cf. Eq. (63))

$$\begin{cases} \mathrm{d}\tilde{X}_t = -\dfrac{\tilde{f}(\tilde{V}_t,t)^2}{2}\,\mathrm{d}t + \tilde{f}(\tilde{V}_t,t)\,\mathrm{d}W_t, \\ \mathrm{d}\tilde{V}_t = -\tilde{\beta}(t)\Big[\Big(\rho_t\dfrac{\mu-r}{f(\tilde{V}_t,t)} + \gamma_t\sqrt{1-\rho_t^2}\Big)\mathrm{d}t - (\rho_t\,\mathrm{d}W_t + \sqrt{1-\rho_t^2}\,\mathrm{d}Z_t)\Big], \end{cases} \tag{68}$$

where $\tilde{\beta}(t) = \mathrm{e}^{-\alpha(T-t)}\beta$, $\tilde{f}(\tilde{V}_t,t) = f(V_t)$ which means that

$$\tilde{f}(v,t) = f\big(m + \mathrm{e}^{\alpha(T-t)}(v-m)\big) \quad \text{for } (v,t) \in \mathbb{R}\times[0,T],$$

$\rho_t = \rho(X_t,V_t) = \tilde{\rho}(\tilde{X}_t,\tilde{V}_t,t)$, and $\gamma_t = \gamma(X_t,V_t) = \tilde{\gamma}(\tilde{X}_t,\tilde{V}_t,t)$. Finally, we define $\tilde{p}$ by

$$\tilde{p}(\tilde{X}_t,\tilde{V}_t,t) = \mathrm{e}^{r(T-t)}A_t^h = \mathrm{e}^{r(T-t)}p(X_t,V_t,t) = \mathbb{E}_{\mathbb{P}}\big[h(X_T)\,\big|\,\mathcal{F}_t\big], \tag{69}$$

so that we have the *arbitrage-free price* relation

$$\tilde{p}(x,v,t) = \mathrm{e}^{r(T-t)}p\big(x - r(T-t), m + \mathrm{e}^{\alpha(T-t)}(v-m), t\big). \tag{70}$$

Similarly as for the pair (X_t, V_t) above, from Eq. (69) and the Feynman–Kac formula (cf. A. FRIEDMAN [26, Chapt. 6] or B. ØKSENDAL [63, Chapt. 8]) we deduce, this time using the Markov property of $(\tilde{X}_t, \tilde{V}_t)$, that $\tilde{p}$ verifies the (terminal value) Cauchy problem

$$\begin{cases} \dfrac{\partial \tilde{p}}{\partial t} + \tilde{\mathcal{G}}_t \tilde{p} = 0, & (x, v, t) \in \mathbb{R}^2 \times (0, T); \\ \tilde{p}(x, v, T) = h(x), & (x, v) \in \mathbb{R}^2, \end{cases} \tag{71}$$

where now $\tilde{\mathcal{G}}_t$ is the (*time-dependent*) "infinitesimal generator" of the time-inhomogeneous Markov process $(\tilde{X}_t, \tilde{V}_t)$. This "generator" is ***not*** infinitesimal in the usual sense, because it depends on time t; cf. A. FRIEDMAN [26, Chapt. 2, p. 30] and the appropriate, more general definition in [26, Eq. (3.7), p. 31]. Note that we still use the pair of variables (x, v), but now this pair corresponds to the process $(\tilde{X}_t, \tilde{V}_t)$. This is a Markov process which is *not* time-homogeneous (i.e., Eq. (71) is *not* autonomous). Setting $u(x, v, t) = \tilde{p}(x, v, T - t)$, we deduce that u solves the Cauchy problem (66) with the initial data $u(x, v, 0) = h(x, v) = h(x)$ and the coefficients $a(x, v, t) \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$, $b(x, v, t) \in \mathbb{R}^2$, and $c(x, v, t) \in \mathbb{R}$ given by

$$a(x, v, T - t) = \frac{1}{2} \begin{pmatrix} \tilde{f}(v, t)^2 & \tilde{\beta}(t) \tilde{f}(v, t) \tilde{\rho}(x, v, t) \\ \tilde{\beta}(t) \tilde{f}(v, t) \tilde{\rho}(x, v, t) & \tilde{\beta}^2(t) \end{pmatrix},$$

$$b(x, v, T - t) = \begin{pmatrix} -\frac{1}{2} \tilde{f}(v, t)^2 \\ -\tilde{\beta}(t)(\tilde{\rho}(x, v, t) \frac{\mu - r}{\tilde{f}(v,t)} + \tilde{\gamma}(x, v, t)\sqrt{1 - \tilde{\rho}(x, v, t)^2}) \end{pmatrix},$$

$$c(x, v, t) = 0.$$

Recall that $\tilde{\beta}(t) = \mathrm{e}^{-\alpha(T-t)}\beta$ for $t \in [0, T]$.

