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Extensions of convex and semiconvex functions and intervally thin sets

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ABSTRACT

We call $A \subset \mathbb{R}^N$ *intervally thin* if for all $x, y \in \mathbb{R}^N$ and $\varepsilon > 0$ there exist $x' \in B(x, \varepsilon)$, $y' \in B(y, \varepsilon)$ such that $[x', y'] \cap A = \emptyset$. Closed intervally thin sets behave like sets with measure zero (for example such a set cannot "disconnect" an open connected set). Let us also mention that if the $(N-1)$-dimensional Hausdorff measure of A is zero, then A is intervally thin. A function f is preconvex if it is convex on every convex subset of its domain. The consequence of our main theorem is the following: *Let U be an open subset of $\mathbb{R}^N$ and let A be a closed intervally thin subset of U. Then every preconvex function $f: U \setminus A \to \mathbb{R}$ can be uniquely extended (with preservation of preconvexity) onto U.* In fact we show that a more general version of this result holds for semiconvex functions.

1. Introduction

Convex functions and their generalizations play an important role in mathematics [8], in particular in optimization [1,2] and optimal control [3]. A special role is played by semiconcave and semiconvex functions considered by S. Cannarsa and C. Sinestrari [3]:

[3, Definition 2.1.1] Let S be a subset of $\mathbb{R}^N$. We say that a function $f: S \to \mathbb{R}$ is *semiconvex* if there exists a nondecreasing upper semicontinuous function $\omega: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\rho \to 0^+} \omega(\rho) = 0$ and

$$f\big(\alpha x + (1-\alpha)y\big) \leqslant \alpha f(x) + (1-\alpha) f(y) + \alpha(1-\alpha)\|x-y\|\omega\big(\|x-y\|\big) \quad \text{for } \alpha \in [0,1],$$

for any points $x, y \in S$ such that

$$[x, y] := \big\{\alpha x + (1-\alpha)y \colon \alpha \in [0,1]\big\} \subset S.$$

We call ω a modulus of semiconvexity of f. We will shortly say that f is ω-semiconvex.

Remark 1.1. We would like to mention that a similar generalization of convexity was independently developed by S. Rolewicz under the name of *paraconvexity* [10,11]. For related results concerning the general theory of *approximately convex functions* see also the results obtained by A. Házy and Zs. Páles [9,5,6,4] and by the authors [12,13].

A semiconvex function is locally Lipschitz on the interior of its domain, see [3, Theorem 2.1.7]. For more information on properties and applications of semiconvex functions we refer the reader to [3]. We reserve the name *preconvex function* for

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a 0-semiconvex function (in the case when the domain is open preconvex functions were considered in [7] under the name of locally convex functions). Let us note that preconvexity does not depend on the norm.

Dealing with spaces of functions we naturally meet the problem of their extensions. More precisely, we ask when the extension with the preservation of its properties to a larger domain exists. Clearly the most satisfactory situation occurs when the extension exists and is uniquely determined. Such is the case for subharmonic functions.

[7, Theorem 3.4.3] *Let U be an open set in $\mathbb{R}^N$, A a closed polar subset and f a subharmonic function in $U \setminus A$ such that f is bounded above in $K \setminus A$ for every compact set $K \subset U$.*

Then f can be uniquely extended to a subharmonic function in U. If U is connected, then $U \setminus A$ is connected.

Preconvex functions can be seen as the “coordinate free” subharmonic functions (a function is preconvex iff it is subharmonic under an arbitrary affine change of coordinates, see [7, Theorem 3.2.28]). Our aim in this paper is to study a similar problem for preconvex and semiconvex functions. To do so we introduce and study intervally thin sets, which play a similar role for semiconvex functions as polar sets for subharmonic ones. Main results of the paper, Theorem 3.1 and Observation 4.2, can be summarized in the following theorem.

Theorem. *Let U be an open set in $\mathbb{R}^N$, A a closed intervally thin subset of U and f a semiconvex function on $U \setminus A$.*

Then f can be uniquely extended with the preservation of the modulus of semiconvexity onto U. If U is connected, then $U \setminus A$ is connected.

2. Semiconvex functions

The role of negligible sets for semiconvex functions is played by intervally thin sets.

