Applied Mathematics and Computation 218 (2011) 3615–3623

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Properties for uniformly starlike and related functions under the Srivastava-Attiya operator

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ARTICLE INFO

Keywords:
Uniformly starlike functions
Convex functions
Close-to-convex functions
Quasi-convex functions
Inclusion relations
Coefficient inequalities
Srivastava–Attiya operator

ABSTRACT

In the present paper, we introduce and investigate classes of analytic functions involving the Srivastava–Attiya operator. Basic properties for β-uniformly starlike functions of order γ are studied, such as inclusion relations, sufficient conditions, coefficient inequalities and distortion inequalities. The results are also extended to β-uniformly convex, close-to-convex, and quasi-convex functions. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}), \tag{1.1}$$

which are *analytic* in the *open* unit disk $\mathbb{U} = \{z : |z| < 1\}$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

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doi:10.1016/j.amc.2011.09.002

For $0 \leqq \gamma < 1$, we denote by $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ the usual subclasses of $\mathcal{A}$ consisting of functions which are, respectively, *starlike of order* γ and *convex of order* γ in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UST}(\beta,\gamma)$, the class of β-uniformly starlike functions of order γ, if f satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma \quad (\beta \geqq 0,\ 0 \leqq \gamma < 1;\ z \in \mathbb{U}). \tag{1.2}$$

Also, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}(\beta,\gamma)$, the class of β-uniformly convex functions of order γ, if f satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right| + \gamma \qquad (\beta \geqq 0,\ 0 \leqq \gamma < 1;\ z \in \mathbb{U}). \tag{1.3}$$

Uniformly starlike and convex functions were first introduced by Goodman [7] and then studied by various authors (see [12,17,18]). $f \in \mathcal{UST}(\beta,\gamma)$ and $f \in \mathcal{UCV}(\beta,\gamma)$ if and only if $p(z) = \frac{zf'(z)}{f(z)}$ and $p(z) = 1 + \frac{zf''(z)}{f'(z)}$, respectively, take all values in the conic domain $R_{\beta,\gamma}$, which defined by

$$R_{\beta,\gamma} = \left\{u + iv :\ u > \beta\sqrt{(u-1)^2 + v^2} + \gamma\right\}.$$

The functions $q_{\beta,\gamma}(z) \in R_{\beta,\gamma}$ were obtained in [2] as follows

$$q_{\beta,\gamma}(z) = \begin{cases} \frac{1+(1-2\gamma)z}{1-z} & \beta = 0, \\ 1 + \frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & \beta = 1, \\ \frac{1-\gamma}{1-\beta^2}\cos\left\{\left(\frac{2}{\pi}\arccos\beta\right)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{\beta^2-\gamma}{1-\beta^2} & 0 < \beta < 1, \\ \frac{1-\gamma}{\beta^2-1}\sin\left(\frac{\pi}{2K(t)}\right)\int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}dx + \frac{\beta^2-\gamma}{\beta^2-1} & \beta > 1, \end{cases}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}},\ t \in (0,1),\ z \in \mathbb{U}$ and t is chosen such that $\beta = \cosh\frac{\pi K'(t)}{4K(t)}, K(t)$ is Legendre's complete elliptic integral of the first kind and $K'(t)$ is complementary integral of $K(t)$.

We may rewrite the conditions (1.2) or (1.3) in the form

$$p(z) \prec q_{\beta,\gamma}(z). \tag{1.4}$$

By virtue of (1.4) and the properties of the domains $R_{\beta,\gamma}$ we have

$$\Re(p(z)) > \Re(q_{\beta,\gamma}(z)) > \frac{\beta+\gamma}{\beta+1}. \tag{1.5}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UCC}(\beta,\gamma)$, the class of β-uniformly close-to-convex functions of order γ, if f satisfies the condition

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \beta\left|\frac{zf'(z)}{g(z)} - 1\right| + \gamma \quad (\beta \geqq 0,\ 0 \leqq \gamma < 1;\ z \in \mathbb{U}) \tag{1.6}$$

for some $g \in \mathcal{UST}(\beta,\gamma)$.

Similarly, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UQC}(\beta,\gamma)$, the class of β-uniformly quasi-convex functions of order γ, if f satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{g'(z)}\right) > \beta\left|\frac{zf''(z)}{g'(z)}\right| + \gamma \qquad (\beta \geqq 0,\ 0 \leqq \gamma < 1;\ z \in \mathbb{U}) \tag{1.7}$$

for some $g \in \mathcal{UCV}(\beta,\gamma)$.

