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A unified presentation of certain meromorphic functions related to the families of the partial zeta type functions and the L-functions

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ABSTRACT

The aim of this paper is to construct a unified family of meromorphic functions, which is related to many known functions such as a unified family of partial zeta type functions, a unified family of L-functions, and so on. We investigate and derive many properties of this family of meromorphic functions. Moreover, we compute the residues of this family of meromorphic functions at their poles. We also give some applications and remarks involving this family of meromorphic functions.

1. Introduction, definitions and preliminaries

The theory of the families of the L-functions and the partial zeta type functions, and also the family of zeta functions themselves, have become a very important part of *Analytic Number Theory*. In this paper, by using a new type of generating functions of the classes of special numbers and polynomials, we construct and investigate various properties of a unified family of meromorphic functions, which is related to many known functions such as a unified family of partial zeta type functions, a unified family of L-functions, and so on. Moreover, we compute the residues of this family of meromorphic functions at their poles and also give some applications and remarks involving this family of meromorphic functions.

Throughout our present investigation, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. We also *tacitly* assume that $\log z$ denotes the *principal branch* of the *multi-valued* function $\log z$ with the imaginary part $\Im(\log z)$ constrained by

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$-\pi < \Im(\log z) \leqq \pi.$

In recent years, many authors introduced and investigated various interesting unifications of the clnouolynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar Apostol type generalizations $\mathcal{B}_n(x;\lambda), \mathcal{E}_n(x;\lambda)$ and $\mathcal{G}_n(x;\lambda)$ $(\lambda \in \mathbb{C})$, which are defined by means of the following generating functions (see, for example, [1,13–15,19,30,31]; see also [22–24,26–28] and the references cited in each of these earlier works):

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\, \frac{t^n}{n!} \tag{1.1}$$

$(|t| < 2\pi \quad \text{when} \quad \lambda = 1; \quad |t| < |\log \lambda| \quad \text{when} \quad \lambda \neq 1),$

$$\frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x;\lambda)\, \frac{t^n}{n!} \quad (|t| < |\log(-\lambda)|) \tag{1.2}$$

and

$$\frac{2te^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{G}_n(x;\lambda)\, \frac{t^n}{n!} \quad (|t| < |\log(-\lambda)|), \tag{1.3}$$

so that, obviously,

$$B_n(x) = \mathcal{B}_n(x;1), \quad E_n(x) = \mathcal{E}_n(x;1) \quad \text{and} \quad G_n(x) = \mathcal{G}_n(x;1).$$

For $x = 0$, (1.1) to (1.3) reduce immediately to the generating functions for the Apostol–Bernoulli numbers $\mathcal{B}_n(\lambda)$, the Apostol–Euler numbers $\mathcal{E}_n(\lambda)$ and the Apostol–Genocchi numbers $\mathcal{G}_n(\lambda)$ $(\lambda \in \mathbb{C})$, respectively, that is,

$$\mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda), \quad \mathcal{E}_n(\lambda) := \mathcal{E}_n(0;\lambda) \quad \text{and} \quad \mathcal{G}_n(\lambda) := \mathcal{G}_n(0;\lambda) \quad (\lambda \in \mathbb{C}).$$

Moreover, for the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n, we have

$$B_n := B_n(0) = \mathcal{B}_n(1), \quad E_n := E_n(0) = \mathcal{E}_n(1) \quad \text{and} \quad G_n := G_n(0) = \mathcal{G}_n(1).$$

Özden [15] (see also Özden et al. [19]) introduced and systematically studied the following family of generating functions which provides a unification of the generating functions (1.1)–(1.3) of the Apostol–Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$, the Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$ and the Apostol–Genocchi polynomials $\mathcal{B}_n(x;\lambda)$ (*cf.* [[19,p. 2779, Eq. (1)]:

$$\mathfrak{g}_\beta(x,t;k,a,b) = \mathfrak{h}_\beta(t;k,a,b) \cdot e^{xt} := \left(\frac{2^{1-k}\, t^k}{\beta^b e^t - a^b}\right) \cdot e^{xt} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x;k,a,b)\, \frac{t^n}{n!} \tag{1.4}$$

$$\left(|t| < 2\pi \quad \text{when} \quad \beta = a; \; |t| < \left| b \log\left(\frac{\beta}{a}\right)\right| \quad \text{when} \quad \beta \neq a; \; k \in \mathbb{N}_0; \; \beta \in \mathbb{C}; \; a, b \in \mathbb{C} \setminus \{0\}\right).$$

