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Cesàro averaging operators

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We define a family of Cesàro operators $\mathcal{C}^{\vec{\gamma}}$ on the polydisc U^n, and consider the question of its boundedness on some spaces of analytic functions.

1 Introduction and preliminaries

The Cesàro operator $\mathcal{C}$ is defined by

$$\mathcal{C}(f)(z) \; = \; \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n \, ,$$

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ being an analytic function on the unit disk U. It is known that the operator $\mathcal{C}$ is bounded on the Hardy spaces for $0 < p < \infty$. The boundedness on $H^p(U)$ for $p = 1$ was given a particularly elegant proof by A. Siskakis [13]. A different proof of the result can be found in [7]. J. Miao [10] extended the result to H^p, $p \in (0,1)$, by following ideas that originate from [9]. For $p \in (1,\infty)$ it is a consequence of a result of G. Hardy [8], see also [11]. The case $p = \infty$ was considered in [4]. $\mathcal{C}$ is also bounded on the Bergman spaces ([14]) as well as on the weighted Bergman spaces ([1]). In [6] P. Galanopoulos proved the boundedness of $\mathcal{C}$ on Dirichlet spaces $\mathcal{D}_{2,a}$, for $a \in (0,1)$. On the other hand $\mathcal{C}$ is not bounded on the classical Dirichlet space $\mathcal{D}_{2,0}$. Indeed, $\mathcal{C}(1)(z) \notin \mathcal{D}_{2,0}$.

For each complex γ with $\Re\gamma > -1$ and k nonnegative integer let A_k^γ be defined as the kth coefficient in the expression

$$\frac{1}{(1-x)^{\gamma+1}} \; = \; \sum_{k=0}^{\infty} A_k^\gamma x^k \, ,$$

so that $A_k^\gamma = \frac{(\gamma+1)\dots(\gamma+k)}{k!}$.

For an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on U, the generalized Cesàro operator is defined by

$$\mathcal{C}^\gamma(f)(z) \; = \; \sum_{n=0}^{\infty} \left(\frac{1}{A_n^{\gamma+1}} \sum_{k=0}^{n} A_{n-k}^\gamma a_k \right) z^n .$$

These operators were introduced in [15] on Hardy spaces and have been subsequently studied and proved its boundedness on all Hardy spaces in [2] and [17]. Note that for $\gamma = 0$ we obtain the classical Cesàro operator $\mathcal{C}^0 = \mathcal{C}$. K. Stempak proved that $\mathcal{C}^\gamma$ is bounded on $H^p(U)$ for $0 < p \le 2$. For $0 < p \le 1$, his method is similar to that of J. Miao; for $p = 2$, it is based on the boundedness of an

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appropriate sequence transformation, and an interpolation then yields the result for $p \in (1,2)$. After that, K. F. Andersen and J. Xiao proved the boundedness of $\mathcal{C}^\gamma$, on $H^p(U)$ for $p > 2$ using different methods. K. F. Andersen proved the stronger and somewhat surprising result that for $\Re\gamma > -1$ and $0 < p < \infty$ the integral mean $M_p(\mathcal{C}^\gamma(f), r)$ on $|z| = r$ of $\mathcal{C}^\gamma(f)$, is dominated by a constant multiple of $M_p(f, r)$, from which the result follows immediately. In fact, he proved the following theorem:

Theorem 1.1 *For $\Re\gamma > -1$ and $0 < p < \infty$, there is a constant $c_{\gamma,p}$ depending only on γ and p such that*

$$M_p(\mathcal{C}^\gamma(f), r) \;\le\; c_{\gamma,p} M_p(f, r)$$

for all $0 < r < 1$ and every function $f \in H(U)$.

The main purpose of this paper is to generalize Theorem 1.1 in the case of analytic functions defined on the polydisk.

The adjoint operator, $\mathcal{A}^\gamma$, of $\mathcal{C}^\gamma$, was considered in [2, 6, 11, 12, 15, 16] and [17].

The integral form of $\mathcal{C}^\gamma$ is (see [15])

$$\mathcal{C}^\gamma(f)(z) \;=\; \frac{\gamma+1}{z^{\gamma+1}} \int_0^z f(\zeta)\, \frac{(z-\zeta)^\gamma}{(1-\zeta)^{\gamma+1}}\, d\zeta\,,$$

or, taking simply as a path the segment joining 0 and z,

$$\mathcal{C}^\gamma(f)(z) \;=\; (\gamma+1) \int_0^1 f(tz)\, \frac{(1-t)^\gamma}{(1-tz)^{\gamma+1}}\, dt\,.$$

Motivated by [15], we define and study a family of Cesàro operators $\mathcal{C}^{\vec{\gamma}}$, for the polydisk U^n.