Recently, Srivastava and Attiya [16] introduced and investigated the linear operator:

$$\mathcal{J}_{s,b}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{1+b}{n+b}\right)^s a_n z^n \quad (z \in \mathbb{U};\ b \in \mathbb{C}\setminus\{0,-1,-2,\ldots\};\ s \in \mathbb{C};\ f \in \mathcal{A}). \tag{1.8}$$

It is readily verified from (1.8) that

$$z(\mathcal{J}_{s,b}f)'(z) = (1+b)\mathcal{J}_{s,b}f(z) - b\mathcal{J}_{s+1,b}f(z). \tag{1.9}$$

As special cases of $\mathcal{J}_{s,b}f$, we obtain the following identities:

(1) $\mathcal{J}_{0,b}f(z) = f(z),\ \mathcal{J}_{-1,0}f(z) = zf'(z),\ \mathcal{J}_{-1,\frac{1}{1-\lambda}}f(z) = \lambda f(z) + (1-\lambda)zf'(z)\ (\lambda \neq 1).$

(2) $\mathcal{J}_{1,0}f(z) = A(f)(z),\ \mathcal{J}_{1,1}f(z) = L(f)(z)$ (see [3,9]).
(3) $\mathcal{J}_{1,\gamma}f(z) = L_\gamma(f)(z)\ (\gamma > -1)$ (see [6]).
(4) $\mathcal{J}_{\sigma,1}f(z) = I^\sigma(f)(z)\ (\sigma > 0)$ (see [8]).
(5) $\mathcal{J}_{-m,0}f(z) = D^m(f)(z)\ (m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ (see [15]).
(6) $\mathcal{J}_{-m,\frac{1-\lambda}{\lambda}}f(z) = D^m_\lambda(f)(z)\ (m \in \mathbb{N}_0,\ \lambda > 0)$ (see [4]).

For $b = 0, \mathcal{J}_{s,0}f(z) = \lim_{b\to 0}\mathcal{J}_{s,b}f(z)$.

Using the operator $\mathcal{J}_{s,b}$, we introduce the subclasses $\mathcal{UST}_{s,b}(\beta,\gamma), \mathcal{UCV}_{s,b}(\beta,\gamma), \mathcal{UCC}_{s,b}(\beta,\gamma)$ and $\mathcal{UQC}_{s,b}(\beta,\gamma)$ of class $\mathcal{A}$, which are defined by

$$\mathcal{UST}_{s,b}(\beta,\gamma) := \{f \in \mathcal{A} : \mathcal{J}_{s,b}f \in \mathcal{UST}(\beta,\gamma)\},$$
$$\mathcal{UCV}_{s,b}(\beta,\gamma) := \{f \in \mathcal{A} : \mathcal{J}_{s,b}f \in \mathcal{UCV}(\beta,\gamma)\},$$
$$\mathcal{UCC}_{s,b}(\beta,\gamma) := \{f \in \mathcal{A} : \mathcal{J}_{s,b}f \in \mathcal{UCC}(\beta,\gamma)\}$$

and

$$\mathcal{UQC}_{s,b}(\beta,\gamma) := \{f \in \mathcal{A} : \mathcal{J}_{s,b}f \in \mathcal{UQC}(\beta,\gamma)\}.$$

It is clear that

$$f \in \mathcal{UCV}_{s,b}(\beta,\gamma) \Longleftrightarrow zf' \in \mathcal{UST}_{s,b}(\beta,\gamma), \tag{1.10}$$

and

$$f \in \mathcal{UQC}_{s,b}(\beta,\gamma) \Longleftrightarrow zf' \in \mathcal{UCC}_{s,b}(\beta,\gamma). \tag{1.11}$$

In order to prove our main results, we need the following lemmas.

