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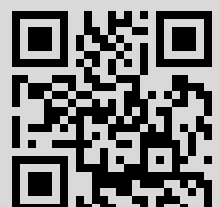
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## UNIValence OF HARMONIC FUNCTIONS, PROBLEM OF PONNUSAMY AND SAIRAM, AND CONSTRUCTIONS OF UNIVALENT POLYNOMIALS<sup>1</sup>

**Abstract.** The criteria of the univalence of a harmonic mapping is obtained in this paper. Particularly, it permits to formulate the hypothesis of the harmonic function classes equality  $S_H^0 = S_H^0(S)$  (hypothesis of Ponnusamy and Sairam), in analytic form. The method of construction of the univalent harmonic polynomials with desired properties, according to a given harmonic function, is obtained by means of the univalence criteria.

**Key words:** *harmonic functions, criteria of the univalence, harmonic univalent polynomials.*

**2010 Mathematical Subject Classification:** *30C55, 30C10, 31A05.*

**1. Introduction.** Let  $S$  be the class of all analytic and univalent functions of the form  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  in the disk  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ .

The problem of determining a necessary coefficient condition in this class was set up by Bieberbach [1]. It consisted of validity of the inequality  $|a_n| \leq n$  for each  $n \in \mathbb{N}$  (equality for Koebe function  $k(z) = z/(1-z)^2$  and its rotations  $k_\theta(z) = e^{-i\theta}k(ze^{i\theta})$ ). The Bieberbach hypothesis contributed largely to the origin and development of a great number of ideas and methods in complex analysis. The full solution of this problem was finally given by de Branges [2].

The theory of univalent harmonic functions began its active development since the eighties of the previous century. Here the main object

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of research is the class  $S_H$  of harmonic univalent and sense-preserving functions  $f$  in  $\Delta$  given by  $f(z) = h(z) + \overline{g(z)}$ , where

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad \overline{g(z)} = \sum_{n=1}^{\infty} a_{-n} \bar{z}^n$$

with  $a_1 = 1$  (see, for example, [3]). The class  $S_H$  is an analog of the class  $S$ . For obtaining information about functions from  $S_H$  it is often convenient to have such information about functions from subclass  $S_H^0 \subset S_H$ , where  $S_H^0 = \{f \in S_H : a_{-1} = 0\}$ . This circumstance explains the interest in studying the class  $S_H^0$ . In [4] Clunie and Sheil-Small formulated the following conjecture (the problem about coefficients in  $S_H^0$ ): for each function  $f \in S_H^0$  and a natural number  $n \in \mathbb{N}$ , the inequalities

$$|a_n| \leq \frac{(2n+1)(n+1)}{6}, \quad |a_{-n}| \leq \frac{(2n-1)(n-1)}{6}, \quad |a_n - a_{-n}| \leq n \quad (1)$$

are true. A great deal of papers are dedicated to this conjecture. Particularly, in the paper of Ponnusamy and Sairam Kaliraj [5] this conjecture, together with some other results were proved for the subclass

$$S_H^0(S) = \{f = h + \bar{g} \in S_H^0 : h + e^{i\phi}g \in S \text{ for some } \phi \in \mathbb{R}\}$$

of the class  $S_H^0$ . Besides, in this paper the authors formulated hypothesis about the equality of these classes, namely,  $S_H^0(S) = S_H^0$ , the proof of which would permit to obtain the full solution of the coefficients problems (1) of Clunie and Sheil-Small. In the present paper criteria of univalence of harmonic functions (Theorem 1) is obtained. With the aid of this, a criteria for functions belonging to the class  $S_H^0(S)$  (Theorem 2) is obtained and several examples are performed. Theorem 3 permits to construct harmonic univalent polynomials for a given  $f \in S_H$ .

**2. Univalence criteria and hypothesis of Ponnusamy and Sairam Kaliraj.** The univalence criteria of arbitrary harmonic function in  $\Delta$  of the form

$$f(z) = \sum_{n=1}^{\infty} (a_n z^n + a_{-n} \bar{z}^n) \quad (2)$$

will be obtained analogously to Bazilevich's [6] univalence criteria for analytic functions:

**Theorem A.** [6] An analytic function  $f(z) = \sum_{n=1}^{\infty} a_n z^n$  in  $\Delta$  is univalent in  $\Delta$  if and only if for each  $z \in \Delta$  and each  $t \in [0, \pi/2]$ ,

$$\sum_{n=1}^{\infty} a_n \frac{\sin nt}{\sin t} z^{n-1} \neq 0, \quad \left( \frac{\sin nt}{\sin t} \right) \Big|_{t=0} = n.$$