Definition 2.1. Let $A \subset \mathbb{R}^N$. We call A *intervally thin* if for all $x, y \in \mathbb{R}^N$ and $\varepsilon > 0$ there exist $x' \in B(x, \varepsilon)$, $y' \in B(y, \varepsilon)$ such that

$$[x', y'] \cap A = \emptyset.$$

Since intervally thin sets play a basic role in our investigation, in the last section of the paper we present a detailed study of their properties.

For $A \subset \mathbb{R}^N$ and $\varepsilon > 0$ we put

$$A_\varepsilon := \{x \in \mathbb{R}^N \colon \operatorname{dist}(x; A) < \varepsilon\}.$$

We begin with the theorem which shows that continuous functions which are semiconvex almost everywhere (with respect to intervally thin sets) are semiconvex.

Theorem 2.1. *Let U be an open subset of $\mathbb{R}^N$, and let $f : U \to \mathbb{R}$ be a continuous function. Suppose that there exists an intervally thin set $A \subset U$ such that $f|_{U \setminus A}$ is ω-semiconvex. Then f is ω-semiconvex.*

Proof. Let $x, y \in U$, $x \neq y$ be arbitrary points such that $[x, y] \subset U$. Since A is intervally thin there exist sequences $x_n \to x$, $y_n \to y$ satisfying

$$[x_n, y_n] \cap A = \emptyset. \tag{1}$$

Let $\varepsilon > 0$ be such that $[x, y]_\varepsilon \subset U$. Without loss of generality we may assume that $\|x_n - x\|, \|y_n - y\| < \varepsilon$. Then

$$[x_n, y_n] \subset [x, y]_\varepsilon \subset U. \tag{2}$$

Consider an arbitrary $\alpha \in [0, 1]$. Making use of continuity of f, (1), (2) and upper semicontinuity of ω we obtain

$$\begin{aligned} f\big(\alpha x + (1-\alpha)y\big) &= \lim_{n\to\infty} f\big(\alpha x_n + (1-\alpha)y_n\big) \\ &\leqslant \limsup_{n\to\infty}\big[\alpha f(x_n) + (1-\alpha) f(y_n) + \alpha(1-\alpha)\|x_n - y_n\|\omega\big(\|x_n - y_n\|\big)\big] \\ &\leqslant \alpha f(x) + (1-\alpha) f(y) + \alpha(1-\alpha)\|x - y\|\omega\big(\|x - y\|\big). \qquad \square \end{aligned}$$

We are going to show that the assumption of the previous theorem that A is intervally thin cannot be much weakened. It occurs that the assertion of Theorem 2.1 fails to hold if A is a ball in hyperplane of codimension one.

Example 2.1. We consider $\mathbb{R}^N$ and take

$$M := \{(x_1, \dots, x_{N-1}, 0)\colon x_1, \dots, x_{N-1} \in \mathbb{R}\}.$$

In $\mathbb{R}^N \simeq M \times \mathbb{R}$ we take the product maximum norm

$$\|(m, r)\| = \max(\|m\|, |r|) \quad \text{for } (m, r) \in M \times \mathbb{R}.$$

Let $p = (0_M, 1)$ and $A = \bar{B}_M(0, 1) \times \{0\}$. We define a function $F : \mathbb{R}^N \to \mathbb{R}$ by the formula

$$F(x) := \operatorname{dist}(x; \{-p, p\}).$$

Evidently F is continuous. We are going to show that $F|_{\mathbb{R}^N \setminus A}$ is preconvex, but F is not convex.

To show that F is preconvex on $\mathbb{R}^N \setminus A$ it is sufficient to prove that for each point $x \in \mathbb{R}^N \setminus A$ there exists a neighbourhood V_x such that F is preconvex on V_x.

So let us take an $x = (m, r) \in \mathbb{R}^N \setminus A$. We consider three cases.

1) $r > 0$. We will show that F is convex on $B(x, r)$. Consider an arbitrary $x' = (m', r') \in B(x, r)$. Clearly $r' > 0$ and we have

$$\begin{aligned} F(x') &= F(m', r') = \min(\|x' - p\|, \|x' + p\|) \\ &= \min(\max(\|m'\|, |r' - 1|), \max(\|m'\|, |r' + 1|)) = \max(\|m'\|, |r' - 1|) = \|x' - p\|. \end{aligned}$$

Since the norm is a convex function, we obtain that F is convex on $B(x, r)$.

2) $r < 0$. By the similar reasoning as in the case 1) we directly obtain that F is convex on $B(x, -r)$.