Lemma 1 *(See [10]). Let $\kappa,\ \gamma \in \mathbb{C}$ let h be univalently convex in the unit disk $\mathbb{U}$ with h(0) = c and $\Re(\kappa h(z) + \gamma) > 0$. Let $g(z) = 1 + \sum_{n=1}^\infty p_n z^n$ be analytic in $\mathbb{U}$. Then*

$$g(z) + \frac{zg'(z)}{\kappa g(z) + \gamma} \prec h(z) \Rightarrow g(z) \prec h(z).$$

Lemma 2 *(See [11]). Let h be convex in the unit disk $\mathbb{U}$ and let $A \geqq 0$. Suppose B(z) is analytic in $\mathbb{U}$ with $\Re(B(z)) \geqq A$. If g is analytic in $\mathbb{U}$ and g(0) = h(0). Then*

$$Az^2g''(z) + B(z)zg'(z) + g(z) \prec h(z) \Rightarrow g(z) \prec h(z).$$

Lemma 3 *(See [13]). Let $h(z) = 1 + \sum_{n=1}^\infty c_n z^n$ be subordinate to $H(z) = 1 + \sum_{n=1}^\infty C_n z^n$ in $\mathbb{U}$. If H(z) is univalent in $\mathbb{U}$ and $H(\mathbb{U})$ is convex, then*

$$|c_n| \leqq |C_1|, \quad (n \in \mathbb{N}).$$

Lemma 4. *Let*

$$C_n(s,b) = \left(\frac{1+b}{n+b}\right)^s \quad (s \in \mathbb{C},\ b \in \mathbb{C} \setminus \{-1,-2,\ldots\};\ n \in \mathbb{N}). \tag{1.12}$$

Suppose also that the sequence $\{A_n\}_{n=1}^\infty$ is defined by

$$A_1 = 1, \quad A_2 = \frac{P_1}{|C_2(s,b)|}$$

and

$$A_n = \frac{P_1}{(n-1)|C_n(s,b)|}\left(1 + \sum_{j=2}^{n-1} A_j|C_j(s,b)|\right) \quad (n \in \mathbb{N} \setminus \{1,2\}). \tag{1.13}$$

Then

$$A_n = \frac{1}{|C_n(s,b)|}\,\frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (n \in \mathbb{N}), \tag{1.14}$$

where

$$(P_1)_n = \begin{cases} 1 & n = 0, \\ P_1(P_1+1)\cdots(P_1+n-1) & n \in \mathbb{N}. \end{cases}$$

Proof. We make use of the principle of mathematical induction to prove the assertion (1.14).

For $n = 3$, we know that

$$A_3 = \frac{P_1}{2|C_3(s,b)|}\left(1 + \frac{P_1}{|C_2(s,b)|}|C_2(s,b)|\right) = \frac{1}{|C_3(s,b)|}\frac{(P_1)_2}{(1)_2},$$

which implies that (1.14) holds for $n = 3$.

We now suppose that (1.14) holds for $n = 3, 4, \ldots, m$. Then

$$A_j = \frac{1}{|C_j(s,b)|}\frac{(P_1)_{j-1}}{(1)_{j-1}} \quad (j = 3, 4, \ldots, m). \tag{1.15}$$

Combining (1.13) and (1.15), we find that

$$A_{m+1} = \frac{P_1}{m|C_{m+1}(s,b)|}\left(1 + \sum_{j=2}^{m} A_j|C_j(s,b)|\right) = \frac{P_1}{m|C_{m+1}(s,b)|}\left(1 + \sum_{j=2}^{m}\frac{1}{|C_j(s,b)|}\frac{(P_1)_{j-1}}{(1)_{j-1}}|C_j(s,b)|\right)$$
$$= \frac{P_1}{m|C_{m+1}(s,b)|}\left(1 + \sum_{j=2}^{m}\frac{(P_1)_{j-1}}{(1)_{j-1}}\right) = \frac{P_1}{m|C_{m+1}(s,b)|}\frac{(P_1+1)(P_1+2)\ldots(P_1+(m-1))}{(m-1)!} = \frac{1}{|C_{m+1}(s,b)|}\frac{(P_1)_m}{(1)_m},$$

which shows that (1.14) holds for $n = m + 1$. The proof of Lemma 4 is evidently completed. □

2. Main results

Firstly, we give several inclusion relationships for analytic function classes defined in the first section, which are associated with the Srivastava–Attiya operator $\mathcal{J}_{s,b}$.