**Theorem 1.** Harmonic sense-preserving function in  $\Delta$ , determined by the formula (2), is univalent in  $\Delta$  if and only if for each  $z \in \Delta \setminus \{0\}$  and each  $t \in (0, \pi/2]$ ,

$$\sum_{n=1}^{\infty} \left[ (a_n z^n - a_{-n} \bar{z}^n) \frac{\sin nt}{\sin t} \right] \neq 0. \tag{3}$$

**Proof.** Let  $f \in S_H$ . Then, for  $z_1, z_2 (z_1 \neq z_2)$  from  $\Delta$ , we have

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} \neq 0.$$

Particularly, for  $z_1 = r e^{i\theta_1} \neq r e^{i\theta_2} = z_2$ ,  $r \in (0, 1)$ ,  $\theta_k \in \mathbb{R}$ , this is equivalent to

$$\begin{aligned} & \frac{f(r e^{i\theta_2}) - f(r e^{i\theta_1})}{r e^{i\theta_2} - r e^{i\theta_1}} = \\ & = \sum_{n=1}^{\infty} r^{n-1} \left( a_n \frac{e^{in\theta_2} - e^{in\theta_1}}{e^{i\theta_2} - e^{i\theta_1}} + a_{-n} \frac{e^{-in\theta_2} - e^{-in\theta_1}}{e^{i\theta_2} - e^{i\theta_1}} \right) \neq 0. \end{aligned} \tag{4}$$

Without loss of generality, we may assume that  $\theta_1 < \theta_2 \leq \theta_1 + \pi$ . Let

$$t = \frac{\theta_2 - \theta_1}{2} \in (0, \pi/2] \quad \text{and} \quad \theta = \frac{\theta_2 + \theta_1}{2} \in \mathbb{R}.$$

Then

$$\frac{e^{in\theta_2} - e^{in\theta_1}}{e^{i\theta_2} - e^{i\theta_1}} = e^{i(n-1)\frac{\theta_2+\theta_1}{2}} \frac{e^{in\frac{\theta_2-\theta_1}{2}} - e^{-in\frac{\theta_2-\theta_1}{2}}}{e^{i\frac{\theta_2-\theta_1}{2}} - e^{-i\frac{\theta_2-\theta_1}{2}}} = e^{i(n-1)\theta} \frac{e^{int} - e^{-int}}{e^{it} - e^{-it}},$$

and

$$\frac{e^{-in\theta_2} - e^{-in\theta_1}}{e^{i\theta_2} - e^{i\theta_1}} =$$

$$= \left( \frac{e^{in\theta_2} - e^{in\theta_1}}{e^{i\theta_2} - e^{i\theta_1}} \right) \frac{e^{-i\theta_2} - e^{-i\theta_1}}{e^{i\theta_2} - e^{i\theta_1}} = e^{-i(n-1)\theta} \frac{e^{int} - e^{-int}}{e^{it} - e^{-it}} (-e^{-2i\theta}).$$

Hence (4) may be represented as

$$\sum_{n=1}^{\infty} \left[ (a_n r^{n-1} e^{i(n-1)\theta} - a_{-n} r^{n-1} e^{-i(n-1)\theta} e^{-2i\theta}) \frac{\sin nt}{\sin t} \right] \neq 0$$

which is equivalent to

$$\sum_{n=1}^{\infty} \left[ (a_n z^{n-1} - a_{-n} \bar{z}^{n-1} e^{-2i\theta}) \frac{\sin nt}{\sin t} \right] \neq 0, \quad (5)$$

where  $z = re^{i\theta} \in \Delta$ ,  $\theta = \arg z$ ,  $t \in (0, \pi/2]$ . But (5)  $\iff$  (3). Let us note, that (5) is fulfilled for  $z = 0$  as well, because  $|a_1| > |a_{-1}|$ , since  $f$  is sense-preserving.

Let us next prove the inverse proposition. Suppose a harmonic function (2) is sense-preserving in  $\Delta$  and condition (3) is fulfilled. According to the accepted designations this is equivalent to fulfilling of the condition (4), i. e.

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} \neq 0 \quad \forall z_1 = re^{i\theta_1} \neq re^{i\theta_2} = z_2, \quad r \in (0, 1), \quad \theta_k \in \mathbb{R}.$$

Thus  $f$  is univalent on any circle  $\{z \in \mathbb{C} : |z| = r\}$ .