3) $r = 0$. We are going to verify that F is convex on $B(x, \varepsilon)$, where $\varepsilon = \frac{\|m\|-1}{2}$.

Since $x = (m, 0) \in \mathbb{R}^N \setminus A$, we know that $\|x\| = \|m\| > 1$. Consider an arbitrary $x' = (m', r') \in B(x, \varepsilon)$. Then clearly $\|m' - m\| < \varepsilon$ and $|r'| < \varepsilon$. Whence we obtain

$$\|m'\| > \|m\| - \varepsilon = \|m\| - \frac{\|m\| - 1}{2} = \frac{\|m\| + 1}{2} > |r'| + 1.$$

Now we get

$$\begin{aligned} F(x') &= \min[\max(\|m'\|, |r' - 1|), \max(\|m'\|, |r' + 1|)] \\ &= \max(\|m'\|, |r' - 1|) = \|x' - p\|, \end{aligned}$$

which directly proves the convexity of F on $B(x, \varepsilon)$.

To verify that F is not convex on $\mathbb{R}^N$ notice that $F(p) = F(-p) = 0$, while $F(\frac{p+(-p)}{2}) = F(0) = 1$.

Example 2.2. Let us notice that the function F from Example 2.1 can be modified in such a way that it is preconvex on $\mathbb{R}^N \setminus A$, but does not have a continuous extension onto $\mathbb{R}^N$. We put

$$F(x_1, \dots, x_n) := \begin{cases} \max(\|(x_1, \dots, x_{n-1})\|, |x_n - 1|) & \text{if } x_n > 0, \\ \|(x_1, \dots, x_{n-1})\| & \text{otherwise.} \end{cases}$$

One can easily check that F is preconvex. However, it does not have a continuous extension onto $\mathbb{R}^N$ since

$$\lim_{h \to 0^+} F(0, \dots, 0, h) = 1 \neq 0 = \lim_{h \to 0^-} F(0, \dots, 0, h).$$

3. Unique extension

Let $S \subset \mathbb{R}^N$. We say that a function $f : S \to \mathbb{R}$ is *locally bounded from below at a point* $a \in \operatorname{cl} S$ if there exists a neighbourhood U of a such that the set $f(U \cap S)$ is bounded from below.

To prove our main theorem we will need a few auxiliary results.

Proposition 3.1. *Let $U \subset \mathbb{R}^N$ be an open set and let A be a closed intervally thin subset of U. Then every ω-semiconvex function $f : U \setminus A \to \mathbb{R}$ is locally bounded from below at every point of U.*

Proof. Let us first observe that since A is intervally thin, $\operatorname{int} A = \emptyset$, and consequently $U \subset \operatorname{cl}(U \setminus A)$.

Now suppose, for an indirect proof, that f is not locally bounded from below at a certain point $a \in U$. Then there exists a sequence $a_n \in U \setminus A$, $a_n \to a$ such that $f(a_n) < -n$. Since f is continuous and $U \setminus A$ is open, there exists a sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$ such that $B(a_n, \varepsilon_n) \subset U \setminus A$ and $f(B(a_n, \varepsilon_n)) \subset (-\infty, -n)$.

Because $\operatorname{int} A = \emptyset$, we can find a point $b \in U \setminus A$ such that $[a, b] \subset U$. Furthermore, as $U \setminus A$ is open, there exists an $\alpha \in (0, 1)$ such that $[\alpha a + (1 - \alpha)b, b] \subset U \setminus A$. Since A is intervally thin, we can find sequences $(\tilde{a}_n)$, (b_n) such that $\tilde{a}_n \in B(a_n, \varepsilon_n)$, $b_n \to b$ and $[\tilde{a}_n, b_n] \subset U \setminus A$. Then we have

$$f(\alpha \tilde{a}_n + (1 - \alpha)b_n) \leqslant \alpha f(\tilde{a}_n) + (1 - \alpha) f(b_n) + \alpha(1 - \alpha)\|\tilde{a}_n - b_n\|\omega(\|\tilde{a}_n - b_n\|),$$

and consequently

$$f(b_n) \geqslant \frac{1}{1-\alpha} f\big(\alpha\tilde{a}_n + (1-\alpha)b_n\big) - \frac{\alpha}{1-\alpha} f(\tilde{a}_n) - \alpha\|\tilde{a}_n - b_n\|\omega\big(\|\tilde{a}_n - b_n\|\big).$$