Theorem 1. *Let* $\Re(b) > -\frac{\beta+\gamma}{\beta+1}$. *Then*

$$\mathcal{UST}_{s,b}(\beta,\gamma) \subseteq \mathcal{UST}_{s+1,b}(\beta,\gamma). \tag{2.1}$$

Proof. Let $f \in \mathcal{UST}_{s,b}(\beta,\gamma)$, by definition, we have $\mathcal{J}_{s,b}f(z) \in \mathcal{UST}(\beta,\gamma)$. Setting

$$p(z) = \frac{z(\mathcal{J}_{s+1,b}f)'(z)}{\mathcal{J}_{s+1,b}f(z)}$$

in (1.9), we can write

$$\frac{\mathcal{J}_{s,b}f(z)}{\mathcal{J}_{s+1,b}f(z)} = \frac{1}{1+b}\left(\frac{(\mathcal{J}_{s+1,b}f)'(z)}{\mathcal{J}_{s+1,b}f(z)} + b\right) = \frac{1}{1+b}(p(z)+b). \tag{2.2}$$

Differentiating (2.2) yields

$$\frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}f(z)} = \frac{z(\mathcal{J}_{s+1,b}f)'(z)}{\mathcal{J}_{s+1,b}f(z)} + \frac{zp'(z)}{p(z)+b} = p(z) + \frac{zp'(z)}{p(z)+b}. \tag{2.3}$$

From (2.3) and the argument given in Section 1 we may write

$$p(z) + \frac{zp'(z)}{p(z)+b} \prec q_{\beta,\gamma}(z).$$

Therefore the theorem follows by Lemma 1 and the condition (1.5) since $q_{\beta,\gamma}$ is univalent and convex in $\mathbb{U}$ and $\Re\left(q_{\beta,\gamma}(z)+b\right) > 0$. □

By similarly applying the method of proof of Theorem 1, we easily get the following result of the class $\mathcal{UCV}_{s,b}(\beta,\gamma)$.

Theorem 2. *Let* $\Re(b) > -\frac{\beta+\gamma}{\beta+1}$. *Then*

$$\mathcal{UCV}_{s,b}(\beta,\gamma) \subseteq \mathcal{UCV}_{s+1,b}(\beta,\gamma). \tag{2.4}$$

Remark 1. If we consider $s = -m$ and $b = \frac{1-\lambda}{\lambda}$ in Theorems 1 and 2, respectively, we obtain the same results in [5].

We next prove the following.

Theorem 3. *Let* $\Re(b) > -\frac{\beta+\gamma}{\beta+1}$. *Then*

$$\mathcal{UCC}_{s,b}(\beta,\gamma) \subseteq \mathcal{UCC}_{s+1,b}(\beta,\gamma). \tag{2.5}$$

Proof. Let $f \in \mathcal{UCC}_{s,b}(\beta,\gamma)$, by definition, we have

$$\frac{z(\mathcal{J}_{s,b}f)'(z)}{\omega(z)} \prec q_{\beta,\gamma}(z)$$

for some $\omega(z) \in \mathcal{UST}(\beta,\gamma)$. For ω so that $\mathcal{J}_{s,b}g(z) = \omega(z)$, we have

$$\frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}g(z)} \prec q_{\beta,\gamma}(z). \tag{2.6}$$

Letting

$$h(z) = \frac{z(\mathcal{J}_{s+1,b}f)'(z)}{\mathcal{J}_{s+1,b}g(z)}, \qquad H(z) = \frac{z(\mathcal{J}_{s+1,b}g)'(z)}{\mathcal{J}_{s+1,b}g(z)}.$$

We observe that h and H are analytic in $\mathbb{U}$ and $h(0) = H(0) = 1$. Now, by Theorem 1, $\mathcal{J}_{s+1,b}g(z) \in \mathcal{UST}(\beta,\gamma)$ and so $\Re(H(z)) > \frac{\beta+\gamma}{\beta+1}$. Also, by noting that

$$z(\mathcal{J}_{s+1,b}f)'(z) = (\mathcal{J}_{s+1,b}g(z))h(z). \tag{2.7}$$

Differentiating both sides of (2.7) yields

$$\frac{z(\mathcal{J}_{s+1,b}(zf'))'(z)}{\mathcal{J}_{s+1,b}g(z)} = \frac{z(\mathcal{J}_{s+1,b}g)'(z)}{\mathcal{J}_{s+1,b}g(z)}h(z) + zh'(z) = H(z)h(z) + zh'(z).$$