The purpose of the change of coordinates in (67) was to make the coefficient $b = \binom{b_1}{b_2}$ in the drift term of Eq. (66) *bounded* (by eliminating the first summand $\alpha(m - v)$ in $b_2(x, v, t)$) and the coefficient c *vanish* entirely. This substitution does not harm the ellipticity or boundedness hypotheses on the coefficients as long as time t stays in the bounded interval $0 \leqslant t \leqslant T$ (or in a bounded complex neighborhood thereof, say, $\mathfrak{T}^{(\delta)}_{-\delta, T+\delta}$ for some $\delta > 0$, defined before Theorem 1.1), thanks to $\partial \tilde{v} / \partial v = \mathrm{e}^{-\alpha(T-t)}$.

We assume that the (original) functions f, γ, and ρ in (63) satisfy the analyticity hypothesis (H4) stated below, after Example 8.3. To formulate this hypothesis, let us define the complex planar domains

$$\nabla^{(r)}_{\vartheta} \stackrel{\text{def}}{=} \{\zeta = \xi \mathrm{e}^{\mathrm{i}\theta} + \mathrm{i}\eta \in \mathbb{C}\colon \xi \in \mathbb{R},\ \eta \in (-r, r),\ \text{and } |\theta| < \vartheta\}, \tag{72}$$

$$\nabla^{(r)}_{0} \stackrel{\text{def}}{=} \{\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}\colon \xi \in \mathbb{R},\ \eta \in (-r, r)\} = \bigcap_{0<\vartheta<\pi/2} \nabla^{(r)}_{\vartheta} = \mathbb{R} + \mathrm{i}(-r, r) \tag{73}$$

for $r \in (0, \infty)$ and $0 < \vartheta < \pi/2$, the latter being a strip in $\mathbb{C}$. Their respective closures in $\mathbb{C}$ are denoted by $\overline{\nabla}^{(r)}_{\vartheta}$ and $\overline{\nabla}^{(r)}_{0}$; both contain the origin $0 \in \mathbb{C}$. Notice the relation $\mathfrak{X}^{(r)} = (\nabla^{(r)}_0)^N \subset \mathbb{C}^N$.

The following is a typical example of a function that is analytic and bounded in $\nabla^{(r_0)}_{\vartheta_0}$, whenever $r_0 \in (0, \infty)$ and $0 < \vartheta_0 < \pi/2$:

Example 8.2. To begin with, let us remark that the domain defined in (72) takes the form

$$\nabla_{\vartheta}^{(r)} = \bigcup_{\eta\in(-r,r)} \left(\mathrm{i}\eta + \nabla_{\vartheta}^{(0)}\right) \subset \mathbb{C} \tag{74}$$

for any given numbers $r \in (0,\infty)$ and $0 < \vartheta < \pi/2$, where

$$\nabla_{\vartheta}^{(0)} \stackrel{\text{def}}{=} \left\{\zeta = \xi \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}\colon\ \xi \in \mathbb{R} \text{ and } |\theta| < \vartheta\right\} = (\Delta_\vartheta) \cup (-\Delta_\vartheta) \cup \{0\} \tag{75}$$

is a *symmetric sector* in $\mathbb{C}$. Recall that the open sector Δ_ϑ has been defined in (6).

Now, as an ***example***, let us consider the function

$$f(v) = (f_+ - f_-)\frac{1}{\pi}\arctan(v) + \frac{1}{2}(f_+ + f_-) \quad \text{defined for every } v \in \mathbb{R},$$

where f_- and f_+ are given constants with $0 < f_- < f_+ < \infty$. Hence, we have $f_- < f(v) < f_+$ together with $\lim_{v\to\pm\infty} f(v) = f_\pm$, respectively, and $f' > 0$ in $\mathbb{R}$. Such function $f : \mathbb{R} \to (f_-, f_+)$ gives a simple *"model" relation* between the stochastic volatility V_t and the instantaneous volatility level $f(V_t)$. Taking advantage of the complex extension of the derivative of f furnished by

$$f'(z) = \frac{f_+ - f_-}{\pi}\cdot\frac{1}{1+z^2} = \frac{f_+ - f_-}{2\pi}\left(\frac{1}{1+\mathrm{i}z} + \frac{1}{1-\mathrm{i}z}\right), \quad z \in \mathbb{C}\setminus\{\pm\mathrm{i}\},$$

we can extend f as a holomorphic function (i.e., analytic and single-valued) via the following formula,

$$f(z) = \frac{\mathrm{i}}{2\pi}(f_+ - f_-)\cdot\log\left(\frac{1-\mathrm{i}z}{1+\mathrm{i}z}\right) + \frac{1}{2}(f_+ + f_-)$$
$$\text{for every } z \in \mathcal{O} \stackrel{\text{def}}{=} \mathbb{C}\setminus\left\{\mathrm{i}y\colon\ y \in (-\infty,-1]\cup[1,\infty)\right\},$$

