Jordan and Julia

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1 Quasiconformal Surgery

Let f be a rational function of degree $d > 1$, and denote by $\mathcal{J}$ and $\mathcal{F}$ its Julia set and Fatou set, respectively. Then $\mathcal{J}$ is always non-empty and compact, and either connected or else has uncountably many connected components. On the other hand, the corresponding Fatou set is either empty or else consists of one, two, or infinitely many connected components – called stable domains or domains of normality. $\mathcal{J}$ is called a *dendrite,* if it is connected and if the Fatou set is non-empty and connected. For standard facts and details the reader is referred to [**?**],[**?**],[**?**].

In this paper we are concerned with the following question:

When is a Julia set a quasiconformal Jordan arc or curve?

We will give a complete answer in both cases. The proofs rely on two facts:

The *first* one is Theorem VI.5.1 in [**?**] and its parabolic counterpart, stated as Theorem 1 below. Both theorems say that it is possible to "fuse" critical points in attracting and parabolic basins, by "quasiconformal surgery".

Theorem 1 *Let f be a rational function with simply connected parabolic basin U, such that the corresponding fixed point has $m = \deg f|_U$ simple preimages on ∂U. Then there exist a compact and connected subset E of $\overline{U}$, which intersects ∂U only in finitely many points, a quasiconformal self-map ϕ of the sphere and a rational function f_0 such that $f \circ \phi = \phi \circ f_0$ holds on $\mathcal{J}_{f_0}$ and on $\widehat{\mathbb{C}} \setminus \phi^{-1}(E)$; f_0 has only one critical point in its parabolic basin $\phi^{-1}(U)$, of order $m - 1$.*

The proof of Theorem 1 will be given in Section 6. It can be viewed as the analytic part of the proofs of Theorems 3 and 4, but might have independent interest.

Secondly, we need a characterization of those rational functions whose Julia sets are Jordan arcs or curves. In the Jordan curve case we simply have to consider all functions whose Fatou sets consist of two components, and hence have to distinguish four cases. The following result gives a complete answer in the Jordan arc case (see [**?**], Theorems 4 and 5 in chapter 5, section 5):

Theorem A *Let f be a rational function, whose Julia set is a dendrite. Then the condition $f(\mathcal{J}\cap\mathcal{C}\cup\{a,b\})=\{a,b\}$, where $\mathcal{C}$ denotes the critical set, is necessary and sufficient for $\mathcal{J}$ to be a Jordan arc – with endpoints a and b. In that case, $\mathcal{J}$ contains $d-1$ distinct and simple critical points, and $f^{-1}(\{a,b\})=\mathcal{C}\cap\mathcal{J}\cup\{a,b\}$ holds. If f is a polynomial of degree d, then f is conjugate to $\pm T_d$, where T_d denotes the d-th Chebychev polynomial. Hence the Julia set is a line segment.*

We will prove that, in the Jordan arc case, $\mathcal{J}$ is a quasiconformal arc if and only if f has a (super-)attracting fixed point or a parabolic fixed point which is an endpoint of $\mathcal{J}$. In the Jordan curve case it is known that $\mathcal{J}$ is a quasicircle if the Fatou set consists of two basins of attraction, i.e., if f is hyperbolic. We prove that the same is true if $\mathcal{F}$ consists of a single Leau flower. In all other cases, $\mathcal{J}$ is, obviously, *not* a quasicircle.

2 Julia Sets and Jordan Arcs

Theorem 2 *Let $\mathcal{J}$ be a Jordan arc, and assume that $U=\mathcal{F}$ is a basin of attraction (Böttcher or Schröder domain). Then $\mathcal{J}$ is a quasiconformal Jordan arc, parametrized by $z=\phi(t)$, $-1\le t\le 1$, where ϕ is a quasiconformal self-map of $\widehat{\mathbb{C}}$. Moreover, the conjugation $f\circ\phi=\phi\circ(\pm T_d)$ holds on some neighbourhood of $\mathcal{J}_{T_d}=[-1,1]$.*

Theorem 3 *Let $\mathcal{J}$ be a Jordan arc, and assume that $U=\mathcal{F}$ is a parabolic basin (Leau domain) whose corresponding fixed point is an endpoint of $\mathcal{J}$. Then $\mathcal{J}$ is a quasiconformal Jordan arc, i.e., the image of the interval $[0,+\infty]$ under a quasiconformal map ϕ of the sphere. Moreover, the conjugation $f\circ\phi=\phi\circ f_0$ holds on $\mathcal{J}_{f_0}=[0,+\infty]$ and even on some neighbourhood of $(0,+\infty)\setminus f_0^{-1}(\{0\})$, and d^2f_0 is analytically conjugate to T_d:*

$$f_0=d^{-2}S^{-1}\circ T_d\circ S\,,\quad S(z)=\frac{1-z}{1+z}\,. \tag{1}$$

Remark For the proofs of both theorems we need only that the condition stated in Theorem A is necessary. Its sufficiency follows from the proofs of Theorems 2 and 3.