Since $f(\alpha\tilde{a}_n + (1-\alpha)b_n) \to f(\alpha a + (1-\alpha)b)$ and $f(\tilde{a}_n) \to -\infty$, we obtain that $f(b_n) \to \infty$, which gives a contradiction as $f(b_n) \to f(b)$. $\square$

Lemma 3.1. *Let $(a_k)_{k=0,\ldots,n} \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$ be such that*

$$a_{k+1} \leqslant \frac{a_k + a_{k+2}}{2} + \alpha \quad \text{for } k = 0, \ldots, n-2. \tag{3}$$

Then

$$a_k \geqslant a_0 + k(a_1 - a_0) - k(k-1)\alpha \quad \text{for } k = 0, \ldots, n. \tag{4}$$

Proof. The proof goes by induction. For $k = 0, 1$ (4) is trivial. Suppose that (4) holds for a given $k \geqslant 1$. Directly from (3) we obtain that

$$a_{k+1} - a_k \geqslant a_k - a_{k-1} - 2\alpha.$$

Applying the above inequalities k-times we obtain

$$(a_{k+1} - a_k) \geqslant (a_k - a_{k-1}) - 2\alpha \geqslant \cdots \geqslant (a_1 - a_0) - 2k\alpha.$$

Adding all the above inequalities up we get

$$a_{k+1} - a_0 = (a_{k+1} - a_k) + \cdots + (a_1 - a_0) \geqslant (k+1)(a_1 - a_0) - (k+1)k\alpha,$$

which directly yields (4).

Lemma 3.2. *Let $U \subset \mathbb{R}^N$ be an open set and let A be a closed intervally thin subset of U. Let $f : U \setminus A \to \mathbb{R}$ be an ω-semiconvex function.*

Then for every $a \in U$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\big|f(b) - f(c)\big| \leqslant \varepsilon \quad \text{for } b, c \in B(a,\delta) \cap (U \setminus A).$$

Proof. Since f is continuous, it is sufficient to consider the case $a \in A$. For an indirect proof suppose that there exist $\varepsilon > 0$ and $a \in A$ such that the assertion of Lemma 3.2 does not hold. Without loss of generality we may assume that $a = 0$.

By Proposition 3.1 we can find $r > 0$ and $M \in \mathbb{R}$ such that $B(0,r) \subset U$ and $f(B(0,r) \setminus A) \subset (M, \infty)$. Because $A \cap B(0,r)$ is a closed nowhere dense subset of $B(0,r)$, we obtain that $(A \cap B(0,r)) \cup (-A \cap B(0,r))$ is a nowhere dense subset of $B(0,r)$, and consequently that there exists an $x \in B(0,r)$, $x \neq 0$ such that $x, -x \notin A$. By continuity of f we can find $r_1 > 0$ and $K \in \mathbb{R}$ such that

$$B(x,r_1), B(-x,r_1) \subset B(0,r) \setminus A,$$

$$f\big(B(x,r_1) \setminus A\big), f\big(B(-x,r_1) \setminus A\big) \subset (-\infty, K).$$

Let us fix an arbitrary $n \in \mathbb{N}$, $n \geqslant 2$ and consider the balls $B_0 := B(0, r_1/(4n))$, $B_1 := B(x/n, r_1/(4n))$. By the assumptions there exist $b, c \in B_0 \setminus A$ such that $|f(b) - f(c)| > \varepsilon$. Let us now take an arbitrary $d \in B_1 \setminus A$. Then either $|f(b) - f(d)| > \varepsilon/2$ or $|f(c) - f(d)| > \varepsilon/2$. Suppose that $|f(b) - f(d)| > \varepsilon/2$ (the other case is analogous). Then two cases may occur: either $f(b) > f(d) + \varepsilon/2$ or $f(d) > f(b) + \varepsilon/2$.