by virtue of the argument formula $\arg(1+\mathrm{i}y) = \arctan(y)$ for $y \in \mathbb{R}$. This extension of f to the domain $\mathcal{O} = \mathbb{C}\setminus \pm\mathrm{i}[1,\infty) \equiv \mathbb{C}\setminus\{\pm\mathrm{i}y\colon\ y \in [1,\infty)\}$ is holomorphic, thanks to the argument restriction $\arg(\frac{1-\mathrm{i}z}{1+\mathrm{i}z}) \in (-\pi,\pi)$ for $z \in \mathcal{O}$. Indeed, notice that in the argument of $\log(\frac{1-\mathrm{i}z}{1+\mathrm{i}z}) = \ln|\frac{1-\mathrm{i}z}{1+\mathrm{i}z}| + \mathrm{i}\cdot\arg(\frac{1-\mathrm{i}z}{1+\mathrm{i}z})$ we have the ratio

$$\frac{1-\mathrm{i}z}{1+\mathrm{i}z} = -\frac{|z|^2-1}{|z|^2+1-2\cdot\Im\mathfrak{m}\,z} - 2\mathrm{i}\cdot\frac{\Re\mathfrak{e}\,z}{|z|^2+1-2\cdot\Im\mathfrak{m}\,z} \tag{76}$$

which yields

$$|z| < 1 \quad\Longrightarrow\quad \arg\left(\frac{1-\mathrm{i}z}{1+\mathrm{i}z}\right) = -\arctan\left(\frac{2\cdot\Re\mathfrak{e}\,z}{1-|z|^2}\right) \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right), \tag{77}$$

$$|z| > 1 \quad\Longrightarrow\quad \arg\left(\frac{1-\mathrm{i}z}{1+\mathrm{i}z}\right) = \begin{cases} -\pi + \arctan(\frac{2\cdot\Re\mathfrak{e}\,z}{|z|^2-1}) & \text{if } \Re\mathfrak{e}\,z > 0; \\ \pi + \arctan(\frac{2\cdot\Re\mathfrak{e}\,z}{|z|^2-1}) & \text{if } \Re\mathfrak{e}\,z < 0. \end{cases} \tag{78}$$

We note the important fact that the domain $\mathcal{O}$ contains the *closure* $\overline{\nabla}_{\vartheta}^{(r)}$ *of the subdomain* $\nabla_{\vartheta}^{(r)}$ whenever $0 < r < 1$ and $0 < \vartheta < \pi/2$ (see Eq. (74)). Furthermore, for any $\delta > 0$ we have

$$M_\delta \stackrel{\text{def}}{=} \sup_{\substack{z\in\mathcal{O} \\ |z+\mathrm{i}|\geqslant\delta,\ |z-\mathrm{i}|\geqslant\delta}} \left|f(z)\right| < \infty. \tag{79}$$

Clearly, we get also $f(0) = \frac{1}{2}(f_+ + f_-)$ together with the limits

$$\begin{cases} f(z) \to f_+ & \text{as } z \to \infty \text{ with } z \in \mathbb{C},\ \mathfrak{Re}\, z > 0; \\ f(z) \to f_- & \text{as } z \to \infty \text{ with } z \in \mathbb{C},\ \mathfrak{Re}\, z < 0, \end{cases} \tag{80}$$

where $z \to \infty$ for $z \in \mathbb{C}$ means precisely $|z| \to \infty$. These claims follow from the fact that in the argument of $\log(\frac{1-\mathrm{i}z}{1+\mathrm{i}z})$ we have the ratio (cf. (76))

$$\frac{1-\mathrm{i}z}{1+\mathrm{i}z} \longrightarrow -1 \quad \text{as } |z| \to \infty,$$

whence the sign of the real part $\mathfrak{Re}\, z$ decides about the limit of $f(z)$ as $|z| \to \infty$, by (78). In particular, the domain $\mathbb{C} \setminus \pm\mathrm{i}[1,\infty)$ contains the strip $\mathbb{R} \times \mathrm{i}(-r_0, r_0)$ whenever $0 < r_0 < 1$. The imaginary part of $f(z)$ is uniformly bounded for $|\mathfrak{Im}\, z| < r_0$.

Example 8.3. Another ***example*** of a suitable simple choice of f in Example 8.2 is the function

$$f(v) = \frac{1}{2}(f_+ - f_-)\cdot \tanh v + \frac{1}{2}(f_+ + f_-) \quad \text{defined for every } v \in \mathbb{R},$$

where f_- and f_+ are given constants with $0 < f_- < f_+ < \infty$. Again, we have $f_- < f(v) < f_+$ together with $\lim_{v\to\pm\infty} f(v) = f_\pm$, respectively, and $f' > 0$ in $\mathbb{R}$.