The local univalence of the function  $f$  implies that  $\partial f(\Delta_r) \subset f(\partial\Delta_r)$ , where  $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ . Then the assumption that  $f$  is not univalent (but locally univalent) in  $\Delta$  implies the existence of a disk  $\Delta_R$  (a disk of  $f$  univalence), in which  $f$  is univalent, but on  $\partial\Delta_R$  there exist points  $z_1 \neq z_2$  such that  $f(z_1) = f(z_2)$ . This contradicts the univalence of  $f$  on the circle  $\{z \in \mathbb{C} : |z| = R\}$ . This contradiction proves Theorem 1.  $\square$

The following theorem represents the criteria of belonging of a function to the class  $S_H^0(S)$ .

**Theorem 2.** *Let  $f \in S_H^0$ . Define*

$$A = A(z, t) = \sum_{n=1}^{\infty} a_n \frac{\sin nt}{\sin t} z^n, \quad B = B(z, t) = \sum_{n=1}^{\infty} \overline{a_{-n}} \frac{\sin nt}{\sin t} z^n,$$

and

$$E = \{(z, t) \in \Delta \times (0, \pi/2] : |A(z, t)| = |B(z, t)|\}.$$

Then  $f \in S_H^0(S)$  if and only if there exists a  $\phi \in [0, 2\pi)$  such that

$$A(z, t) \neq -e^{i\phi}B(z, t) \quad \forall (z, t) \in E.$$

**Proof.** Let  $f = h + \bar{g} \in S_H^0(S)$  and it be determined by (2). According to the definition,  $S_H^0(S) \ni f$  if and only if there exists a  $\phi \in [0, 2\pi)$  such that  $h + e^{i\phi}g \in S$ . Theorem A implies that

$$\sum_{n=1}^{\infty} (a_n + e^{i\phi} \bar{a}_{-n}) \frac{\sin nt}{\sin t} z^n \neq 0, \quad \text{for } z \in \Delta \setminus \{0\}, t \in (0, \pi/2],$$

$$\iff A(z, t) \neq -e^{i\phi}B(z, t) \quad \forall (z, t) \in E.$$

This completes the proof.  $\square$

**Corollary 1.** Let  $f \in S_H^0$  and  $E$  be the set defined in Theorem 2. If  $E = \emptyset$ , then  $f \in S_H^0(S)$ .

**Remark 1.** Denote  $q(z) = \sum_{n=1}^{\infty} \frac{\sin nt}{\sin t} z^n$ . Then

$$A(z, t) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} h\left(\frac{z}{\zeta}\right) q(\zeta) \frac{d\zeta}{\zeta}, \quad B(z, t) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} g\left(\frac{z}{\zeta}\right) q(\zeta) \frac{d\zeta}{\zeta}.$$

Hence the statement of Theorem 2 may be represented by means of these integrals.

**Example.** For fixed  $R \in (0, 1)$ , consider the harmonic function  $f = h + \bar{g}$  in  $\Delta$ , where

$$h(z) = \frac{z}{(1 - Rz)^2} \quad \text{and} \quad g(z) = kz^2, \quad k \in \mathbb{R}.$$

By means of Theorem 1 we determine for which values of  $k$  and  $R$ ,  $f \in S_H^0$ . Applying Theorem 2, let us show that for these values of parameters, the function  $f \in S_H^0(S)$ .

Firstly we need to determine, for which values of the parameters, the function  $f$  is sense-preserving in  $\Delta$ . The condition of sense-preservation means the validity for any  $z \in \Delta$ , the inequality

$$|h'| - |g'| = \left| \frac{1 + Rz}{(1 - Rz)^3} \right| - |2kz| > 0$$

which holds if

$$\min_{|z|=r} \left| \frac{1 + Rz}{(1 - Rz)^3} \right| = \frac{1 - Rr}{(1 + Rr)^3} > 2|k|r, \quad \forall r \in [0, 1).$$

Since the function  $\frac{1 - Rr}{r(1 + Rr)^3}$  decreases with respect to  $r$ , the latter inequality gives

$$2|k| \leq \frac{1 - R}{(1 + R)^3}. \quad (6)$$

The values of parameters for which the function  $f$  is sense-preserving in  $\Delta$  is determined from (6). Further, let the condition (6) be valid. According to Theorem 1,  $f$  is univalent if and only if  $A(z, t) \neq \overline{B(z, t)}$  (designations from Theorem 2) in  $(\Delta \setminus \{0\}) \times (0, \pi/2]$ . Let us show that the equality  $A(z, t) = \overline{B(z, t)}$  is not possible. We see that