Suppose that $f(b) > f(d) + \varepsilon/2$. Consider the line l from b through d (that is the affine function such that $l(0) = b$, $l(1) = d$). Thus our inequality means that $f(l(0)) > f(l(1)) + \varepsilon/2$. Since $l(0) \in B_0$, $l(1) \in B_1$, we obtain that

$$l(-n) \in B(-x, r_1).$$

Because A is intervally thin and f is continuous, we can find a line l' such that $l'([-n,1]) \subset U \setminus A$ and that l' is so close to l that

$$l'(-n) \in B(-x,r_1), \qquad l'(0) \in B_0, \qquad l'(1) \in B_1, \qquad f\big(l'(0)\big) > f\big(l'(1)\big) + \varepsilon/2.$$

Then $\|l'(1) - l'(0)\| \leqslant (\|x\| + r_1/2)/n =: s/n$. Consider the sequence $a_k = f(l'(1-k))$ for $k = 0, \ldots, n+1$. Then by the ω-semiconvexity of f we have

$$\begin{aligned} a_{k+1} - \frac{a_k + a_{k+2}}{2} &= f\big(l'(-k)\big) - \frac{f(l'(1-k)) + f(l'(-1-k))}{2} \\ &\leqslant \frac{1}{2}\left(1 - \frac{1}{2}\right)\omega\big(\big\|l'(1-k) - l'(-1-k)\big\|\big) \cdot \big\|l'(1-k) - l'(-1-k)\big\| \\ &= \frac{1}{2}\omega\big(2\big\|l'(1) - l'(0)\big\|\big) \cdot \big\|l'(1) - l'(0)\big\| \leqslant \omega(2s/n) \cdot \frac{s}{2n}. \end{aligned}$$

Directly from the definition of the sequence a_k we also get

$$a_1 - a_0 = f\big(l'(0)\big) - f\big(l'(1)\big) > \varepsilon/2.$$

Applying Lemma 3.1 with $\alpha = \omega(2s/n)s/(2n)$ we obtain that

$$\begin{aligned} K > f\big(l'(-n)\big) &= a_{n+1} \geqslant a_0 + (n+1)(a_1 - a_0) - n(n+1)\alpha \\ &\geqslant M + (n+1)\varepsilon/2 - (n+1)\omega(2s/n)s/2 = M + (n+1)\big[\varepsilon - \omega(2s/n)s\big]/2. \end{aligned}$$

Now we discuss the case $f(d) > f(b) + \varepsilon/2$. Since it is similar to the previous one we just show the main steps. Consider the line l from b through d. Thus our inequality means that $f(l(1)) > f(l(0)) + \varepsilon/2$. Clearly, as $l(0) \in B_0$, $l(1) \in B_1$, we obtain that

$$l(n) \in B(x, r_1).$$

Since A is intervally thin and f is continuous, we can find a line l' such that $l'([0,n]) \subset U \setminus A$ and that

$$l'(n) \in B(x, r_1), \qquad l'(0) \in B_0, \qquad l'(1) \in B_1, \qquad f\big(l'(1)\big) > f\big(l'(0)\big) + \varepsilon/2.$$

We consider the sequence $a_k = l'(k)$, $k = 0, \ldots, n$. Analogously as before, making use of Lemma 3.1, one can show that $K > M + n[\varepsilon - \omega(2s/n)s]/2$.

Thus in both cases we have obtained that

$$K > M + n\big[\varepsilon - \omega(2s/n)s\big]/2 \quad \text{for } n \in \mathbb{N},$$

which yields a contradiction as $\varepsilon > 0$ and $\omega(2s/n) \to 0$ as $n \to \infty$. □

We are ready to prove the main result of the paper.

Theorem 3.1. *Let U be an open subset of $\mathbb{R}^N$ and let A be a closed intervally thin subset of U. Let $f : U \setminus A \to \mathbb{R}$ be an ω-semiconvex function. Then f has a unique ω-semiconvex extension onto U.*

Proof. Since $U \setminus A$ is an open set, the function f is continuous. It follows from Lemma 3.2 that f has a continuous extension F onto U. Since A is nowhere dense this extension is unique. By Theorem 2.1 we obtain that F is ω-semiconvex. □

At the end of this section we would like to pose the following problem.

Problem 3.1. Let A be a closed subset of $\mathbb{R}^N$. Suppose that for an arbitrary open set U containing A, every preconvex function $f : U \setminus A \to \mathbb{R}$ has a unique convex extension onto U. Is A an intervally thin set?

4. Intervally thin sets

As we have seen, closed intervally thin sets play the role of “negligible sets” for semiconvex functions. Since in our opinion this family is important in this section we investigate its properties.

We are going to show that the family of intervally thin sets have properties consistent with intuition of “small sets”. Subset of an intervally thin set is obviously an intervally thin set. It is also evident that this family is invariant with respect to affine isomorphisms. We show that the union of a finite family of closed intervally thin sets is an intervally thin set.