This choice of function f has analogous properties as in Example 8.2. Here, it is helpful to write $\tanh v = \frac{z^2-1}{z^2+1} = -\frac{(1-z)(1+z)}{(1-\mathrm{i}z)(1+\mathrm{i}z)}$ with $z = \mathrm{e}^v$ for $v \in \overline{\nabla}_{\vartheta}^{(r)} \subset \mathbb{C}$. In particular, we have $\tanh v \to 1$ as $|z| = \mathrm{e}^{\mathfrak{Re}\, v} \to \infty$ (i.e., as $\mathfrak{Re}\, v \to +\infty$), whereas $\tanh v \to -1$ as $|z| = \mathrm{e}^{\mathfrak{Re}\, v} \to 0$ (i.e., as $\mathfrak{Re}\, v \to -\infty$). □

In both Examples 8.2 and 8.3, the restriction $f|_{\mathbb{R}} : \mathbb{R} \to (f_-, f_+)$ of the complex function $f : \overline{\nabla}_{\vartheta}^{(r)} \subset \mathbb{C} \to \mathbb{C}$ to the real line is real-analytic, bounded, and strictly monotone increasing, thus reflecting the fact that the *diffusion coefficient* $\sigma_t = f(V_t)$ (i.e., the *instantaneous volatility level*) in the first equation of system (63) is a real-analytic, bounded, and strictly monotone increasing function of the *stochastic volatility* V_t. In contrast, in [36,42,68] the function f is neither bounded nor is it smooth at zero (see [22, Table 2.1, p. 42]).

Motivated by these two examples, we impose the following analyticity hypothesis on f, γ, and ρ:

Hypothesis.

(H4) All functions $f, \gamma, \rho : \mathbb{R}^2 \to \mathbb{R}$ of the pair of variables $(x, v) \in \mathbb{R}^2$ can be extended to bounded holomorphic functions $f, \gamma, \rho : \Omega \to \mathbb{C}$ on the complex domain $\Omega =$

$\nabla_0^{(r_0)} \times \nabla_{\vartheta_0}^{(r_0)} \subset \mathbb{C}^2$ for some $r_0 \in (0, \infty)$ and $0 < \vartheta_0 < \pi/2$. Moreover, for the images of $\mathbb{R}^2$ we assume $f(\mathbb{R}^2) \subset (f_-, f_+)$ for some $-\infty < f_- < f_+ < \infty$ and $\gamma(\mathbb{R}^2) \subset (\gamma_-, \gamma_+)$ for some $-\infty < \gamma_- < \gamma_+ < \infty$, and the real part of ρ satisfies $\Re\mathfrak{e}\, \rho(\Omega) \subset (-\rho_0', \rho_0')$ for some $\rho_0' \in (\rho_0, 1)$.

Last but not least, we would like to mention that negative values of ρ are *not* unusual in a volatile market: asset prices tend to go down when volatility goes up (see [22, p. 41]).

Example 8.4. A canonical choice suggested in [22, p. 46] is to take both γ and ρ constant (determined from common experience and empirical studies), and the function f from Example 8.2 above, so that f is analytic and bounded in the domain $\nabla_{\vartheta_0}^{(r_0)}$ for some $0 < r_0 < 1$ and $0 < \vartheta_0 < \pi/2$. (This domain contains the strip $\nabla_0^{(r_0)}$.)

In order to guarantee that the holomorphic extension of the function f in Example 8.2 (denoted by f again) stays in the correct domain, i.e., is holomorphic (meaning analytic and single-valued) in $\mathbb{C} \setminus \pm\mathrm{i}[1, \infty) = \mathbb{C} \setminus \{\pm\mathrm{i}y\colon\ y \in [1, \infty)\}$, we need to take the angle $\vartheta_0 > 0$ of the time sector Δ_{ϑ_0} (cf. (6) and (7)) to be sufficiently small, such that the argument restriction $\arg(\frac{1-\mathrm{i}z}{1+\mathrm{i}z}) \in (-\pi, \pi)$ for $z \in \mathbb{C} \setminus \pm\mathrm{i}[1, \infty)$ stays preserved also for the product $f(v)\mathrm{e}^{\alpha(T-t)}$ whenever $t \in [0, T]$. In contrast, if $\vartheta_0 \in (0, \pi/2)$ is arbitrary, but fixed, we need to take the constant $\alpha > 0$ (which is the *rate of mean reversion*) small enough, such that the argument restriction above again stays preserved. (Thus, fixing first ϑ_0 ($0 < \vartheta_0 < \pi/2$) would force an unpleasant smallness restriction on the rate of mean reversion $\alpha > 0$.) More precisely, in the transformation (67) we need to perform also the change of variables $\tilde{V}_t = m + \mathrm{e}^{-\alpha(T-t)}(V_t - m)$. Notice that for $t = \varrho \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}$ with $0 < \varrho < T/\cos\vartheta_0$ and $\theta \in (-\vartheta_0, \vartheta_0)$ we have