$$\begin{aligned} A(z, t) &= \sum_{n=1}^{\infty} nR^{n-1} z^n \frac{e^{int} - e^{-int}}{2i \sin t} = \\ &= \frac{1}{2i \sin t} \left[ \frac{ze^{it}}{(1 - Rze^{it})^2} - \frac{ze^{-it}}{(1 - Rze^{-it})^2} \right] = \\ &= \frac{z(1 - R^2 z^2)}{(1 - 2Rz \cos t + R^2 z^2)^2} \end{aligned}$$

and

$$B(z, t) = 2\bar{k}z^2 \cos t.$$

Now we show that for  $t \in (0, \pi/2)$  and  $z \in \Delta \setminus \{0\}$ , the equation

$$\frac{z(1 - R^2 z^2)}{(1 - 2Rz \cos t + R^2 z^2)^2} = 2k\bar{z}^2 \cos t \quad (7)$$

has no solution. It suffices to show that in (7) the absolute values of the left hand and right hand sides are not equal. If

$$|2k| = \left| \frac{1 - R^2 z^2}{z \cos t (1 - 2R \cos t z + R^2 z^2)^2} \right| = L(z, t),$$

then for some  $t$ ,

$$|2k| \geq \min_{0 < |z| \leq 1} L(z, t) = \min_{|z|=1} L(z, t) = L(-1, t) =$$

$$= \frac{1 - R^2}{\cos t(1 + 2R \cos t + R^2)^2} > \frac{1 - R}{(1 + R)^3}$$

for values of  $t$  under consideration. Thus a contradiction with (6) is obtained. Therefore, if the condition (6) is fulfilled, then all functions  $f$  from this example are univalent. As shown above,  $|A(z, t)| \neq |B(z, t)|$ , in  $(\Delta \setminus \{0\}) \times (0, \pi/2]$  and therefore the set  $E$  defined in Theorem 2 is empty. Hence, for parameters values satisfying the inequality (6),  $f \in S_H^0(S)$ .

**3. Univalent harmonic polynomials** are functions of the following form  $P = h + \bar{g}$ , where  $h$  and  $g$  are classic polynomials in  $z$ . Generally speaking, there is not much information about univalent harmonic polynomials than about other functions from  $S_H$  (here we speak about univalence in  $\Delta$ ) (see [7]).

For example, in a survey paper [8] the authors note: “Finding a method of constructing sense-preserving univalent harmonic polynomials is another important problem”. In the analytic case, Bazilevich [6] proposed a method of construction of univalent polynomials, associated with a given function from  $S$ . Further, his idea has been transferred to harmonic case in Theorem 3. Moreover, unlike with analytic case, the proof will be constructive. Thus, Theorem 3 gives an opportunity to construct harmonic univalent polynomials of sufficiently high power for any function  $f \in S_H^0$ .

**Lemma 1.** *If  $f \in S_H^0, r \in (0, 1), t, \phi \in \mathbb{R}$ , then*

$$\left| \frac{f(re^{it}) - f(re^{i\phi})}{re^{it} - re^{i\phi}} \right| \geq \frac{1 - r}{4\alpha r} \left( \frac{1 - r}{1 + r} \right)^\alpha \left[ 1 - \left( \frac{1 - r}{1 + r} \right)^{2\alpha} \right],$$

where  $\alpha (= \text{ord}S_H) \stackrel{\text{def}}{=} \sup_{f \in S_H} |a_2|$ .

**Proof.** If  $f = h + \bar{g} \in S_H^0$ , then the linear invariance of the class  $S_H$  (see [9]) implies that

$$F(z) = \frac{f\left(\frac{z+a}{1+\bar{a}z}\right) - f(a)}{h'(a)(1-|a|^2)} \in S_H \quad \forall a \in \Delta.$$

Let

$$a = re^{i\phi}, \quad \frac{z+a}{1+\bar{a}z} = re^{it}, \quad \text{i. e.} \quad z = \frac{re^{it} - a}{1 - \bar{a}re^{it}}.$$



From the affine invariance (see. [9]) of the class  $S_H$ , it follows that the function

$$\psi(z) = \frac{F(z) - a_{-1}\overline{F(z)}}{1 - |a_{-1}|^2} \in S_H^0, \quad \text{where } a_{-1} = \frac{\partial F}{\partial \bar{z}}(0) = \frac{\overline{g'(a)}}{h'(a)}.$$