Now using the identity (1.9) we obtain

$$\begin{aligned}\frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}g(z)} &= \frac{\mathcal{J}_{s,b}(zf')(z)}{\mathcal{J}_{s,b}g(z)} = \frac{z(\mathcal{J}_{s+1,b}(zf'))'(z) + b\mathcal{J}_{s+1,b}(zf')(z)}{z(\mathcal{J}_{s+1,b}g)'(z) + b\mathcal{J}_{s+1,b}g(z)} = \frac{\frac{z(\mathcal{J}_{s+1,b}(zf'))'(z)}{\mathcal{J}_{s+1,b}g(z)} + b\frac{\mathcal{J}_{s+1,b}(zf')(z)}{\mathcal{J}_{s+1,b}g(z)}}{\frac{z(\mathcal{J}_{s+1,b}g)'(z)}{\mathcal{J}_{s+1,b}g(z)} + b} \\ &= \frac{H(z)h(z) + zh'(z) + bh(z)}{H(z) + b} = h(z) + \frac{zh'(z)}{H(z) + b}.\end{aligned} \tag{2.8}$$

Combining (2.6) and (2.8) we conclude that

$$h(z) + \frac{zh'(z)}{H(z) + b} \prec q_{\beta,\gamma}(z).$$

If we letting $A = 0$ and $B(z) = \frac{1}{H(z)+b}$, we obtain

$$\Re(B(z)) = \frac{1}{|H(z) + b|^2}\Re(H(z) + b) > 0.$$

The above inequality satisfies the conditions required by Lemma 2. Hence $h(z) \prec q_{\beta,\gamma}(z)$ and so the proof is completed. □

By similarly applying the method of proof of Theorem 3, we easily get the following result of the class $\mathcal{UQC}_{s,b}(\beta,\gamma)$.

Theorem 4. *Let* $\Re(b) > -\frac{\beta+\gamma}{\beta+1}$. *Then*

$$\mathcal{UQC}_{s,b}(\beta,\gamma) \subseteq \mathcal{UQC}_{s+1,b}(\beta,\gamma). \tag{2.9}$$

Remark 2. If we consider $s = \sigma$, $b = 1$ or $s = 1$, $b = \gamma$ in Theorems 1–4, respectively, we obtain the same results in [2].

From (1.2), (1.3), (1.6) and (1.7), we note the following for $0 \leqq \beta_2 \leqq \beta_1$ and $0 \leqq \gamma_2 \leqq \gamma_1 < 1$.

$$\mathcal{UST}_{s,b}(\beta_1,\gamma_1) \subseteq \mathcal{UST}_{s,b}(\beta_2,\gamma_2), \quad \mathcal{UCV}_{s,b}(\beta_1,\gamma_1) \subseteq \mathcal{UCV}_{s,b}(\beta_2,\gamma_2)$$

and

$$\mathcal{UCC}_{s,b}(\beta_1,\gamma_1) \subseteq \mathcal{UCC}_{s,b}(\beta_2,\gamma_2), \quad \mathcal{UQC}_{s,b}(\beta_1,\gamma_1) \subseteq \mathcal{UQC}_{s,b}(\beta_2,\gamma_2).$$

Next, we obtain some sufficient conditions for the classes $\mathcal{UST}_{s,b}(\beta,\gamma)$ and $\mathcal{UCV}_{s,b}(\beta,\gamma)$.

Theorem 5. *If $f \in \mathcal{A}$ satisfies the inequality:*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\beta+\gamma)]|C_n(s,b)||a_n| \leqq 1-\gamma, \tag{2.10}$$

where $C_n(s,b)$ is given by (1.12), then $f \in \mathcal{UST}_{s,b}(\beta,\gamma)$.

Proof. It suffices to show that

$$\beta\left|\frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}f(z)}-1\right|-\Re\left(\frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}f(z)}-1\right)<1-\gamma.$$

We have

$$\beta\left|\frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}f(z)}-1\right|-\Re\left(\frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}f(z)}-1\right) \leqq (1+\beta)\left|\frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}f(z)}-1\right| \leqq \frac{(1+\beta)\sum_{n=2}^{\infty}(n-1)|C_n(s,b)||a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}|C_n(s,b)||a_n||z|^{n-1}}$$
$$< \frac{(1+\beta)\sum_{n=2}^{\infty}(n-1)|C_n(s,b)||a_n|}{1-\sum_{n=2}^{\infty}\left|\left(\frac{1+b}{n+b}\right)^s\right||a_n|}.$$

The last expression is bounded above by $(1-\gamma)$ if (2.11) is satisfied. □

Remark 3. If we consider $\beta = 0$ or $s = -m, b = 0$ in Theorem 5, we obtain the same results in [14] and [12], respectively.