$$\mathrm{e}^{-\alpha(T-t)} = \mathrm{e}^{-\alpha T}\mathrm{e}^{\alpha\varrho\cos\theta}\mathrm{e}^{\mathrm{i}\alpha\varrho\sin\theta} = \mathrm{e}^{\alpha(\varrho\cos\theta - T)}\big[\cos(\alpha\varrho\sin\theta) + \mathrm{i}\sin(\alpha\varrho\sin\theta)\big]$$

with the imaginary part of the last term being arbitrarily small provided the product $\alpha \cdot \sin\theta$ is small enough. Given any fixed constant $\alpha > 0$, this condition can be fulfilled by assuming $|\theta| < \vartheta_0$ with $\vartheta_0 \equiv \vartheta_0(\alpha) > 0$ small enough, as indicated above. □

We note that for the discussion below we could further generalize the model by taking α, m, and β to be functions of (X_t, V_t) and, in fact, we could let all the functions depend on (X_t, V_t, t) while keeping analyticity in $\Omega \times \Delta_{\vartheta_0}^{T_0,T} = \nabla_0^{(r_0)} \times \nabla_{\vartheta_0}^{(r_0)} \times \Delta_{\vartheta_0}^{T_0,T} \subset \mathbb{C}^3$, for some $r_0 \in (0, \infty)$, $0 < \vartheta_0 < \pi/2$, and $0 < T' \leqslant T < \infty$, in accordance with the analyticity hypothesis (H4). (Recall that $\Delta_{\vartheta_0}^{T_0,T} \subset \mathbb{C}$ has been defined in Eq. (8).)

As we have already mentioned above, our present motivation to study the space–time analyticity of solutions to problem (1) comes from *Mathematical Finance*. More precisely, we are interested in market completeness when the set of tradable assets includes not only the underlying stock and a bond, but also some *options* written in the market, i.e., we want to understand whether dynamic trading in such an enlarged set of assets allows (theoretically) to perfectly hedge risk associated with any potential financial instrument introduced in the market. In a recent paper, M.H.A. DAVIS and J. OBŁÓJ [16] studied such a setup treated earlier in M. ROMANO and N. TOUZI [65, Sect. 3] for European options in a stochastic volatility model. They have essentially shown that the market is completed by *tradable options* if their prices are real analytic functions of space and time [16, Proposition 5.1, p. 55]. Their proof is based on a more general result [16, Theorem 4.1, p. 53] that the market is complete if and only if a certain matrix G is non-degenerate, where the entries of G are given by the first partial derivatives of the option prices with respect to the stock price and the stochastic volatility. Under analyticity assumption on option prices, the determinant $\det G$ itself is real analytic and hence its set of zeros is either Lebesgue negligible (i.e., of zero Lebesgue measure) or else it is the whole domain $\mathbb{R}^N$ (cf. S.G. KRANTZ and H.R. PARKS [54, p. 83]). In consequence it suffices to examine $\det G$ in an arbitrarily small neighborhood of a single ("central") point. The present work complements results in [16] in that it answers the fundamental question posed in [16, Sect. 7] whether the option prices, which are solutions to problem (1) via the Feynman–Kac formula, are ***real analytic*** in $(x, t) \in \mathbb{R}^N \times (0, \infty)$.

The definition of a ***complete market*** in the *"martingale model"* is given in [16, Definition 3.1, p. 51] in probabilistic (and measure-theoretic) terms: In our stochastic volatility model (63), *every contingent claim can be replicated by a self-financing trading strategy in the stock and bond* (contingent claims can be perfectly hedged against risks). Ref. [16] also provides two types of necessary and sufficient conditions for a market to be *complete*, one probabilistic and the other analytic, respectively, see Theorems 3.2 (p. 51), and 4.1 (p. 53, with Corollary 4.2). We will show that all hypotheses of [16, Corollary 4.2, p. 53] are satisfied in our model, cf. [16, Proposition 5.1, p. 55]. Under quite different sufficient conditions, a related result on market completeness is established in [65, Theorem 3.1, p. 406]: *A single European call option completes the market when there is stochastic volatility driven by one extra Brownian motion* (under some additional assumptions; see [65, pp. 404–407]).

To this end, for any fixed time $t \in (0, T)$, let us consider the Jacobian matrix

$$G(x, v, t) = \begin{pmatrix} 1, & 0 \\ \frac{\partial \tilde{p}}{\partial x}(x, v, t), & \frac{\partial \tilde{p}}{\partial v}(x, v, t) \end{pmatrix}$$

of the mapping $(x, v) \mapsto (x, \tilde{p}(x, v, t)) : \mathbb{R}^2 \to \mathbb{R}^2$ defined in Eq. (70), whose determinant is thus given by $\det G(x, v, t) = \frac{\partial \tilde{p}}{\partial v}(x, v, t)$. By the main result of this article, Theorem 3.3 (Section 3), function $\tilde{p} : \mathbb{R}^2 \times (0, T) \to \mathbb{R}$ can be (uniquely) extended to a holomorphic function in the domain $\Gamma_T^{(T_0)}(\kappa_0, \nu_0) \subset \mathbb{C}^2 \times \mathbb{C}$ and, consequently, the same is true of $\det G : \mathbb{R}^2 \times (0, T) \to \mathbb{R}$.