Hence,

$$|F(z)|(1 + |a_{-1}|) \geq |F(z) - a_{-1}\overline{F(z)}| \geq |\psi(z)|(1 - |a_{-1}|^2),$$

so that  $|F(z)| \geq |\psi(z)|(1 - |a_{-1}|)$ . Since  $f \in S_H^0$ , then  $\frac{\partial f}{\partial \bar{z}}(0) = 0$ , i. e.  $g'(0) = 0$ . Since  $f$  is sense-preserving in  $\Delta$ , we have  $|g'(z)/h'(z)| < 1$  in  $\Delta$ . Therefore, according to Schwarz's lemma,  $|a_{-1}| = |g'(a)/h'(a)| \leq r$  and

$$|F(z)| \geq |\psi(z)|(1 - r). \quad (8)$$

For any function  $\psi = H + \bar{G} \in S_H^0$  and any  $z \in \Delta$ , one has [9]:

$$\frac{1}{2\alpha} \left[ 1 - \left( \frac{1 - |z|}{1 + |z|} \right)^\alpha \right] \leq |\psi(z)| \quad \text{and} \quad \frac{(1 - |z|)^{\alpha-1}}{(1 + |z|)^{\alpha+1}} \leq |H'(z)|,$$

from which with regard to (8) we obtain

$$\begin{aligned} |f(re^{it}) - f(re^{i\phi})| &\geq \frac{1}{2\alpha} \left[ 1 - \left( \frac{1 - |z|}{1 + |z|} \right)^\alpha \right] (1 - r)(1 - r^2) |h'(re^{i\phi})| \geq \\ &\geq \frac{1 - r}{2\alpha} \left( \frac{1 - r}{1 + r} \right)^\alpha \left[ 1 - \left( \frac{1 - |z|}{1 + |z|} \right)^\alpha \right], \end{aligned} \quad (9)$$

where

$$z = \frac{r(e^{it} - e^{i\phi})}{1 - r^2 e^{i(t-\phi)}}.$$

Denote by  $w = e^{is} = e^{i(t-\phi)}$  and  $\zeta = \frac{1 - w}{1 - r^2 w}$ . Then

$$w = \frac{1 - \zeta}{1 - r^2 \zeta}, \quad 0 \leq |z| = r|\zeta| \leq \frac{2r}{1 + r^2}, \quad |1 - w| = \frac{|\zeta|(1 - r^2)}{|1 - r^2 \zeta|}.$$

Represent (9) as follows

$$\left| \frac{f(re^{it}) - f(re^{i\phi})}{re^{it} - re^{i\phi}} \right| \geq \frac{1 - r}{2\alpha} \left( \frac{1 - r}{1 + r} \right)^\alpha \left[ 1 - \left( \frac{1 - r|\zeta|}{1 + r|\zeta|} \right)^\alpha \right] \frac{|1 - r^2 \zeta|}{r|\zeta|(1 - r^2)} \geq$$

$$\geq \frac{1}{2\alpha} \left(\frac{1-r}{1+r}\right)^\alpha \left[1 - \left(\frac{1-x}{1+x}\right)^\alpha\right] \frac{1-rx}{x(1+r)}, \tag{10}$$

$x = r|\zeta| \in [0, 2r/(1+r^2)]$ .

Define

$$u(x) = \frac{1}{x} \left[1 - \left(\frac{1-x}{1+x}\right)^\alpha\right], \quad x \in \left[0, \frac{2r}{1+r^2}\right].$$

We show that  $u$  is decreasing on  $(0, 2r/(1+r^2))$ . Then,

$$\begin{aligned} x^2 u'(x) \leq 0 &\iff \left(\frac{1-x}{1+x}\right)^{\alpha-1} \frac{2\alpha x}{(1+x)^2} - 1 + \left(\frac{1-x}{1+x}\right)^\alpha \leq 0 \\ &\iff 2\alpha x + 1 - x^2 \leq (1-x^2) \left(\frac{1+x}{1-x}\right)^\alpha \iff \\ &\iff u_1(x) \leq u_2(x), \end{aligned} \tag{11}$$

where

$$u_1(x) = \ln(1+2\alpha x - x^2) \quad \text{and} \quad u_2(x) = \alpha(\ln(1+x) - \ln(1-x)) + \ln(1-x^2).$$

Note that  $u_1(0) = 0 = u_2(0)$  and

$$u'_1(x) = \frac{2(\alpha-x)}{1+2\alpha x - x^2} \leq \frac{2(\alpha-x)}{1-x^2} = u'_2(x),$$

since (see [9])  $3 \leq \alpha < 48,9$ . This proves the validity of (11) and hence  $u(x)$  is decreasing on  $(0, 2r/(1+r^2))$  and hence

$$\min_{x \in [0, 2r/(1+r^2)]} u(x) = u\left(\frac{2r}{1+r^2}\right) = \frac{1+r^2}{2r} \left[1 - \left(\frac{1-r}{1+r}\right)^{2\alpha}\right].$$

Then from (10) we obtain the desired inequality in Lemma 1. The proof of the lemma is complete.  $\square$

In Lemma 2 below the estimates of coefficients of functions from  $S_H^0$  will be obtained. The estimates are not exact, but they are sufficient to achieve the goal which we set up in this section (see Theorem 3).