Remark 1. Upon comparing the generating function (1.4) with the generating functions (1.1)–(1.3), we are led easily to the following relationships with the Apostol–Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$, the Apostol–Euler polynomials $\mathcal{E}_n(x;\lambda)$ and the Apostol-Genocchi polynomials $\mathcal{B}_n(x;\lambda)$:

$$\mathcal{B}_n(x;\lambda) = \mathcal{Y}_{n,\lambda}(x;1,1,1), \tag{1.5}$$

$$\mathcal{E}_n(x;\lambda) = \mathcal{Y}_{n,\lambda}(x;0,-1,1) \tag{1.6}$$

and

$$\mathcal{G}_n(x;\lambda) = \mathcal{Y}_{n,\lambda}(x;1,-1,1). \tag{1.7}$$

Many other closely-related recent works on this subject include (for example) [3,4,7,10,16–18,21] and [29,34–36].

Remark 2. The *generalized* Apostol type polynomials

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) \quad (\lambda, \mu \in \mathbb{C}; \; \nu \in \mathbb{N}_0)$$

of order α are defined by means of the following generating function (see, for details, [14,31]):

$$\left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)\, \frac{t^n}{n!} \quad (|t| < |\log(-\lambda)|), \tag{1.8}$$

where it is *tacitly* assumed that the parameters α and ν are restricted in such a way that the generating function on the left-hand side of (1.8) is analytic within the disk

$$|t| < |\log(-\lambda)| \quad (\lambda \in \mathbb{C}).$$

In terms of the similarly generalized Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α, Apostol–Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α and Apostol–Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of order α, it readily follows from the definition (1.8) that

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = (-1)^{\alpha}\mathcal{F}_n^{(\alpha)}(x;-\lambda;0;1), \tag{1.9}$$

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1;0) \tag{1.10}$$

and

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1;1), \tag{1.11}$$

so that, obviously,

$$\mathcal{B}_n(x;\lambda) = \mathcal{B}_n^{(1)}(x;\lambda), \quad \mathcal{E}_n(x;\lambda) = \mathcal{E}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{G}_n(x;\lambda) = \mathcal{G}_n^{(1)}(x;\lambda) \tag{1.12}$$

for the Apostol–Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$, the Apostol–Euler polynomials $\mathcal{E}_n(x;\lambda)$ and the Apostol–Genocchi polynomials $\mathcal{G}_n(x;\lambda)$, which are defined by means of the generating function (1.1)–(1.3), respectively. Moreover, by comparing the generating functions (1.4) and (1.18), we have the following relationship:

$$\mathcal{Y}_{n,\beta}(x;k,a,b) = -\frac{1}{a^b}\mathcal{F}_n^{(1)}\left(x;-\frac{\beta^b}{a^b};1-k;k\right). \tag{1.13}$$

We next recall here the general Hurwitz–Lerch zeta function $\Phi(z,s,a)$ defined by (see, for details, [32, p. 121 *et seq.*] and [[33, p. 194 *et seq.*] see also [5–8,12,20,30])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.14}$$

$$\left(a \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z|<1;\ \Re(s)>1 \quad \text{when} \quad |z|=1\right),$$

which contains, as its *special* cases, not only the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,a)$ (see, for details, [33, Chapter 2]; see also [41, p. 265 *et seq.*]):

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s} = \Phi(1,s,1) \quad \text{and} \quad \zeta(s,a) := \sum_{n=0}^{\infty}\frac{1}{(n+a)^s} = \Phi(1,s,a) \tag{1.15}$$

and the Lerch zeta function $\ell_s(\xi)$ defined by

$$\ell_s(\xi) := \sum_{n=1}^{\infty}\frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\Phi\left(e^{2\pi i\xi},s,1\right) \quad (\xi \in \mathbb{R};\ \Re(s)>1), \tag{1.16}$$

but also such other important functions of *Analytic Number Theory* as the Polylogarithmic function (or *de Jonquière's function*) $\mathrm{Li}_s(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty}\frac{z^n}{n^s} = z\Phi(z,s,1) \tag{1.17}$$

$$(s \in \mathbb{C} \quad \text{when} \quad |z|<1;\ \Re(s)>1 \quad \text{when} \quad |z|=1)$$