Example 1 $f(z)=a+\dfrac{1+a}{1-a}\left(z-2+\dfrac{1}{z}\right)$ with Re $a>0$ and $a\neq 1$ has two critical points $z=\pm1$; $z=1$ belongs to $\mathcal{J}$, this following from

$1 \mapsto a \mapsto b = 1/a \mapsto b$ and $f'(b) = (1+a)^2$. Hence the basin of attraction at ∞ is simply connected and completely invariant, and so coincides with the Fatou set. This shows that the Julia set is a dendrite and in fact a quasiconformal Jordan arc by Theorem 2; compare the figure for $a = 0.1 + i$. For a real the Julia set is the interval $[a, b]$ or $[b, a]$, respectively. Otherwise $\mathcal{J}$ has nowhere a tangent. In all cases, f is quasiconformally conjugate to T_2 on a neighbourhood of $[-1, 1]$.

The Julia set of

$$f(z) = a + \frac{1+a}{1-a}\left(z - 2 + \frac{1}{z}\right),$$

$a = 0.1 + i$, is a quasiconformal Jordan arc.

Example 2 The fixed points 0 and ∞ of $f(z) = z\left(\dfrac{cz+1}{z+1}\right)^2$, $0 < |c| < 1$, are rationally indifferent and repelling, respectively, and hence belong to $\mathcal{J}$. The critical points $-1/c$ and -1 are mapped onto 0 and ∞ . Thus $\mathcal{J}$ is a quasiconformal Jordan arc if it is a dendrite. This depends on the critical points different from $-1/c$ and -1: $\mathcal{J}$ is a dendrite if and only if they belong to the parabolic basin associated with the fixed point $z = 0$; f is quasiconformally conjugate to $f_0(z) = \dfrac{z}{9}\left(\dfrac{z-3}{3z-1}\right)^2$, at least on $\mathcal{J}_{f_0} = [0, +\infty]$ and on some open neighbourhood of $(0, 3) \cap (3, +\infty]$.

3 Julia Sets and Jordan Curves

If the Julia set of f is a Jordan curve, then $\mathcal{F}$ consists of two connected components U and V , say. The converse is also true[1], and there are four possibilities as follows:

[1] The case where both domains are parabolic is contained in a more general result due to Ch. Mattler (1994).

(A) U and V are attracting.

(B) U is attracting and V is parabolic.

(C) U and V form a Leau flower.

(D) U and V are parabolic with different fixed points.

Case (A) is well-known (due to D. Sullivan, see [?], Theorem VI.2.1): The Julia set is a quasicircle, and f is quasiconformally conjugate to $z \mapsto z^{\pm d}$ on a neighbourhood of $\partial\mathbb{D}$.

In case (B), $\mathcal{J}$ is a Jordan curve, but not a quasicircle, since it has a cusp at the parabolic fixed point (and hence has cusps at the points of a dense subset). It is easily seen, by applying [?], Theorem VI.5.1 and Theorem 1, that f is quasiconformally conjugate to $f_0(z) = d^{-1}(z^d + d - 1)$, at least on $\mathcal{J}_{f_0}$ and on some open neighbourhood of $\mathcal{J}_{f_0} \setminus \{z : z^d = 1\}$.

Case (C) is covered by Theorem 4.

Example 3 By conjugating Example 2 by z^2 we obtain $f(z) = z\dfrac{cz^2+1}{z^2+1}$. If the Fatou set consists simply of the Leau flower about $z = 0$ (this depends on $c \neq 0$, $|c| < 1$), then, by Theorem 4, the Julia set is a quasicircle, and f is quasiconformally conjugate to $z \mapsto \dfrac{z}{3}\dfrac{z^2-3}{3z^2-1}$, representing the functions of degree 3 of type (C). A possibly simpler representative is the Blaschke product $B(z) = \dfrac{2z^3+1}{2+z^3}$.

In case (D), $\mathcal{J}$ is also Jordan curve, but not a quasicircle. A simple application of Theorem 1 leads to the quasiconformal representatives $z \mapsto \pm S_A\big(\big(S_a(z)\big)^d\big)$, where $S_a(z) = \dfrac{a-z}{1-az}$, $A = a^d$, and a, $|a| > 1$, is any solution of the equation $a^d - a^{-d} = \pm d(a - a^{-1})$. We note that these functions are invariant under the conjugation $z \mapsto 1/z$. We do not know whether they belong to different quasiconformal equivalence classes for different parameters.