By virtue of (1.10) and Theorem 5, we have the following.

Theorem 6. *If $f \in \mathcal{A}$ satisfies the inequality:*

$$\sum_{n=2}^{\infty} n[n(1+\beta)-(\beta+\gamma)]|C_n(s,b)||a_n| \leqq 1-\gamma, \tag{2.11}$$

where $C_n(s,b)$ is given by (1.12), then $f \in \mathcal{UCV}_{s,b}(\beta,\gamma)$.

At last, we derive some coefficient inequalities of series expansion of functions belonging to these classes.

Theorem 7. *If $f \in \mathcal{A}$ and $f \in \mathcal{UST}_{s,b}(\beta,\gamma)$, then*

$$|a_n| \leqq \frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (n \in \mathbb{N}), \tag{2.12}$$

where $C_n(s,b)$ is given by (1.12) and

$$P_1 := P_1(\beta,\gamma) = \begin{cases} \frac{8(1-\gamma)(\arccos\beta)^2}{\pi^2(1-\beta^2)} & 0 \leqq \beta < 1, \\ \frac{8(1-\gamma)}{\pi^2} & \beta = 1, \\ \frac{\pi^2(1-\gamma)}{4\sqrt{t}(\beta^2-1)K^2(t)(1+t)} & \beta > 1. \end{cases} \tag{2.13}$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty}a_n z^n$ and $f \in \mathcal{UST}_{s,b}(\beta,\gamma)$. By (1.4), we obtain

$$\frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}f(z)} \prec q_{\beta,\gamma}(z).$$

Define

$$p(z) = \frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}f(z)} = 1 + \sum_{n=1}^{\infty}c_n z^n. \tag{2.14}$$

By means of Lemma 3, we have

$$|c_n| \leqq P_1, \quad (n \in \mathbb{N}), \tag{2.15}$$

where $P_1 := P_1(\beta,\gamma)$ is given by (2.13). It follows from (2.14) that

$$z(\mathcal{J}_{s,b}f)'(z) = p(z)\mathcal{J}_{s,b}f(z)$$

and comparing coefficients of z^n on both sides, we get

$$(n-1)a_n C_n(s,b) = \sum_{j=1}^{n-1} c_{n-j} a_j C_j(s,b), \quad a_1 = 1. \tag{2.16}$$

Combining (2.15) and (2.16), we get

$$a_1 = 1, \quad |a_2| = \frac{|c_1|}{|C_2(s,b)|} \leqq \frac{P_1}{|C_2(s,b)|} \tag{2.17}$$

and

$$\begin{aligned} |a_n| &\leqq \frac{1}{(n-1)|C_n(s,b)|}\left(|c_1| + \sum_{j=2}^{n-1} |c_{n-j}||a_j||C_j(s,b)|\right) \\ &\leqq \frac{P_1}{(n-1)|C_n(s,b)|}\left(1 + \sum_{j=2}^{n-1} |a_j||C_j(s,b)|\right) \qquad (k \in \mathbb{N} \setminus \{1,2\}). \end{aligned} \tag{2.18}$$

Next, we define the sequence $\{A_k\}_{k=1}^{\infty}$ as follows:

$$\begin{cases} A_1 = 1, & A_2 = \frac{P_1}{|C_2(s,b)|}, \\ A_n = \frac{P_1}{(n-1)|C_n(s,b)|}\left(1 + \sum_{j=2}^{n-1} A_j |C_j(s,b)|\right) & (n \in \mathbb{N} \setminus \{1,2\}). \end{cases} \tag{2.19}$$

In order to prove that

$$|a_k| \leqq A_k \quad (k \in \mathbb{N}),$$

we make use of the principle of mathematical induction. By noting that

$$|a_1| \leqq A_1 = 1, \quad |a_2| \leqq A_2 = \frac{P_1}{|C_2(s,b)|}.$$

Therefore, assuming that

$$|a_l| \leqq A_l \quad (l = 1,2,\ldots,n;\ n \in \mathbb{N}).$$

In view of Lemma 4, from (2.17) and (2.18), we get

$$|a_{n+1}| \leqq \frac{P_1}{n|C_{n+1}(s,b)|}\left(1 + \sum_{j=2}^{n} |a_j||C_j(s,b)|\right) \leqq \frac{P_1}{n|C_{n+1}(s,b)|}\left(1 + \sum_{j=2}^{n} A_j|C_j(s,b)|\right) = A_{n+1}.$$

Hence, by the principle of mathematical induction, we have

$$|a_k| \leqq A_k \quad (k \in \mathbb{N}) \tag{2.20}$$

as desired. By means of Lemma 4 and (2.20), we know that (2.12) holds true. □

By virtue of (1.10) and Theorem 7, we have the following.