and the Lipschitz–Lerch zeta function $\phi(\xi,a,s)$ [32,p. 122, Eq. 2.5(11)]:

$$\phi(\xi,a,s) := \sum_{n=0}^{\infty}\frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(e^{2\pi i\xi},s,a\right) =: L(\xi,s,a) \tag{1.18}$$

$$\left(a \in \mathbb{C}\setminus\mathbb{Z}_0^-;\Re(s)>0 \quad \text{when} \quad \xi \in \mathbb{R}\setminus\mathbb{Z};\Re(s)>1 \quad \text{when} \quad \xi \in \mathbb{Z}\right),$$

which was first studied by Rudolf Lipschitz (1832–1903) and Matyáš Lerch (1860–1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions (see also [39].

Remark 3. A systematic study of the extended (multi-parameter) Hurwitz–Lerch zeta function

$$\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a)$$

defined by

$$\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}(z,s,a) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}} \frac{z^n}{(n+a)^s} \tag{1.19}$$

$$\Big(p,q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\,(j=1,\ldots,p);\ a,\mu_j \in \mathbb{C}\setminus Z_0^-\ (j=1,\ldots,q);$$

$$\rho_j,\sigma_k \in \mathbb{R}^+\ (j=1,\ldots,p;\ k=1,\ldots,q); \Delta > -1 \quad \text{when} \quad s,z \in \mathbb{C};$$

$$\Delta = -1 \quad \text{and} \quad s \in \mathbb{C} \quad \text{when} \quad |z| < \nabla^*;\ \Delta = -1 \quad \text{and} \quad \Re(\Xi) > \frac{1}{2} \quad \text{when} \quad |z| = \nabla^*\Big)$$

was initiated recently by Srivastava et al. [37] in which one can also find many references to the earlier attempts at unification (and generalization) of the familiar Hurwitz–Lerch zeta function $\Phi(z,s,a)$ defined by (1.14) (see also [11] and the references cited therein). Here, as usual, $(\lambda)_\nu$ $(\lambda,\nu \in \mathbb{C})$ denotes the Pochhammer symbol defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{1.20}$$

it being assumed *conventionally* that $(0)_0 := 1$ and understood *tacitly* that the Γ-quotient exists,

$$\Delta := \sum_{j=1}^{q}\sigma_j - \sum_{j=1}^{p}\rho_j \quad \text{and} \quad \nabla^* := \left(\prod_{j=1}^{p}\rho_j^{-\rho_j}\right)\cdot\left(\prod_{j=1}^{q}\sigma_j^{\sigma_j}\right). \tag{1.21}$$

Now, in terms of a Dirichlet character χ with conductor $f \in \mathbb{N}$, Özden et al. [19] introduced and studied such χ-extended polynomials and χ-extended numbers as those associated with the generating function (1.4). Each of these χ-extended polynomials and χ-extended numbers are potentially useful in many different areas of Mathematics and Mathematical Physics.

Definition 1 (Özden et al. [19, p. 2783, Definition 2]). Let χ be a Dirichlet character with conductor $f \in \mathbb{N}$. Then the aforementioned χ-extended generalized Bernoulli–Euler–Genocchi numbers $\mathcal{Y}_{n,\chi,\beta}(k,a,b)$ and the aforementioned χ-extended generalized Bernoulli–Euler–Genocchi polynomials $\mathcal{Y}_{n,\chi,\beta}(x;k,a,b)$ are given by the following generating functions:

$$\mathfrak{F}_{\chi,\beta}(t;k,a,b) := \left(\frac{2^{1-k}\,t^k}{\beta^{bf}e^{ft} - a^{bf}}\right)\cdot\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj} e^{jt} = \sum_{n=0}^{\infty}\mathcal{Y}_{n,\chi,\beta}(k,a,b)\,\frac{t^n}{n!} \tag{1.22}$$

$$\left(|t| < 2\pi \quad \text{when} \quad \beta = a;\ |t| < \left|b\log\left(\frac{\beta}{a}\right)\right| \quad \text{when} \quad \beta \neq a;\ k \in \mathbb{N}_0;\ \beta \in \mathbb{C}\ (|\beta| < 1);\ a,b \in \mathbb{C}\setminus\{0\}\right)$$

and

$$\mathfrak{H}_{\chi,\beta}(x,t;k,a,b) := \mathfrak{F}_{\chi,\beta}(t;k,a,b)\cdot e^{xt} = \sum_{n=0}^{\infty}\mathcal{Y}_{n,\chi,\beta}(x;k,a,b)\,\frac{t^n}{n!} \tag{1.23}$$

$$\left(|t| < 2\pi \quad \text{when} \quad \beta = a;\ |t| < \left|b\log\left(\frac{\beta}{a}\right)\right| \quad \text{when} \quad \beta \neq a;\ k \in \mathbb{N}_0;\ \beta \in \mathbb{C}\ (|\beta| < 1);\ a,b \in \mathbb{C}\setminus\{0\}\right).$$