Example 4 [2] $f(z) = z\dfrac{z^2+cz+1}{z^2-cz+1}$, $0 < c < 2$, has two completely invariant parabolic basins, U_0 and U_∞, say, associated with the fixed points 0 and ∞, respectively. Since f is real, U_0 and U_∞ are symmetric with respect to the real axis: U_0 contains two critical points z_0 and $\overline{z}_0$, while the critical points $-1/z_0$ and $-1/\overline{z}_0$ are contained in U_∞; f is quasiconformally conjugate to $z \mapsto z\dfrac{z^2 - i\sqrt{2}\,z - 1}{z^2 + i\sqrt{2}\,z - 1}$, at least on $\mathcal{J}$ and on some open neighbourhood of $\mathcal{J} \setminus f^{-1}(\{0\})$. This function represents the class of degree-3-functions of type (D). The Julia set is a Jordan curve, but not a quasicircle. Each parabolic basin contains one critical point of order 2.

Theorem 4 *Suppose that f is a rational function of degree $d > 1$, whose Fatou set $\mathcal{F}$ consists of a Leau flower. Then $\mathcal{J}$ is a quasicircle, and f is*

[2] Due to W. Bergweiler and A. Eremenko.

quasiconformally conjugate to the Blaschke product

$$B(z) = \frac{z^d + \frac{d-1}{d+1}}{1 + \frac{d-1}{d+1} z^d} \,.$$

There exists a quasiconformal self-map of $\widehat{\mathbb{C}}$ *such that the conjugation* $f \circ \phi = \phi \circ B$ *holds on* $\partial\mathbb{D}$ *and on some neighbourhood of* $\partial\mathbb{D} \setminus \{z : z^d = 1\}$ *. The Julia set of* f *has the parametrization* $\zeta \mapsto \phi(\zeta)\,, |\zeta| = 1$ *.*

Remark We have tacitely assumed that U and V are fixed under f ; if not, f has to be replaced by f^2 .

Theorems 3 and 4 are equivalent, provided $\mathcal{F}$, in Theorem 4, is symmetric with respect to the parabolic fixed point. If we place the fixed point at $z = 0$, and if f satisfies the hypothesis of Theorem 3, then $\sqrt{f(z^2)}$ satisfies the hypothesis of Theorem 4 and is odd. Conversely, if, in Theorem 4, f is odd, then $(f(\sqrt{z}\,))^2$ satiesfies the hypothesis of Theorem 3.

4 Proof of Theorems 2 and 3

Let $U = \mathcal{F}$ be a simply connected basin of attraction. Then f maps U onto itself with maximal degree d . By [?], Theorem VI.5.1, f is quasiconformally conjugate to a polynomial of degree d , i.e., there exists a quasiconformal self-map of the sphere, say ψ , a compact set $E \subset U$ and a polynomial P , such that $f \circ \psi = \psi \circ P$ holds on $\widehat{\mathbb{C}} \setminus \psi^{-1}(E)$. By hypothesis, the Julia set of P is a Jordan arc. Then Theorem A says that P is analytically conjugate to $\pm T_d$, and so f is quasiconformally conjugate to $\pm T_d$, $f \circ \phi = \phi \circ (\pm T_d)$ on some neighbourhood of $[-1, 1]$; $\mathcal{J}$ is the quasiconformal Jordan arc parametrized by $z = \phi(t)$, $-1 \le t \le 1$. □

The parabolic case (Theorem 3) can be treated in a similar way. We assume that f has a parabolic fixed point at $z = 0$ and that $\mathcal{J}$ is a Jordan arc (starting at $z = 0$ and) ending at $z = \infty$. Then, by Theorem A, f is given by

$$f(z) = z(Q(z))^2 \,,$$

where $Q(0) = 1$, $Q(\infty) \ne \infty$, and Q has only simple zeros and poles. Hence

$$f(z^2) = (g(z))^2$$

holds with $g(z) = zQ(z^2)$. Since the Fatou set of g consists of two completely invariant parabolic basins V and $-V$ with common parabolic fixed point zero, the Julia set, which is the common boundary, is a Jordan curve; it contains no critical points.

To continue the proof of Theorem 3 we apply Theorem 1 to g and, simultaneously, to V and $-V$, and obtain an odd quasiconformal map ψ and an odd rational function g_0 with $g \circ \psi = \psi \circ g_0$ on $\mathcal{J}_{g_0}$, such that each corresponding parabolic basin V_0 and $-V_0$ of g_0 contains exactly one critical point of order $d-1$. We may assume that the critical points are located at $\pm i$.