Theorem 8. *If $f \in \mathcal{A}$ and $f \in \mathcal{UCV}_{s,b}(\beta,\gamma)$, then*

$$|a_n| \leqq \frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_n} \quad (n \in \mathbb{N}), \tag{2.21}$$

where $C_n(s,b)$ and $P_1 = P_1(\beta,\gamma)$ are given by (1.12) and (2.13), respectively.

Remark 4. If we consider $s = -m$ and $b = \frac{1-\lambda}{\lambda}$ in Theorems 7 and 8, respectively, we obtain the same results in [5].

Theorem 9. *If $f \in \mathcal{A}$ and $f \in \mathcal{UCC}_{s,b}(\beta,\gamma)$, then*

$$|a_n| \leqq \frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (n \in \mathbb{N}), \tag{2.22}$$

where $C_n(s,b)$ and $P_1 = P_1(\beta,\gamma)$ are given by (1.12) and (2.13), respectively.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{UCC}_{s,b}(\beta,\gamma)$, by definition, we have

$$\frac{z(\mathcal{J}_{s,b}f)'(z)}{\omega(z)} \prec q_{\beta,\gamma}(z)$$

for some $\omega(z) \in \mathcal{UST}(\beta,\gamma)$. For ω so that $\mathcal{J}_{s,b}g(z) = \omega(z), g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we have

$$\frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}g(z)} \prec q_{\beta,\gamma}(z).$$

Letting

$$p(z) = \frac{z(\mathcal{J}_{s,b}f)'(z)}{\mathcal{J}_{s,b}g(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{2.23}$$

By means of Lemma 3, we have

$$|c_n| \leqq P_1, \quad (n \in \mathbb{N}) \tag{2.24}$$

and in view of Theorem 7, we have

$$|b_n| \leqq \frac{1}{|C_n(s,b)|} \frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (n \in \mathbb{N}), \tag{2.25}$$

where $P_1 := P_1(\beta,\gamma)$ is given by (2.13). It follows from (2.23) that

$$z(\mathcal{J}_{s,b}f)'(z) = p(z)\mathcal{J}_{s,b}g(z)$$

and comparing coefficients of z^n on both sides, we get

$$nC_n(s,b)a_n = C_n(s,b)b_n + \sum_{j=1}^{n-1} c_{n-j}C_j(s,b)b_j, \quad a_1 = 1. \tag{2.26}$$

Combining (2.24), (2.26) and (2.27), we get

$$|a_2| \leqq \frac{|c_1| + |C_2(s,b)||b_2|}{|C_2(s,b)|} \leqq \frac{P_1}{|C_2(s,b)|}, \tag{2.27}$$

and

$$|a_n| \leqq \frac{1}{n|C_n(s,b)|}\left(|c_1| + \sum_{j=2}^{n-1} |c_{n-j}||C_j(s,b)||b_j| + |C_n(s,b)||b_n|\right) \leqq \frac{P_1}{n|C_n(s,b)|}\left(1 + \sum_{j=2}^{n-1} \frac{(P_1)_{j-1}}{(1)_{j-1}} + \frac{(P_1)_{n-1}}{P_1(1)_{n-1}}\right)$$
$$= \frac{1}{|C_n(s,b)|} \frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (k \in \mathbb{N} \setminus \{1,2\}). \qquad \Box \tag{2.28}$$

By virtue of (1.11) and Theorem 9, we have the following.

Theorem 10. *.If $f \in \mathcal{A}$ and $f \in \mathcal{UQC}_{s,b}(\beta,\gamma)$, then*

$$|a_n| \leqq \frac{1}{|C_n(s,b)|} \frac{(P_1)_{n-1}}{(1)_n} \quad (n \in \mathbb{N}), \tag{2.29}$$

where $C_n(s,b)$ and $P_1 = P_1(\beta,\gamma)$ are given by (1.12) and (2.13), respectively.