Observation 4.1. Let $A_1, \ldots, A_n$ be closed intervally thin sets. Then $A_1 \cup \cdots \cup A_n$ is a closed intervally thin set.

Proof. It is enough to show that the union of two intervally thin closed sets is an intervally thin set. Let $A, B \subset \mathbb{R}^N$ be closed intervally thin sets. Consider arbitrary $x, y \in \mathbb{R}^N$ and $\varepsilon > 0$. We can find $x_1 \in B(x, \varepsilon/2)$, $y_1 \in B(y, \varepsilon/2)$ such that $[x_1, y_1] \cap A = \emptyset$. Since A is closed, there exists a $\delta \in (0, \varepsilon/2)$ such that

$$[x_1, y_1]_\delta \cap A = \emptyset.$$

Now we can find $x' \in B(x_1, \delta) \subset B(x', \varepsilon)$, $y' \in B(y_1, \delta) \subset B(y', \varepsilon)$ satisfying

$$[x', y'] \cap B = \emptyset.$$

Since $[x', y'] \subset [x_1, y_1]_\delta \subset \mathbb{R}^N \setminus A$, we obtain that $[x', y'] \cap A = \emptyset$. □

As a consequence we obtain that being a closed intervally thin set is a "local" property.

Corollary 4.1. *Let A be a closed subset of $\mathbb{R}^N$. Then A is intervally thin iff for every $a \in A$ there exists an $\varepsilon > 0$ such that $A \cap \overline{B}(a, \varepsilon)$ is intervally thin.*

Proof. Assume that for every $a \in A$ there exists an $\varepsilon > 0$ such that $A \cap \overline{B}(a, \varepsilon)$ is intervally thin. To show that A is intervally thin, it is clearly enough to prove that $A^r := A \cap \overline{B}(0, r)$ is intervally thin for every $r > 0$.

By the assumptions, for every point $a \in A^r$ there exists an $\varepsilon_a > 0$ such that $B(a, \varepsilon_a) \cap A^r$ is intervally thin. Applying compactness of A^r and the previous observation we get that A^r is a finite union of intervally thin sets, and consequently A^r is intervally thin. □

It occurs that closed intervally thin sets cannot "disconnect" open connected sets.

Observation 4.2. Let A be a closed intervally thin subset of an open connected set $U \subset \mathbb{R}^N$. Then $U \setminus A$ is connected.

Proof. Consider arbitrary points $x, y \in U \setminus A$. We are going to show that x can be connected with y by a "broken line" in $U \setminus A$.

Since U is open and connected there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ such that

$$[x_i, x_{i+1}] \subset U \quad \text{for } i = 0, \ldots, n-1.$$

Now we can find $r > 0$ with the property

$$[x_i, x_{i+1}]_r \subset U \quad \text{for } i = 0, \ldots, n-1. \tag{5}$$

Without loss of generality we may assume that

$$B(x, r) \subset U \setminus A, \qquad B(y, r) \subset U \setminus A. \tag{6}$$

Because A is intervally thin, we can find $x'_0 \in B(x_0, r)$, $x'_1 \in B(x_1, r)$ such that

$$[x'_0, x'_1] \cap A = \emptyset.$$

Since A is closed, there exists a $\delta_1 > 0$ such that

$$B(x'_1, \delta_1) \subset B(x_1, r) \setminus A.$$

Now we choose $x''_1 \in B(x'_1, \delta_1)$, $x'_2 \in B(x_2, r)$ in such a way that

$$[x''_1, x'_2] \cap A = \emptyset.$$

We continue the above procedure and get $\delta_1, \ldots, \delta_{n-1} > 0$ and points $x'_0, \ldots, x'_n$, $x''_1, \ldots, x''_{n-1}$ such that

$$\begin{aligned} &x'_k \in B(x_k, r) \setminus A \quad \text{for } k = 0, \ldots, n, \\ &x''_k \in B(x'_k, \delta_k) \subset B(x_k, r) \setminus A \quad \text{for } k = 1, \ldots, n-1, \\ &[x''_k, x'_{k+1}] \cap A = \emptyset \quad \text{for } k = 1, \ldots, n-1. \end{aligned} \tag{7}$$

In view of (5), (6) and (7) the "broken line" with vertices

$$x = x_0, x'_0, x'_1, x''_1, \ldots, x''_{n-1}, x'_n, x_n = y$$

lies entirely in $U \setminus A$. □

It is obvious that every intervally thin set has empty interior. However, the opposite need not be true. We will give a criterion for a set to be intervally thin.