Since g_0 is odd, the z^2-conjugation may be reversed to obtain a rational function f_0 satisfying $f_0(z^2) = (g_0(z))^2$, and f and f_0 are quasiconformally conjugate, at least on $\mathcal{J}$ and on some open neighbourhood of $\mathcal{J} \setminus f^{-1}(\{0\})$, by the following reasons. Since ψ is odd, $\phi(z) = (\psi(\sqrt{z}\,))^2$ is well defined on $\widehat{\mathbb{C}}$, and quasiregular on $\mathbb{C} \setminus \{0\}$. Obviously, ϕ is injective, and so a quasiconformal self-map of the sphere, and $f \circ \phi = \phi \circ f_0$ holds on $\mathcal{J}_{f_0}$ and on some open neighbourhood of $\mathcal{J}_{f_0} \setminus f_0^{-1}(\{0\})$.

The Julia set of f_0 is a Jordan arc, and its Fatou set consists of a single parabolic basin U_0 which contains exactly one critical point $z = -1$, of order $d - 1$, with critical value $w_0 = f_0(-1)$.

Remark The reader might wonder why we did not apply Theorem 1 immediately to f . The reason is that the proof of Theorem 1 does not work in this situation, since the parabolic fixed point 0 is also a critical value.

Since $\Big(f_0(\frac{1+z}{1-z}) - w_0\Big)^{-1}$ is a polynomial of degree d , we may write

$$f_0(z) = w_0 \, \frac{P(\frac{1-z}{1+z}) - 1}{P(\frac{1-z}{1+z}) + 1} \, . \tag{2}$$

By Theorem A, f_0 has only doubly poles and zeros, except at $z = 0$ and $z = \infty$, and hence the polynomial P assumes the values -1 and 1 always with multiplicity 2 , except at $\zeta = 1$ and $\zeta = -1$: $P(1) = 1$ and $P(-1) = \pm 1$, $P'(\pm 1) \neq 0$. This, however, implies that $\dfrac{(P'(\zeta))^2}{(P(\zeta))^2 - 1}(\zeta^2 - 1) = d^2$, and $P(1) = 1$ gives $P = T_d$.

Differentiation of eq. (**??**) leads to $1 = f_0'(0) = -w_0 T_d'(1) = -w_0 d^2$. In particular, w_0 is real, and hence f_0 is a real rational function. Thus the Julia set of f_0 is a Jordan arc, symmetric with respect to the real axis, and starting at 0 and ending at ∞ . Since $-1 \notin \mathcal{J}_{f_0}$ we conclude that $\mathcal{J}_{f_0} = [0, +\infty]$, and, moreover,

$$f_0 = d^{-2} S \circ T_d \circ S$$

with $S(z) = \dfrac{1-z}{1+z} = S^{-1}(z)$. The conjugation $f \circ \phi = \phi \circ f_0$, where ϕ is quasiconformal on $\widehat{\mathbb{C}}$, holds on some open neighbourhood of $(0, +\infty) \setminus f_0^{-1}(\{0\})$, and, by continuity, also on $\mathcal{J}_{f_0} = [0, +\infty]$. □

5 Proof of Theorem 4

We apply Theorem 1 twice (or simultaneously) to f and its parabolic basins U and V to obtain a quasiconformal map $\phi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$, such that $F = \phi^{-1} \circ f \circ \phi$ is rational and has a Leau flower about the parabolic fixed point $z = 0$, each basin containing one critical point of order $d - 1$. We may place these points at $z = \pm i$, and hence obtain

$$\frac{F(z)/F(i) - 1}{F(z)/F(-i) - 1} = \left(\frac{1 + iz}{1 - iz}\right)^d . \tag{3}$$

An easy calculation, using (**??**) and the necessary condition $F(z) = z + O(z^3)$ as $z \to 0$, shows that $F(\pm i) = \pm i/d$. Hence F is a real rational function, whose Julia set is the extended real axis.