Let $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$, $\Re\gamma_j > -1$, $j = 1, \ldots, n$. The *generalized Cesàro* operator $\mathcal{C}^{\vec{\gamma}}$ is defined by

$$\mathcal{C}^{\vec{\gamma}}(f)(z) \;=\; \sum_{|\alpha|=0}^{\infty} \left(\frac{\sum_{\beta \le \alpha} a_{\alpha-\beta} \prod_{j=1}^n A_{\beta_j}^{\gamma_j}}{\prod_{j=1}^n A_{\alpha_j}^{\gamma_j+1}} \right) z^\alpha\,,$$

whenever

$$f(z) \;=\; \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$$

is an analytic function on U^n $\big(\alpha$ and β are multi–indices from $\mathbb{Z}_+^n\big)$. A simple calculation with power series then gives

$$\mathcal{C}^{\vec{\gamma}}(f)(z) \;=\; \prod_{j=1}^n (\gamma_j+1) \int_0^1 \cdots \int_0^1 \frac{f(\tau_1 z_1, \ldots, \tau_n z_n)}{\prod_{j=1}^n \big(1 - \tau_j z_j\big)^{\gamma_j+1}} \prod_{j=1}^n \big(1-\tau_j\big)^{\gamma_j} d\tau\,,$$

where $d\tau = d\tau_1 \ldots d\tau_n$.

In what follows, for $z, w \in \mathbb{C}^n$ we write $z \cdot w = (z_1 w_1, \ldots, z_n w_n)$; $e^{i\theta}$ is an abbreviation for $\big(e^{i\theta_1}, \ldots, e^{i\theta_n}\big)$; $d\tau = d\tau_1 \ldots d\tau_n$; $d\theta = d\theta_1 \ldots d\theta_n$ and r, τ are vectors in $\mathbb{C}^n$. If we write $0 \le r < 1$, where $r = (r_1, \ldots, r_n)$ it means $0 \le r_j < 1$ for $j = 1, \ldots, n$.

In order to prove the main result we need several auxiliary results which are incorporated in the following lemmas.

Lemma 1.2 ([3].) *Let $0 < p < \infty$ and $0 \le r < 1$. Then there is a constant C independent of f and r such that*

$$\int_{[0,2\pi]^n} \sup_{0 \le \tau < 1} \big|f\big(\tau \cdot r \cdot e^{i\theta}\big)\big|^p\, d\theta \;\le\; C \int_{[0,2\pi]^n} \big|f\big(r \cdot e^{i\theta}\big)\big|^p\, d\theta$$

for all $f \in H(U^n)$.

Lemma 1.3 ([5, p. 65].) *For each $1 < a < \infty$ there is a positive constant $C = C(a)$ such that*

$$\int_{-\pi}^{\pi} \left|1-\rho e^{i\theta}\right|^{-a} d\theta \ \le\ C(1-\rho)^{1-a}\,, \quad \text{if} \quad 0 \ \le\ \rho \ <\ 1\,.$$

The following lemma is a well–known generalization of a theorem in [9].

Lemma 1.4 *Let $0 < p < \infty$, $1 < a < \infty$ and $0 \le r < 1$. Then there is a constant C independent of f and r such that*

$$\int_{[0,1)^n}\left(\int_{[0,2\pi]^n}\left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|^{pa} d\theta\right)^{1/a}\prod_{j=1}^{n}\left(1-\tau_j\right)^{-1/a} d\tau \ \le\ C\int_{[0,2\pi]^n}\left|f\left(r\cdot e^{i\theta}\right)\right|^{p} d\theta\,,$$

for all $f \in H(U^n)$.

The main result in this paper is the following theorem:

Theorem 1.5 *Let $0 < p \le 1$, $\gamma = (\gamma_1, \dots, \gamma_n)$ such that $\Re\gamma_j > -1$, $j = 1, \dots, n$ and $0 \le r < 1$. Then there is a constant C independent of f and r such that*

$$\int_{[0,2\pi]^n}\left|\mathcal{C}^{\vec{\gamma}}(f)\left(r\cdot e^{i\theta}\right)\right|^{p} d\theta \ \le\ C\int_{[0,2\pi]^n}\left|f\left(r\cdot e^{i\theta}\right)\right|^{p} d\theta\,,$$

for all $H(U^n)$.