Finally, we can apply Proposition 5.1 (and its proof) from [16, p. 55] to conclude that our stochastic volatility model (63) with a European call option is ***complete***:

Theorem 8.5. *Assume that functions $f, \gamma, \rho : \mathbb{R}^2 \to \mathbb{R}$ satisfy the analyticity hypothesis* (H4). *Furthermore, assume that the payoff function $h = \hat{h} \circ \exp$ is* not *affine, that is, $h''(x) = 0$ does not hold for every $x \in \mathbb{R}$. Then the stochastic volatility model* (63) *with a European call option yields a* complete market.

Remark 8.6. Although our Theorem 3.3 (Section 3) allows to consider directly only a payoff function $h = \hat{h} \circ \exp \in L^2(\mathbb{R})$, the procedure indicated in Remark 8.1 allows for more general, even unbounded payoff functions with no faster than linear growth near infinity, cf. [5, Chapt. 4] or [18, Chapt. 5].

Remark 8.7 *(Other stochastic volatility models and complete markets).* (i) As we have already mentioned at the beginning of this section, there are a few other stochastic volatility models that may be even more popular than the one (system (63)) we have investigated in this section. Some of them are listed in [22, Table 2.1, p. 42] including those in [36,42,68], others can be found in [55,72]. Our main analyticity result, Theorem 3.3 (Section 3), does not seem to be directly applicable to the stochastic volatility models in [36,55,68,72]. However, we are able to apply it to [42, Eq. (5), p. 283], in the special case when $\rho = 0$ (i.e., the volatility is *uncorrelated* with the stock price, studied also in [42]) and $\mu = 0$ (i.e., the volatility has zero drift rate, which is less realistic). Without any substantial restrictions, our analyticity result applies to [40, Proposition 2 and Eq. (4)]. In [52] the authors are interested only in the analyticity with respect to the time variable $t \in (0, 1]$. In their setting, if an appropriate analyticity hypothesis with respect to the space variable $x \in \mathbb{R}^N$ is added, then our main result, Theorem 3.3, yields all time analyticity results (established or quoted) in [52, Sect. 4]. In contrast, R. CARMONA and S. NADTOCHIY [13] and J. JACOD and P. PROTTER [43] work with martingales rather than with parabolic partial differential equations (of Black–Scholes-type) which makes it difficult to compare their results with ours.

(ii) Another alternative approach to market completeness, which yields *“endogenous completeness”* of a diffusion driven equilibrium market, has been investigated recently in J. HUGONNIER, S. MALAMUD, and E. TRUBOWITZ [40] and D. KRAMKOV and S. PREDOIU [52]. Also this approach imposes analyticity hypotheses on the *drift* and the *volatility coefficients* in order to obtain a complete market, the former [40] in both, space and time variables, the latter [52] in the time variable only. *“Endogenous completeness”* (called also *“dynamic completeness”*) has been introduced in the pioneering work by R.M. ANDERSON and R.C. RAIMONDO [2]. It is based on the existence of a dynamic equilibrium state in an Arrow–Debreu market model where agents are allowed to shift consumption across states and times by trading a complete set of Arrow–Debreu contingent claims.

9. Historical remarks and comments

It is remarkable that the questions we have studied in this paper have not been answered before, given numerous studies concerned with the regularity of solutions to linear parabolic systems. Maybe, because our motivation comes from new problems in *mathematical finance* and it is this new motivation which prompts us to ask questions somewhat different from the historically relevant ones. Broadly speaking, the existing literature on linear equations and systems of type (1) (or (3)) can be classified into two main streams: The first one has looked at general (interior) smoothness of *all* solutions to the Cauchy problem (1). Here, in the most regular cases one can prove that the solutions are real analytic in the space variables and of Gevrey class 2 in the time variable (not necessarily real analytic); see S.D. EIDEL′MAN [17, Theorem II.7.2] (and the references therein), L. HÖRMANDER [39, Sect. 11.4], and D. KINDERLEHRER and L. NIRENBERG [49, Theorem 1 (p. 285) and Theorem 2 (p. 306)]. It has been known since the work by E. HOLMGREN [37] that even the heat equation (in one space dimension!) has solutions that are *not* real analytic in the time variable (cf. G.G. BILODEAU [6, pp. 124–125]). This phenomenon is due to a possibly very rapid growth of the solutions as the spatial variable $x \in \mathbb{R}$ escapes to $\pm\infty$; to eliminate it one needs to restrict the functional space, where the solutions are considered at each time moment $t \in \mathbb{R}_+$, in order to prevent a too rapid growth of the solutions as $x \to \pm\infty$. This is precisely what has been done in the second main stream to which also our present article belongs.