Now let T be a Möbius transform mapping the upper half-plane onto the unit disc such that $z = i$ and $z = 0$ correspond to $w = 0$ and $w = 1$, respectively. Then $B = T \circ F \circ T^{-1}$ is a Blaschke product with $(d-1)$-fold critical points at $z = 0$ and $z = \infty$, and parabolic fixed point $z = 1$. Since the critical values of F are on the imaginary axis and symmetric with respect to the real axis, the critical values of B are real and symmetric with respect to the unit circle, say $z = a$ and $z = 1/a$. This leads to

$$B(z) = \frac{z^d + a}{1 + az^d},$$

and $B'(1) = 1$ implies $a = \frac{d-1}{d+1}$. Thus, Theorem 4 is completely proved. □

6 Proof of Theorem 1

To fix ideas we assume that

$$f(z) = z - z^{s+1} + \cdots \tag{4}$$

holds near $z = 0$, and denote by U the (simply connected) parabolic basin associated with the fixed point $z = 0$, which contains some interval $(0, \delta)$. It is not hard to construct a smooth and piecewise analytic Jordan curve α in $U \cup \{0\}$ with the following properties:

(a) α is the boundary of a forward invariant petal $\Delta \subset U$: $f(\overline{\Delta}) \subset \Delta \cup \{0\}$, which contains all critical values of f arising from critical points in U.

(b) Near $z = 0$, α has the representation

$$\alpha(t) = f^{-n}((it)^{1/s}) = (it)^{1/s} + n(it)^{1+1/s} + \cdots$$

for some (large) n; t is real and $\zeta = \tau^{1/s}$ maps $|\arg \tau| < \pi$ conformally onto $|\arg \zeta| < \pi/s$.

Let Δ_1 be the connected component of $f^{-1}(\Delta)$ which contains Δ. Then $f : \Delta_1 \longrightarrow \Delta$ is a proper map of degree m, and Δ_1 is bounded by a Jordan curve $\beta \subset \overline{U}$, which intersects the boundary of U at $z_1, \ldots, z_{m-1}$ and $z_m = 0$, with $f(z_\mu) = 0$. The domain $D = \Delta_1 \setminus \overline{\Delta}$ is simply connected. The points z_μ divide β into smooth and piecewise analytic Jordan arcs β_μ, starting at $z_{\mu-1}$ and ending at z_μ, where $z_0 = z_m = 0$. Since f is conformal on β_μ and maps β_μ onto α, β has vertices only at z_μ, where the respective tangents intersect with angle π/s. In particular, β has the parametrization

$$\beta(t) = \alpha(t) + (\alpha(t))^{s+1} + \cdots = (it)^{1/s} + (n+1)(it)^{1+1/s} + \cdots$$

near $z = 0$.

Let h be a conformal map of Δ onto the lower half-plane $\mathbb{H}$, such that $z = 0$ corresponds to $w = \infty$. Since, locally, α is mapped onto a segment of the imaginary axis by $z \mapsto (f^n(z))^s = z^s - nsz^{2s} + \cdots$, h can be normalized in such a way that

$$h(z) = \frac{-i/s}{z^s}(1 + nsz^s + c_1 z^{s+1} + \cdots) \tag{5}$$

holds near $z = 0$. As a function of arc-length, h is continuously differentiable (and piecewise real-analytic) on $\alpha \setminus \{0\}$, and the same holds for h^{-1} on $\mathbb{R}$. Our aim is to extend h to the domain Δ_1 to map D quasiconformally onto the parallel-strip $S = \{w : 0 < \text{Im } w < 1\}$, such that

$$h \circ f = T \circ h \quad \text{holds on } \beta\,, \tag{6}$$

where T , to be specified later, is a proper map of degree m from the half-plane $\mathbb{H}_1 = \{w : \text{Im } w < 1\}$ onto $\mathbb{H}$, which has only one critical point, of order $m-1$, in $\mathbb{H}_1$. Then g , defined by $g = h^{-1} \circ T \circ h$ on Δ_1 , and $g = f$ elsewhere, is analytic on $\widehat{\mathbb{C}} \setminus \overline{D}$, and a quasiregular[3] self-map of the sphere, whose sequence of iterates (g^n) has bounded dilatation. This follows from the fact that every orbit $O^+(z)$ intersects $\overline{D}$ at most twice, except possibly when $z \in f^{-n}(\{0\})$. Thus g^n is at most K^2-quasiregular, provided g is K-quasiregular, since $f^{-n}(\{0\})$ is a finite set. By a result of L. Geyer [**?**], Satz 2.7, which generalizes Shishikura's *"qc-lemma"*, g is quasiconformally conjugate to a rational function f_0 of degree d: there exists a quasiconformal self-map of $\widehat{\mathbb{C}}$ with $\phi(0) = 0$, such that $g \circ \phi = \phi \circ f_0$ holds on $\widehat{\mathbb{C}}$, and hence $f \circ \phi = \phi \circ f_0$ holds on $\widehat{\mathbb{C}} \setminus \phi^{-1}(\Delta_1)$. Obviously, g has only one critical point in U , and $g^n = h^{-1} \circ T^n \circ h$ tends to zero on Δ . Hence f_0^n tends to zero on $\phi^{-1}(\Delta) \supset f_0(\phi^{-1}(\Delta))$, and $0 \in \partial\phi^{-1}(\Delta)$ is a fixed point of f_0 , which can neither be attracting nor repelling. By Fatou's *"snail-lemma"* (see, e.g., [**?**], p. 57, Lemma 1, for a proof), 0 is a fixed point of f_0 with multiplier 1 , and $\phi^{-1}(\Delta)$ is a subdomain of the parabolic basin $U_0 = \phi^{-1}(U)$ of f_0 . Obviously, f_0 has only one critical point, of order $m-1$, in U_0 .