Let M be an affine subspace of $\mathbb{R}^N$. If $a \in M$ then by $B_M(a, r)$ we denote the open ball in M centered at a with radius r. If $A \subset M$ then by $\operatorname{int}_M A$ we understand the relative interior of A in M. By $p_M : \mathbb{R}^N \to M$ we denote the orthogonal projection onto M.

Proposition 4.1. *Let $A \subset \mathbb{R}^N$. We assume that $\operatorname{int}_M(p_M(A)) = \emptyset$ for every $(N-1)$-dimensional subspace M of $\mathbb{R}^N$. Then A is intervally thin.*

Proof. Consider arbitrary $x, y \in \mathbb{R}^N$ and $\varepsilon > 0$. Clearly it is enough to consider the case when $x \neq y$. Let $M := (x-y)^{\perp}$. We are going to prove that there exists an $h \in B_M(0,\varepsilon)$ such that $[x+h, y+h] \cap A = \emptyset$. Suppose, for an indirect proof, that this is not the case, i.e. that

$$[x+h, y+h] \cap A \neq \emptyset \quad \text{for } h \in B_M(0,\varepsilon).$$

Then we get

$$p_M(x) + B_M(0,\varepsilon) \subset A,$$

which means that $\operatorname{int}_M(p_M(A)) \neq \emptyset$, a contradiction. □

As a consequence of the above result we will obtain that the sets with the $(N-1)$-dimensional Hausdorff measure zero are intervally thin. For the convenience of the reader we recall [14] that a subset A of a metric space X has an *s-dimensional measure zero* (where $s > 0$), which we write $\lambda_s(A) = 0$, if

$$\forall \varepsilon > 0\ \exists (w_k)_{k\in\mathbb{N}} \subset X, \quad (r_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+ \colon A \subset \bigcup_{k\in\mathbb{N}} B(w_k, r_k) \quad \text{and} \quad \sum_{k\in\mathbb{N}} r_k^s \leqslant \varepsilon.$$

Example 4.1. Let M be an affine subspace of $\mathbb{R}^N$ of codimension 1 and let $a \in M$, $r > 0$ be arbitrary. Then $B_M(a,r)$ is not intervally thin. To notice this choose an arbitrary $v \perp M$, $v \neq 0$ and consider $x = a + v$, $y = a - v$. One can easily see that for the pair x, y the condition in the definition of intervally thin set does not hold.

We will need the following well-known facts:

- if X, Y are metric spaces, $f : X \to Y$ is Lipschitz and $A \subset X$ is such that $\lambda_s(A) = 0$, then $\lambda_s(f(A)) = 0$;
- $\lambda_k(B_M(a,r)) > 0$ for $a \in M$, $r > 0$, where M is an affine subspace of $\mathbb{R}^N$ of dimension k.

Now we are ready to show

Observation 4.3. Let $A \subset \mathbb{R}^N$ be such that $\lambda_{N-1}(A) = 0$. Then A is intervally thin.

Proof. Let M be an arbitrary $(N-1)$-dimensional subspace of $\mathbb{R}^N$. Since $\lambda_{N-1}(A) = 0$, we obtain that $\lambda_{N-1}(p_M(A)) = 0$, which consequently yields that $\operatorname{int}_M(p_M(A)) = \emptyset$. Proposition 4.1 completes the proof. □

As a trivial corollary of the above observation and Theorem 3.1 we get the following result.

Corollary 4.2. *Let U be an open convex subset of $\mathbb{R}^N$ and let A be a closed subset of U such that $\lambda_{N-1}(A) = 0$. Let $f : U \setminus A \to \mathbb{R}$ be preconvex. Then f has a unique convex extension onto U.*

Remark 4.1. To construct compact intervally thin set $A \subset \mathbb{R}^N$ such that $\lambda_{N-1}(A) > 0$ it is sufficient to take any compact nowhere dense set $B \subset \mathbb{R}^{N-1}$ with $\lambda_{N-1}(B) > 0$ and put $A = B \times \{0\}$.

On the other hand there exists a compact set $A \subset \mathbb{R}^N$, with $\lambda_{N-1}(A) < \infty$, which is not intervally thin, see Example 2.2.

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