Here, the emphasis is on the analytic dependence in time t and the Cauchy problem (1) is viewed as an evolutionary equation in some suitable functional space, e.g., $L^2(\mathbb{R})$ or $L^2(\mathbb{R}^N)$. Consequently, the solution is viewed as a vector-valued function $u : (0, T) \to L^2(\mathbb{R}^N)$ and, thus, regularity results (including analyticity results) have been obtained in this setting; see, for example, T. KATO and H. TANABE [48], H. KOMATSU [50], F.J. MASSEY III [58], K. MASUDA

[60], and in particular H. TANABE [71] and the references therein. In the present work we have used the Hardy space $H^2(\mathfrak{X}^{(r)})$ defined in Section 5, Eq. (48), to which the (classical) solution belongs for each fixed $t > 0$. If the initial data are complex analytic in $H^2(\mathfrak{X}^{(r_0)})$, for some constant $0 < r_0 < \infty$, then the solution belongs to the same space $H^2(\mathfrak{X}^{(r_0)})$ for every $t \in \mathbb{R}_+$, cf. Proposition 6.1. Of course, if the initial data u_0 are not complex analytic, say, only in $L^2(\mathbb{R}^N)$, then the constant $r = r(t) > 0$ depends on time $t > 0$ and, inevitably, we must have either $r(t) \to 0+$ as $t \to 0+$, or else $r_0 = \limsup_{t\to 0+} r(t) > 0$ and the H^2-norms $\|u(\cdot,t)\|_{H^2(\mathfrak{X}^{(r_0)})}$ are unbounded as $t \to 0+$. Namely, if, by contradiction, $r_0 = \limsup_{t\to 0+} r(t) > 0$ and the H^2-norms $\|u(\cdot,t)\|_{H^2(\mathfrak{X}^{(r_0)})}$ are bounded for all $t \in (0, t_0)$ $(0 < t_0 < \infty)$, then we would have complex analytic initial data $u_0 \in H^2(\mathfrak{X}^{(r_0)})$, by virtue of

$$\big\|u(\cdot + \mathrm{i}y, t) - u_0(\cdot + \mathrm{i}y)\big\|_{L^2(\mathbb{R}^N)} \longrightarrow 0 \quad \text{as } t \to 0+, \text{ for every } y \in Q^{(r_0)}.$$

We refer to J.L. BONA and F.B. WEISSLER [11] for a detailed treatment of a closely related phenomenon.

The idea of estimating (from below) the *radius of convergence* of the complex power series of the solution of a partial differential equation, by proving that it belongs to a Hardy space of type $H^2(\mathfrak{X}^{(r)})$, is due to J.L. BONA, Z. GRUJIĆ, and H. KALISCH [9, Theorem 4, p. 795], [10, Theorem 1, p. 187] and has been used also in P. TAKÁČ et al. [70]. However, the idea of using Banach spaces of holomorphic functions (over a temporal complex triangle $\Delta_\vartheta^{(T)}$ or a spatial complex strip $\mathfrak{X}^{(r)}$, respectively) may be traced back to the work by T. KATO and K. MASUDA [47] and N. HAYASHI [31–34]. The former, [47], apparently motivated by the general theory of holomorphic semigroups, has used complex analytic extension to the temporal triangle $\Delta_\vartheta^{(T)} \subset \mathbb{C}$ (i.e., with respect to the time variable, t) in order to verify the convergence of a complex power series. The latter, [31–34], has introduced the use of Bergman and Szegő spaces of holomorphic functions over a "sectorial" complex domain containing the spatial strip $\mathfrak{X}^{(r)} \subset \mathbb{C}^N$ (i.e., with respect to the space variable, x) for the Korteweg–de Vries (KdV, for short) and nonlinear Schrödinger (NLS) equations. The use of *Gevrey spaces* of real-analytic functions can be traced to C.S. KAHANE [45] and C. FOIAS and R. TEMAM [20,21]. We refer to A. DE BOUARD, N. HAYASHI, and K. KATO [12, p. 675] for a rigorous definition of a Gevrey space G_σ of order $\sigma \geqslant 1$. Somewhat more general *Bourgain–Gevrey spaces* X_σ have been introduced into the subject in [9, p. 785]; they facilitate better, more precise tracking of the radius of convergence mentioned above.