The proof of Theorem 1 is thus complete if we are able to extend h quasiconformally to Δ_1 such that (**??**) holds. The extension of h will be determined in several steps.

1° To make a definite choice, we set

$$T(w) = \frac{mi}{2}\frac{\left(\frac{2w-i}{2w-3i}\right)^m + 1}{\left(\frac{2w-i}{2w-3i}\right)^m - 1} = w - i - \sum_{\mu=1}^{m-1} \frac{C_\mu}{w - u_\mu - i}\,, \quad C_\mu > 0\,.$$

We note that T has m analytic inverse functions $\tau_\mu : \mathbb{R} \longrightarrow i + I_\mu$, where $I_1 = (-\infty, u_1)$, $I_m = (u_{m-1}, +\infty)$ and $I_\mu = (u_{\mu-1}, u_\mu)$ for $1 < \mu < m$, such that

$$\tau_1(\xi) = \xi + i + O(|\xi|^{-1})\,,$$
$$\tau_\mu(\xi) = u_{\mu-1} + i - C_{\mu-1}\xi^{-1} + O(\xi^{-2})\,,\ 1 < \mu \le m\,,$$

as $\xi \to -\infty$, and

$$\tau_\mu(\xi) = u_\mu + i - C_\mu \xi^{-1} + O(\xi^{-2})\,,\ 1 \le \mu < m\,,$$

[3] The analytic subarcs of α and β , followed by the remaining endpoints of these arcs, are removable; see,e.g., [**?**].

$$\tau_m(\xi) = \xi + i + O(\xi^{-1})$$

as $\xi \to +\infty$. Furthermore, $\tau'_\mu(\xi)$ is bounded.

2° Let D^+ and D^- be subdomains of D, bounded by subarcs of α and β and certain analytic Jordan arcs $\gamma^\pm$ joining α and β in D. If the arcs $\gamma^\pm$ are chosen appropriately in $\{z : |z| < \delta, \pm\mathrm{Im}\, z < 0\}$, then h maps the respective domains $D^\pm$ conformally onto semistrip-like domains

$$S^\pm = \{w = u + iv : \pm u > u_0 \,, 0 < v < 1 + \varrho(u)\}\,,$$

where ϱ is given by $\varrho(u) = c_1^\pm |u|^{-1/s} + c_2^\pm |u|^{-2/s} + \cdots \quad$ for $\pm u \geq u_0$.

Now $Q = D \setminus \overline{(S^+ \cup S^-)}$ and $Q' = \{u + iv : |u| < u_0 \,, 0 < v < 1 + \varrho_1(u)\}$, where $\varrho_1 : \mathbb{R} \longrightarrow \mathbb{R}$ is any smooth extension of ϱ with $1 + \varrho_1 > 0$, may be looked upon as quadrilaterals. We note that Q and Q' have the same angles at respective vertices. Thus, by the Ahlfors-Beurling Extension Theorem, see, e.g., [**?**], there exists a regular quasiconformal extension of h, say h_1, mapping Q onto Q' and D onto $S_1 = \{w = u + iv : 0 < v < 1 + \varrho_1(u)\}$, with the following properties:

(i) as a function of arc-length, h_1 *is smooth on* $\beta \setminus \{z_0, \ldots, z_{m-1}\}$, *and,*

(ii) at z_μ, *has the asymptotics*

$$h_1(z) \sim w_\mu + a_\mu (z - z_\mu)^s + \widetilde{a}_\mu (\overline{z - z_\mu})^s \,, |a_\mu| > |\widetilde{a}_\mu|\,,$$

this following from the fact that ϱ_1 is smooth and $w = (f^n)^s \circ f(z) = c_\mu (z - z_\mu)^s + \cdots$ maps a neighbourhood (in D) of $z = z_\mu$ conformally onto a neighbourhood (in Re $w > 0$) of $w = 0$.