**Lemma 2.** *If  $\alpha = \text{ord}S_H$ ,  $S_H^0 \ni f$  with series expansion (2), then the following inequality is true:*

$$|a_{\pm n}| < \frac{(2e^2)^\alpha}{2\alpha} n^\alpha, \quad n \in \mathbb{N}.$$

**Proof.** Sheil-Small [9] proved that for  $f \in S_H^0$ , the inequality

$$|f(z)| \leq \frac{1}{2\alpha} \left[ \left( \frac{1+|z|}{1-|z|} \right)^\alpha - 1 \right], \quad z \in \Delta.$$

Hence for  $n \in \mathbb{N}$  and  $r \in (0, 1)$ ,

$$|a_{-n}| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)d\bar{z}}{\bar{z}^{n+1}} \right| \leq \frac{1}{2\alpha r^n} \left( \frac{1+r}{1-r} \right)^\alpha = \frac{\psi(r)}{2\alpha}.$$

The same estimate is also true for  $|a_n|$ ,  $n \in \mathbb{N}$ . In order to find a minimum of the right hand side of inequality we find the point of minimum of the function  $\ln \psi(r)$ :

$$(\ln \psi(r))' = \frac{2\alpha}{1-r^2} - \frac{n}{r} = 0 \iff r^2 + \frac{2\alpha}{n}r - 1 = 0.$$

Therefore,  $r_0 = \sqrt{\alpha^2/n^2 + 1} - \alpha/n$  is the point of minimum and

$$|a_n| \leq \frac{\psi(r_0)}{2\alpha} \quad \text{for all integer } n,$$

where

$$\begin{aligned} \psi(r_0) &= \\ &= \left( \sqrt{\frac{\alpha^2}{n^2} + 1} + \frac{\alpha}{n} \right)^n \left[ \frac{n}{2\alpha} \left( 1 + \sqrt{\frac{\alpha^2}{n^2} + 1} + \frac{\alpha}{n} \right) \left( 1 + \sqrt{\frac{\alpha^2}{n^2} + 1} - \frac{\alpha}{n} \right) \right]^\alpha. \end{aligned}$$

Using inequality  $\sqrt{1+x} \leq 1 + \sqrt{x}$ ,  $x > 0$ , we obtain

$$\begin{aligned} \psi(r_0) &\leq \left( \frac{n}{\alpha} \right)^\alpha \left[ \left( 1 + \frac{2\alpha}{n} \right)^{\frac{n}{2\alpha}} \right]^{2\alpha} \left( 2 + \frac{\alpha}{n} \right)^\alpha = \\ &= (2n)^\alpha \left( \frac{1}{2n} + \frac{1}{\alpha} \right)^\alpha \left[ \left( 1 + \frac{2\alpha}{n} \right)^{n/(2\alpha)} \right]^{2\alpha}. \end{aligned} \quad (12)$$

Introduce

$$\Psi(y) = y \ln \left( 1 + \frac{1}{y} \right).$$

Then  $\sup_{y>0} \Psi(y) = 1$ , since  $\lim_{y \rightarrow +0} \Psi(y) < 1$ ,  $\lim_{y \rightarrow +\infty} \Psi(y) = 1$ , and, if  $\Psi(y)$  has maximum on the interval  $(0, \infty)$  at the point  $y_0$ , then  $\Psi'(y_0) = 0$ . This gives

$$\ln\left(1 + \frac{1}{y_0}\right) = \frac{1}{1 + y_0}$$

which implies that

$$\Psi(y_0) = \frac{y_0}{1 + y_0} < 1.$$

Hence from (12) we have

$$\psi(r_0) < (2n)^\alpha e^{2\alpha} \quad \text{and} \quad |a_{\pm n}| < \frac{(2e^2)^\alpha}{2\alpha} n^\alpha$$

and the proof of Lemma 2 is complete.  $\square$

**Theorem 3.** Let  $f \in S_H^0$  and have the series expansion (1),  $\alpha = \text{ord}S_H$ ,  $s = [\alpha + 2]$ , where  $[\cdot]$  denotes the integer part of a number. Let  $m \in \mathbb{N}$ ,  $\epsilon \in (0, (2e^2)^\alpha / (2\alpha))$  and

$$r \in \left( \left[ 1 - \frac{2\alpha\epsilon}{(2e^2m)^\alpha} \right]^{1/m}, 1 \right) \tag{13}$$

let  $N \in \mathbb{N}$  be so large that

$$\frac{2}{|\ln r|} (\alpha(2 + \ln 2) + \ln(s!) - (s + 1) \ln |\ln r| - \ln(1 - r) - \alpha \ln \frac{1 - r}{1 + r} - \ln \left[ 1 - \left( \frac{1 - r}{1 + r} \right)^{2\alpha} \right] + 2 < N, \quad m < N, \quad \frac{\ln N}{N} \leq \frac{|\ln r|}{2[\alpha + 2]}.$$