2 Proofs of the main results

We now prove the result, emphasizing that in the proof we use Miao's arguments, which are modifications of the corresponding arguments used in the case of the unit disk. Throughout the following proof C will denotes a constant which may change from line to line.

P r o o f. In what follows, for the sake of simplicity, we assume that γ_j, $j = 1, \dots, n$, are real numbers such that $\gamma_j > -1$. Let $f \in H(U^n)$ and $t_k = 1 - 2^{-k}$, $k \in \mathbf{N} \cup \{0\}$. Let

$$I \ =\ M_p^p\left(\mathcal{C}^{\vec{\gamma}}(f), r\right) \ =\ \int_{[0,2\pi]^n}\left|\mathcal{C}^{\vec{\gamma}}(f)\left(r\cdot e^{i\theta}\right)\right|^{p} d\theta\,.$$

By Lemma 1.2 and some simple calculations, we obtain

$$\begin{aligned} I \ &\le\ C\int_{[0,2\pi]^n}\left(\int_{[0,1)^n}\frac{\left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|}{\left|\prod_{j=1}^{n}\left(1-\tau_j r_j e^{i\theta_j}\right)^{\gamma_j+1}\right|}\prod_{j=1}^{n}\left(1-\tau_j\right)^{\gamma_j} d\tau\right)^{p} d\theta \qquad (2.1)\\ &\le\ C\sum_{k_1,\dots,k_n=1}^{\infty}\int_{[0,2\pi]^n}\left(\int_{t_{k_1-1}}^{t_{k_1}}\cdots\int_{t_{k_n-1}}^{t_{k_n}}\frac{\left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|}{\left|\prod_{j=1}^{n}\left(1-\tau_j r_j e^{i\theta_j}\right)^{\gamma_j+1}\right|}\right.\\ &\qquad\qquad\left.\times\prod_{j=1}^{n}\left(1-\tau_j\right)^{\gamma_j} d\tau\right)^{p} d\theta\\ &\le\ C\sum_{k_1,\dots,k_n=1}^{\infty}\frac{1}{2^{p\sum_{j=1}^{n}k_j(\gamma_j+1)}}\int_{[0,2\pi]^n}\sup_{t_{\mathbf{k}-1}<\tau<t_{\mathbf{k}}}\left(\frac{\left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|}{\left|\prod_{j=1}^{n}\left(1-\tau_j r_j e^{i\theta_j}\right)^{\gamma_j+1}\right|}\right)^{p} d\theta\\ &\le\ C\sum_{k_1,\dots,k_n=1}^{\infty}\frac{1}{2^{p\sum_{j=1}^{n}k_j(\gamma_j+1)}}\int_{[0,2\pi]^n}\sup_{0\le\tau<t_{\mathbf{k}}}\left(\frac{\left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|}{\left|\prod_{j=1}^{n}\left(1-\tau_j r_j e^{i\theta_j}\right)^{\gamma_j+1}\right|}\right)^{p} d\theta \ \le \end{aligned}$$

$$
\begin{aligned}
&\le C \sum_{k_1,\dots,k_n=1}^{\infty} \frac{1}{2^{p\sum_{j=1}^{n} k_j(\gamma_j+1)}} \int_{[0,2\pi]^n} \left(\frac{\left|f\left(t_{\mathbf{k}}\cdot r\cdot e^{i\theta}\right)\right|}{\left|\prod_{j=1}^{n}\left(1-t_{k_j} r_j e^{i\theta_j}\right)^{\gamma_j+1}\right|} \right)^p d\theta \\
&\le C \sum_{k_1,\dots,k_n=1}^{\infty} \int_{t_{k_1}}^{t_{k_1+1}} \cdots \int_{t_{k_n}}^{t_{k_n+1}} \int_{[0,2\pi]^n} \left(\frac{\left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|}{\left|\prod_{j=1}^{n}\left(1-\tau_j r_j e^{i\theta_j}\right)^{\gamma_j+1}\right|} \right)^p d\theta \\
&\qquad\qquad \times \prod_{j=1}^{n}\left(1-\tau_j\right)^{p(\gamma_j+1)-1} d\tau \\
&\le C \int_{[0,1)^n} \int_{[0,2\pi]^n} \left(\frac{\left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|}{\left|\prod_{j=1}^{n}\left(1-\tau_j r_j e^{i\theta_j}\right)^{\gamma_j+1}\right|} \right)^p d\theta \prod_{j=1}^{n}\left(1-\tau_j\right)^{p(\gamma_j+1)-1} d\tau\,.
\end{aligned}
$$

Here $t_{\mathbf{k}}$ denotes $(t_{k_1},\dots,t_{k_n})$.