Remark 5. If we consider $s = -m$ and $b = \frac{1-\lambda}{\lambda}$ in Theorems 9 and 10, respectively, we obtain the same results in [1].

From Theorems 7–10, by using the method of [14], we can easily get the following distortion inequalities for functions in the classes defined in the first section.

Corollary 1. *If $f \in \mathcal{A}$ and $f \in \mathcal{UST}_{s,b}(\beta,\gamma)$, then*

$$r - r^2 \sum_{n=2}^{\infty} \frac{1}{|C_n(s,b)|} \frac{(P_1)_{n-1}}{(1)_{n-1}} \leqq |f(z)| \leqq r + r^2 \sum_{n=2}^{\infty} \frac{1}{|C_n(s,b)|} \frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (|z| = r < 1)$$

and

$$1 - r \sum_{n=2}^{\infty} \frac{n}{|C_n(s,b)|} \frac{(P_1)_{n-1}}{(1)_{n-1}} \leqq |f'(z)| \leqq 1 + r \sum_{n=2}^{\infty} \frac{n}{|C_n(s,b)|} \frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (|z| = r < 1),$$

where $C_n(s,b)$ and $P_1 = P_1(\beta,\gamma)$ are given by (1.12) and (2.13), respectively.

Remark 6. If we consider $\beta = 0$ in Corollary 1, we obtain the same results in [14].

Corollary 2. *If* $f \in \mathcal{A}$ *and* $f \in \mathcal{UCV}_{s,b}(\beta,\gamma)$*, then*

$$r - r^2\sum_{n=2}^{\infty}\frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_n} \leqq |f(z)| \leqq r + r^2\sum_{n=2}^{\infty}\frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_n} \quad (|z| = r < 1)$$

and

$$1 - r\sum_{n=2}^{\infty}\frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_{n-1}} \leqq |f'(z)| \leqq 1 + r\sum_{n=2}^{\infty}\frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (|z| = r < 1),$$

where $C_n(s,b)$ *and* $P_1 = P_1(\beta,\gamma)$ *are given by* (1.12) *and* (2.13)*, respectively.*

Corollary 3. *If* $f \in \mathcal{A}$ *and* $f \in \mathcal{UCC}_{s,b}(\beta,\gamma)$*, then*

$$r - r^2\sum_{n=2}^{\infty}\frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_{n-1}} \leqq |f(z)| \leqq r + r^2\sum_{n=2}^{\infty}\frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (|z| = r < 1)$$

and

$$1 - r\sum_{n=2}^{\infty}\frac{n}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_{n-1}} \leqq |f'(z)| \leqq 1 + r\sum_{n=2}^{\infty}\frac{n}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (|z| = r < 1),$$

where $C_n(s,b)$ *and* $P_1 = P_1(\beta,\gamma)$ *are given by* (1.12) *and* (2.13)*, respectively.*

Corollary 4. *If* $f \in \mathcal{A}$ *and* $f \in \mathcal{UQC}_{s,b}(\beta,\gamma)$*, then*

$$r - r^2\sum_{n=2}^{\infty}\frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_n} \leqq |f(z)| \leqq r + r^2\sum_{n=2}^{\infty}\frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_n} \quad (|z| = r < 1)$$

and

$$1 - r\sum_{n=2}^{\infty}\frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_{n-1}} \leqq |f'(z)| \leqq 1 + r\sum_{n=2}^{\infty}\frac{1}{|C_n(s,b)|}\frac{(P_1)_{n-1}}{(1)_{n-1}} \quad (|z| = r < 1),$$

where $C_n(s,b)$ *and* $P_1 = P_1(\beta,\gamma)$ *are given by* (1.12) *and* (2.13)*, respectively.*

Acknowledgements

The authors would like to express their gratitudes to the referees for the comments and suggestions. The present investigation was supported by the National Natural Science Foundation under Grants 11101053 and 11026040, the Key Project of Chinese Ministry of Education under Grant 211118, the Excellent Youth Foundation of Educational Committee of Hunan Province under Grant 10B002, the Foundation of Educational Committee of Hunan Province under Grant 11C0982, the Open Fund Project of Key Research Institute of Philosophies and Social Sciences in Hunan Universities under Grant 11FEFM02 and the Scientific Research Fund of Huaihua University under Grant HHUY2010-01 of the People's Republic of China.

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