Then the harmonic polynomial

$$P(z) = \sum_{k=1}^N (c_k z^k + c_{-k} \bar{z}^k) = \sum_{k=1}^N (a_k r^k z^k + a_{-k} r^k \bar{z}^k)$$

is univalent in  $\Delta$  and moreover  $|a_{\pm k} - c_{\pm k}| < \epsilon$  for all  $k = 1, \dots, m$ .

**Proof.** Let us note that  $2\alpha\epsilon / (2e^2)^\alpha < 1$  for the indicated values  $\epsilon$ . Hence  $\frac{2\alpha\epsilon}{(2e^2k)^\alpha} < 1$  for  $k = 1, \dots, m$ .

The function  $\phi(x) = (1 - x^\alpha(2\alpha\epsilon)/(2e^2)^\alpha)^x$  decreases on  $(0, 1]$ , since  $\frac{d \ln \phi(x)}{dx} < 0$  for  $x \in (0, 1]$ . Therefore,  $\phi(1/k) \leq \phi(1/m)$  for each  $k = 1, \dots, m$ , and (13) implies the inequality  $\phi(1/k) \leq \phi(1/m) < r$ . Hence

$$(1 - r^k) \frac{(2e^2 k)^\alpha}{2\alpha} < \epsilon \quad \text{for } k = 1, \dots, m.$$

From these inequalities and Lemma 2 we obtain inequalities for the first coefficients  $c_{\pm k} = r^k a_{\pm k}$  of the polynomial  $P(z)$ :

$$|a_{\pm k} - c_{\pm k}| = |a_{\pm k} - r^k a_{\pm k}| < \epsilon, \quad k = 1, \dots, m.$$

Let us verify, by the method of mathematical induction, that

$$|\sin nt| \leq n \sin t \quad \forall t \in (0, \pi/2] \text{ and } \forall n \in \mathbb{N}.$$

The inequality is true for  $n = 1$ . Assume that the inequality is true for  $(n - 1)$ . Then

$$\begin{aligned} |\sin nt| &\leq |\sin(n - 1)t| \cos t + |\cos(n - 1)t| \sin t \leq \\ &\leq \sin t [(n - 1) \cos t + |\cos(n - 1)t|] \leq n \sin t. \end{aligned}$$

Let us show under the hypothesis of the theorem that the polynomial  $P(z)$  is univalent in  $\Delta$ . For  $z \in \Delta$ , let us estimate the remainder  $R_N$  of the series. We have

$$|R_N| = \sum_{n=N+1}^{\infty} \left[ (a_n z^n - a_{-n} \bar{z}^n) r^n \frac{\sin nt}{\sin t} \right] \leq \frac{(2e^2)^\alpha}{\alpha} \sum_{n=N+1}^{\infty} n^{\alpha+1} r^n$$

according to Lemma 2. By hypotheses,  $N |\ln r| \geq 2s \ln N > \alpha + 1$ , and therefore,  $N > \frac{\alpha + 1}{|\ln r|}$ . But for  $x \in \left[ \frac{\alpha + 1}{|\ln r|}, \infty \right)$ , the function  $T(x) = x^{\alpha+1} r^x$  decreases. Hence with the increasing values of  $n$ , the terms of the series  $\sum_{n=N}^{\infty} n^{\alpha+1} r^n$  decreases and therefore,

$$\sum_{n=N+1}^{\infty} n^{\alpha+1} r^n < \int_N^{\infty} T(x) dx \leq \int_N^{\infty} x^s r^x dx.$$

Integrating by parts successfully  $s$  times, we obtain

$$\int_N^\infty x^s r^x dx = \frac{N^s r^N}{|\ln r|} + \frac{sN^{s-1} r^N}{|\ln r|^2} + \frac{s(s-1)N^{s-2} r^N}{|\ln r|^3} + \dots + \frac{s! r^N}{|\ln r|^{s+1}}.$$