Choose $a>1$ such that $\max_{j=1,..,n}\{1-p(\gamma_j+1)\}<1/a$. Then by Hölder's inequality with exponents a and $b=a/(a-1)$, and using Lemma 1.3 we obtain

$$
\begin{aligned}
&\int_{[0,2\pi]^n} \left(\frac{\left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|}{\left|\prod_{j=1}^{n}\left(1-\tau_j r_j e^{i\theta_j}\right)^{\gamma_j+1}\right|} \right)^p d\theta \\
&\le \left(\int_{[0,2\pi]^n} \left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|^{pa} d\theta \right)^{1/a} \left(\int_{[0,2\pi]^n} \frac{d\theta}{\left|\prod_{j=1}^{n}\left(1-\tau_j r_j e^{i\theta_j}\right)^{\gamma_j+1}\right|^{pb}} \right)^{1/b} \qquad (2.2)\\
&\le C \left(\int_{[0,2\pi]^n} \left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|^{pa} d\theta \right)^{1/a} \prod_{j=1}^{n}\left(1-\tau_j r_j\right)^{-(\gamma_j+1)p+1-1/a} \\
&\le C \left(\int_{[0,2\pi]^n} \left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|^{pa} d\theta \right)^{1/a} \prod_{j=1}^{n}\left(1-\tau_j\right)^{-(\gamma_j+1)p+1-1/a}.
\end{aligned}
$$

From (2.1), (2.2) and by Lemma 1.4, we obtain

$$
\begin{aligned}
M_p^p\left(\mathcal{C}^{\vec{\gamma}}(f), r\right) &\le C \int_{[0,1)^n} \left(\int_{[0,2\pi]^n} \left|f\left(\tau\cdot r\cdot e^{i\theta}\right)\right|^{pa} d\theta \right)^{1/a} \prod_{j=1}^{n}\left(1-\tau_j\right)^{-1/a} d\tau \\
&\le C \int_{[0,2\pi]^n} \left|f\left(r\cdot e^{i\theta}\right)\right|^{p} d\theta\,,
\end{aligned}
$$

as desired. □

The Hardy space $H^p(U^n)\,(0<p<\infty)$ is defined on U^n by

$$
H^p(U^n) \;=\; \{\, f \mid f\in H(U^n) \text{ and } ||f||_{H^p(U^n)}<\infty \,\}\,,
$$

where

$$
||f||^p_{H^p(U^n)} \;=\; \sup_{0\le r_j<1,\ j=1,\dots,n} \int_{[0,2\pi]^n} \left|f\left(r_1 e^{i\theta_1},\dots,r_n e^{i\theta_n}\right)\right|^p d\theta\,.
$$

From Theorem 1.5 we obtain the following corollaries.

Corollary 2.1 *The generalized Cesàro operator is bounded on $H^p(U^n)$ for $p\in(0,1]$.*

Given $0 < p, q < \infty$, and positive Borel measures μ_j, $j = 1, \ldots, n$ on $r_j \in (0,1)$, the weighted space $\mathcal{A}^{p,q}_{\mu}(U^n)$ consists of those functions f analytic on U^n for which

$$||f||_{\mathcal{A}^{p,q}_{\mu}(U^n)} = \left[\int_{(0,1)^n}\left(\int_{[0,2\pi]^n}\left|f\left(r\cdot e^{i\theta}\right)\right|^p d\theta\right)^{\frac{q}{p}}\prod_{j=1}^{n} d\mu_j\left(r_j\right)\right]^{1/q} < \infty .$$

Of particular interest are the absolutely continuous measures of the form $d\mu_j\left(r_j\right) = \left(1-r_j\right)^a r_j^b dr_j$, the spaces obtained include the Bergman spaces.

Corollary 2.2 *The generalized Cesàro operator is bounded on* $\mathcal{A}^{p,q}_{\mu}(U^n)$ *for* $p\in(0,1]$ *and* $0 < q < \infty$. *Moreover, there is a constant* C *independent of* f, *such that*

$$\left|\left|\mathcal{C}^{\vec{\gamma}}(f)\right|\right|_{\mathcal{A}^{p,q}_{\mu}(U^n)} \leq C\left|\left|f\right|\right|_{\mathcal{A}^{p,q}_{\mu}(U^n)} .$$

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