In general terms, the investigation of the *smoothing* (or *regularizing*) effect in evolutionary equations of parabolic type has a long history; see e.g. S.D. EIDEL′MAN [17], A. FRIEDMAN [23–25], A. PAZY [64], and H. TANABE [71] and numerous references therein. ***Analytic*** smoothing (or regularizing) effects, similar to those treated in our present article, in the space (x) and/or time (t) variable(s), have been obtained somewhat later, beginning with the theory of analytic semigroups (in an abstract Banach space), see e.g. the monographs by T. KATO [46], J.-L. LIONS [56], A. PAZY [64], and H. TANABE [71], and applying (extending) it to nonautonomous analytic evolutionary equations, see e.g. T. KATO and H. TANABE [48], H. KOMATSU [50], K. MASUDA [60], and H. TANABE [71]. Evolutionary equations exhibiting analytic smoothing effects may be split into the following two classes: ***dissipative*** and ***dispersive***. Practically all references mentioned above, [17,23–25,46,48,50,56,64,71], cover only the class of dissipative evolutionary equations. Their results establish only analyticity with respect to the time variable $t \in (0, T) \subset \mathbb{R}$. N. HAYASHI and K. KATO [35] establish an analogous time-analyticity result for the nonlinear

Schrödinger equation (NLS). The early (general) treatments on the analytic smoothing effect with respect to the space variable $x \in \mathbb{R}^N$ are given in C.S. KAHANE [45] and C. FOIAS and R. TEMAM [20,21]. They have inspired a number of later analyticity results, among them a rather general result obtained in P. TAKÁČ et al. [70, Theorem 2.1, p. 429] (see also references therein).

Our present method of verifying the analytic smoothing effect (and the method in [70], as well) is much closer related to the previous work on *dispersive* evolutionary equations of Mathematical Physics, such as the Korteweg–de Vries equation (KdV), the nonlinear Schrödinger equation (NLS), the Benjamin–Ono equation, the Boussinesq equation, the Kadomtsev–Petviashvili equation, and some of their generalized forms, as well. This work starts with the smoothing effect in classical (L^2-type) Sobolev spaces by P. CONSTANTIN and J.-C. SAUT [15]. It continues with the analytic smoothing effect with respect to the space variable $x \in \mathbb{R}^N$ in *Gevrey* and *Bourgain–Gevrey* spaces; see A. DE BOUARD, N. HAYASHI, and K. KATO [12], N. HAYASHI [31–34], all for the (NLS) and (KdV) equations, and the later work on the (generalized) (KdV) equation in J.L. BONA and Z. GRUJIĆ [8], J.L. BONA, Z. GRUJIĆ, and H. KALISCH [9,10], J.L. BONA and F.B. WEISSLER [11], Z. GRUJIĆ and H. KALISCH [27], and H. HANNAH, A.A. HIMONAS, and G. PETRONILHO [29,30]. The significant advantage of using spaces of analytic functions with respect to the space variable $x \in \mathbb{R}^N$ (of Bergman, Szegő, Gevrey, Bourgain–Gevrey, or Hardy type) is clearly shown by the continuity properties of the nonlinear terms in a particular dispersive equation.

In contrast with the *dissipative* evolutionary equations where, typically, there is no restriction on the size of the initial data (in the norm of a given abstract Banach space), nor on the length $T \in (0, \infty)$ of the time interval $(0, T)$ for the time-analyticity of the solution, the corresponding analyticity results for *dispersive* evolutionary equations (mentioned above) usually carry one of the following *two restrictions*: (a) sufficiently small initial data (in the norm of a given Banach space of analytic functions); or (b) sufficiently short length T ($2T$, respectively) of the time interval $[0, T)$ ($(-T, T) \subset \mathbb{R}$) in which the solution exists (i.e., locally in time and globally in space ($\mathbb{R}^N$)).

Finally, analyticity of solutions to elliptic and parabolic problems in a *bounded* spatial domain $\Omega \subset \mathbb{R}^N$ (with analytic boundary $\partial\Omega$) has been established in G. KOMATSU [51]. Analyticity in the space variable x and 2-nd Gevrey class regularity (weaker than analyticity) in the time variable t are established in A. CAVALLUCCI [14, Teorema 6.1, p. 166] for linear parabolic equations. Some results about the analyticity of solutions of nonlinear parabolic systems, which are related to ours, are stated in A. FRIEDMAN [23, Theorems 3 and 4] without proofs, and for linear elliptic systems in C.B. MORREY, JR., and L. NIRENBERG [62]. For the Navier–Stokes equations, such analyticity results have been established in K. MASUDA [61] and, with respect to the space variable $x \in \mathbb{R}^N$ only, earlier in C.S. KAHANE [45] and K. MASUDA [59]. These results state local analyticity of infinitely differentiable solutions without any description of their domain of holomorphy (i.e., domain of complex analyticity). Our present article provides such description in Theorem 3.3 and so do Refs. [9,10]. More results of global nature on the space analyticity can be found in C. BARDOS and S. BENACHOUR [4], C. FOIAS and R. TEMAM [20,21] and Z. GRUJIĆ and I. KUKAVICA [28].

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