Let us represent the condition  $N|\ln r| \geq 2s \ln N$  of the theorem as  $r^{\frac{N}{2}} N^s \leq \leq 1$ . Hence  $r^N N^j \leq r^{N/2}$  for any  $j = 1, \dots, s$ . Taking into account of these inequalities, we have the estimate

$$\begin{aligned} & \int_N^\infty x^s r^x dx \leq \\ & \leq \frac{s! r^{N/2}}{|\ln r|^{s+1}} \left[ \frac{|\ln r|^s}{s!} + \frac{|\ln r|^{s-1}}{(s-1)!} + \dots + \frac{|\ln r|}{1!} + r^{N/2} \right] < \frac{r^{N/2-1} s!}{|\ln r|^{s+1}}. \end{aligned}$$

Hence

$$|R_N| \leq \frac{(2e^2)^\alpha r^{N/2-1} s!}{\alpha |\ln r|^{s+1}}. \tag{14}$$

From (14), Theorem 1 and Lemma 1 we obtain

$$\begin{aligned} & \left| \sum_{n=1}^N \left[ (a_n z^n - a_{-n} \bar{z}^n) r^n \frac{\sin nt}{\sin t} \right] \right| \geq \\ & \geq \left| \sum_{n=1}^\infty \left[ (a_n z^n - a_{-n} \bar{z}^n) r^n \frac{\sin nt}{\sin t} \right] \right| - |R_N| \geq \\ & \geq \frac{1-r}{4\alpha} \left( \frac{1-r}{1+r} \right)^\alpha \left[ 1 - \left( \frac{1-r}{1+r} \right)^{2\alpha} \right] - \frac{(2e^2)^\alpha r^{N/2-1} s!}{\alpha |\ln r|^{s+1}} = Q. \end{aligned}$$

If now  $Q > 0$ , then the univalence of the polynomial  $P$  follows from Theorem 1. The latter inequality is equivalent to

$$\begin{aligned} & \frac{2}{|\ln r|} (\alpha(2 + \ln 2) + \ln(s!) - (s+1) \ln |\ln r| - \ln(1-r) - \alpha \ln \frac{1-r}{1+r} - \\ & - \ln \left[ 1 - \left( \frac{1-r}{1+r} \right)^{2\alpha} \right] + 2 < N, \end{aligned}$$

which proves Theorem 3.  $\square$

The proof of Theorem 3 implies the following result.

**Corollary 2.** Let  $f \in S_H^0$  and  $f$  have the expansion (1),  $\alpha = \text{ord}S_H$ ,  $r \in (0, 1)$  and  $0 \neq m \in \mathbb{Z}$ . Then the  $m$ -th coefficient  $c_m = a_m r^m$  of the univalent function

$$f(rz) = \sum_{n=1}^{\infty} (c_n z^n + c_{-n} \bar{z}^n)$$

may be replaced by  $c_m + \lambda$ , where  $\lambda \in \mathbb{C}$ , and

$$|\lambda| < \frac{1-r}{4\alpha|m|} \left( \frac{1-r}{1+r} \right)^\alpha \left[ 1 - \left( \frac{1-r}{1+r} \right)^{2\alpha} \right]$$

without loss of univalence.

**Proof.** According to Lemma 1,

$$\left| \sum_{n=1}^{\infty} \left[ (a_n z^n - a_{-n} \bar{z}^n) r^n \frac{\sin nt}{\sin t} \right] \right| \geq \frac{1-r}{4\alpha} \left( \frac{1-r}{1+r} \right)^\alpha \left[ 1 - \left( \frac{1-r}{1+r} \right)^{2\alpha} \right].$$

By Theorem 1, the condition of the univalence of the new function, obtained by the variation of the  $m$ -th coefficient  $c_m$  gives

$$\left| \sum_{n=1}^{\infty} \left[ (a_n z^n - a_{-n} \bar{z}^n) r^n \frac{\sin nt}{\sin t} \right] + \sigma \lambda \rho^{|m|} \frac{\sin mt}{\sin t} \right| > 0, \quad z \in \Delta \setminus 0, \quad \rho = |z|,$$

$t \in (0, \pi/2]$ , where  $|\sigma| = 1$ . It is fulfilled provided

$$\left| \lambda \frac{\sin mt}{\sin t} \right| < \frac{1-r}{4\alpha} \left( \frac{1-r}{1+r} \right)^\alpha \left[ 1 - \left( \frac{1-r}{1+r} \right)^{2\alpha} \right].$$

Hence it is certainly fulfilled if the inequality concerning  $|\lambda|$ , from the statement of Lemma, is true.  $\square$

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