A simple calculation now shows that

$$h_2(w) = w - \frac{iv\varrho_1(u)}{1 + \varrho_1(u)}\,, \quad w = u + iv\,,$$

is a regular quasiconformal map of S_1 onto $S = \{\zeta : 0 < \mathrm{Im}\, \zeta < 1\}$. In particular we have $h_2(w) = w + O(|w|^{-1/s})$, $(h_2)_w(w) = 1 + O(|w|^{-1/s})$ and $(h_2)_{\overline{w}}(w) = O(|w|^{-1/s})$ as $w \to \infty$ in S_1, and $h_2|_{\mathbb{R}} = \mathrm{id}$. On combination with h_1 we obtain a regular quasiconformal extension $H = h_2 \circ h_1$ of h, mapping D onto S, such that

$$\begin{array}{rcl} H(z) &=& h(z) + O(|z|) \\ H_z(z) &=& h'(z) + O(|z|^{-s}) = h'(z)(1 + O(|z|)) \\ H_{\overline{z}}(z) &=& O(|z|^{-s}) \end{array} \qquad (7)$$

hold as $z \to 0$.

3° Of course, H will not necessarily satisfy $H \circ f = T \circ H$ on β. We are thus looking for a quasiconformal self-map Φ of the strip S such that

$$\Phi = \mathrm{id} \quad \text{holds on } \mathbb{R}$$

and

$$T \circ \Phi \circ H = \Phi \circ H \circ f \quad \text{holds on } \beta\,. \qquad (8)$$

Then $\Phi \circ H$ is the desired quasiconformal extension of h to the domain D.

Suppose that (**??**) determines a piecewise C^1-homeomorphism Φ of $\mathbb{R} + i$ onto itself with strictly increasing real-part $\Psi(\xi) = \Phi(\xi + i) - i$. Then

$$\Phi(\xi + i\eta) = (1 - \eta)\xi + \eta\Psi(\xi) + i\eta$$

is an orientation-preserving and piecewise regular homeomorphism of S onto itself with correct boundary values. Since

$$2\Phi_\zeta(\xi + i\eta) = 2 + \eta(\Psi'(\xi) - 1) + i(\xi - \Psi(\xi))$$

and

$$2\Phi_{\overline{\zeta}}(\xi + i\eta) = \eta(\Psi'(\xi) - 1) - i(\xi - \Psi(\xi)),$$

whenever Ψ is differentiable, Φ is even quasiconformal, provided $\Psi - \mathrm{id}$, Ψ' and $1/\Psi'$ are bounded on $\mathbb{R}$.

4° Condition (**??**) makes some difficulties, it is not a priori clear whether there exists any map Φ which satisfies (**??**).

Since $\Phi = \mathrm{id}$ on $\mathbb{R}$, $h = H$ on $\alpha = f(\beta)$ and $h \circ f(\beta) = \mathbb{R} \cup \{\infty\}$, (**??**) is equivalent with

$$T \circ \Phi = h \circ f \circ H^{-1} \quad \text{on } i + \mathbb{R}. \tag{9}$$

Set $\xi_\mu + i = H(z_\mu)$, $1 \le \mu < m$, $\xi_1 < \xi_2 < \cdots < \xi_{m-1}$, where $z_\mu \neq 0$ are the pre-images on ∂U of $z = 0$ under f, and consider the intervals $J_1 = (-\infty, \xi_1)$, $J_m = (\xi_{m-1}, +\infty)$ and $J_\mu = (\xi_{\mu-1}, \xi_\mu)$ for $1 < \mu < m$. Then, for $\xi \in J_\mu$, define Φ by

$$\Phi(\xi + i) = \tau_\mu \circ h \circ f \circ H^{-1}(\xi + i) \tag{10}$$

to fulfill (**??**), and hence (**??**). It is obvious that $\xi \mapsto \Phi(\xi + i) - i = \Psi(\xi)$ is an increasing diffeomorphism from J_μ onto I_μ, and we thus have only to look what happens as $\xi \to \infty$ and $\xi \to \xi_\mu$ from both sides.

We first consider the limit $\xi \to +\infty$. Then (**??**), (**??**) and (**??**) together yield $h \circ f \circ H^{-1}(\xi + i) = \xi + O(\xi^{-1/s})$ as $\xi \to +\infty$, and so

$$\Psi(\xi) = \xi + O(\xi^{-1/s}) \tag{11}$$

follows from $\tau_m(\xi) = \xi + i + O(\xi^{-1})$ as $\xi \to \infty$. Also (**??**), written as

$$\Phi \circ H = \tau_m \circ h \circ f,$$

gives

$$\Phi'(H(z))H_z(z) = h'(z) + O(|z|^{-1})$$

as $z \to 0$, and so (**??**) leads to

$$\Psi'(\xi) = 1 + O(\xi^{-1/s}) \quad \text{as } \xi \to +\infty. \tag{12}$$

In a similar way we deal with the limit $\xi \to \xi_\mu + 0$, say. From (ii) and the smoothness of h_2 easily follows that

$$H^{-1}(\xi + i) \sim z_\mu + b_\mu(\xi - \xi_\mu)^{1/s}, \quad b_\mu \neq 0,$$

as $\xi \to \xi_\mu + 0$. Since $h \circ f(z) \sim c_\mu(z - z_\mu)^{-s}$ as $z \to z_\mu$, $c_\mu \neq 0$, and $\tau_\mu(x) \sim u_\mu + i - C_\mu x^{-1}$ as $x \to +\infty$, $C_\mu > 0$, we conclude from (**??**) that

$$\Psi(\xi) \sim u_\mu - \frac{C_\mu}{c_\mu}\, b_\mu^s(\xi - \xi_\mu) \quad \text{as } \xi \to \xi_\mu + 0\,, \tag{13}$$

and so $\Psi'(\xi)$ has a non-zero and hence positive limit. Thus, by (**??**), (**??**) and (**??**), and similar relations in the cases $\xi \to -\infty$ and $\xi \to \xi_\mu - 0$, Φ has all required properties and the proof of Theorem 1 is complete. □

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References

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Note added in Proof

In the proof of Theorem 3 we initially assumed tacitly that the parabolic fixed point is always an endpoint of $\mathcal{J}$. This, however, need not to be true. To determine the quasiconformal representatives in the other case, we first look for an odd rational function g of degree $d > 2$ with the following properties:

(a) $g(0) = 0$ *and* $g(\infty) = \infty$ *or* $g(\infty) = 0$. [4]

(b) g *has two critical points* $\pm a$ *of multiplicity* $d-1$, *with corresponding critical values* $\pm A$.

(c) g *has a parabolic fixed point at* $z = \pm 1$.

We set

$$\frac{A - g(z)}{A + g(z)} = c\left(\frac{a - z}{a + z}\right)^d$$

[4] We note that the condition *(a')* $g(0) = \infty$ *and* $g(\infty) = 0$ leads to similar results, but works only in the case d odd. The simplest case is $d = 3$, where we get $g(z) = z\frac{z^2 - a}{z^4 - bz^2}$ and $f(z) = \left(g(\sqrt{z}\,)\right)^2 = z\left(\frac{z-a}{z^2-bz}\right)^2$ with $a = \frac{1}{9}(1 + i\sqrt{8}\,)$, $b = 9a = 1 + i\sqrt{8}$ and $c = \frac{1-b}{1-a}$.

to fulfill condition (b). Then $g(0) = 0$ if and only if $c = 1$, while $g(\infty) = \infty$ is equivalent with $c = (-1)^{d+1}$ and $g(\infty) = 0$ is equivalent with $c = (-1)^d$, so that (the only possible choice) $c = 1$ leads to $g(\infty) = \infty$ if d is odd and $g(\infty) = 0$ if d is even.

In any case g is an odd function.

Condition (c) is equivalent with

$$\frac{A-1}{A+1} = \left(\frac{a-1}{a+1}\right)^d \tag{A}$$

and

$$\frac{A}{(A+1)^2} = \frac{da}{(a+1)^2}\left(\frac{a-1}{a+1}\right)^{d-1} . \tag{B}$$

Now (A) can be solved for A,

$$A = \frac{\left(\frac{a+1}{a-1}\right)^d + 1}{\left(\frac{a+1}{a-1}\right)^d - 1}, \tag{C}$$

and hence (c) is fulfilled for each a satisfying

$$(a+1)^{2d} - (a-1)^{2d} = 4da(a^2-1)^{d-1}, \; a \neq 0, \tag{D}$$

provided A is defined by (C).

Now

$$f(z) = \left(g(\sqrt{z}\,)\right)^2$$

is rational of degree d, its Julia set is a Jordan arc with endpoints $z = 0$ and $z = \infty$, and f has a parabolic fixed point at $z = 1$, which is an "inner point" of the Jordan arc $\mathcal{J}$. We note, however, that this is only possible for $d \geq 4$, since, for $d = 3$, equation (D) has no (non-zero) solution. For $d = 4$, (D) is equivalent with

$$5a^4 + 2a^2 + 1 = 0,$$

while for $d = 5$ and $d = 6$ we obtain

$$5a^4 + 6a^2 + 5 = 0$$

and

$$35a^8 + 84a^6 + 114a^4 + 20a^2 + 3 = 0,$$

respectively.