Definition 2 (Özden et al. [19, p. 2785, Definition 3]). For

$$\beta \in \mathbb{C} \quad (|\beta| < 1) \quad \text{and} \quad s \in \mathbb{C},$$

a unification $\zeta_\beta(s,x;k,a,b)$ of the zeta type functions is given by

$$\zeta_\beta(s,x;k,a,b) := \left(-\frac{1}{2}\right)^{k-1}\sum_{n=0}^{\infty}\frac{\beta^{bn}}{a^{b(n+1)}(n+x)^s} \quad (\beta \in \mathbb{C}\ (|\beta| < 1);\ \Re(s) > 1), \tag{1.24}$$

which, in the special case when $x = 1$, yields the following unification of the Riemann type zeta functions:

$$\zeta_\beta(s;k,a,b) := \left(-\frac{1}{2}\right)^{k-1}\sum_{n=1}^{\infty}\frac{\beta^{b(n-1)}}{a^{bn}\,n^s} \quad (\beta \in \mathbb{C}\ (|\beta| < 1);\ \Re(s) > 1). \tag{1.25}$$

Remark 4. In terms of the general Hurwitz–Lerch zeta function $\Phi(z,s,a)$ defined by (1.13) and the Polylogarithmic (or *de Jonquère's*) function $\mathrm{Li}_s(z)$ defined by (1.17), the above-defined zeta type functions $\zeta_\beta(s,x;k,a,b)$ and $\zeta_\beta(s;k,a,b)$ can easily be rewritten as follows:

$$\zeta_\beta(s,x;k,a,b) = \left(-\frac{1}{2}\right)^{k-1} a^{-b}\Phi\left(\frac{\beta^b}{a^b},s,x\right) \tag{1.26}$$

and

$$\zeta_\beta(s;k,a,b) = \left(-\frac{1}{2}\right)^{k-1} \beta^{-b}\mathrm{Li}_s\left(\frac{\beta^b}{a^b}\right). \tag{1.27}$$

The relationship (1.26) appeared *erroneously* in the aforecited work by Özden et al [19, p. 2786, Eq. (4.14)].

Lastly, we present the following potentially useful results (see also [15,18,19]).

Theorem 1. *Let χ be a Dirichlet character with conductor $f \in \mathbb{N}$. Suppose also that*

$$\mathcal{Y}_{n,\beta}(k,a,b) := \mathcal{Y}_{n,\beta}(0;k,a,b) \quad \text{and} \quad \mathcal{Y}_{n,\chi,\beta}(k,a,b) := \mathcal{Y}_{n,\chi,\beta}(0;k,a,b). \tag{1.28}$$

Then

$$\mathcal{Y}_{n,\beta}(x;k,a,b) = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}\mathcal{Y}_{j,\beta}(k,a,b), \tag{1.29}$$

$$\mathcal{Y}_{n,\chi,\beta}(x;k,a,b) = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}\mathcal{Y}_{j,\chi,\beta}(k,a,b) \tag{1.30}$$

and

$$\mathcal{Y}_{n,\chi,\beta}(k,a,b) = f^{n-k}\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathcal{Y}_{n,\beta^f}\left(\frac{j}{f};k,a^f,b\right). \tag{1.31}$$

The last assertion (1.31) of Theorem 1 can be proven fairly easily by first combining the definitions in (1.4) and (1.22) as follows:

$$\mathfrak{F}_{\chi,\beta}(t;k,a,b) = \frac{1}{f^k}\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathfrak{g}_{\beta^f}\left(\frac{j}{f},ft;k,a^f,b\right),$$

which, in light of the generating functions in (1.4) and (1.22), yields

$$\sum_{n=0}^{\infty}\mathcal{Y}_{n,\chi,\beta}(k,a,b)\,\frac{t^n}{n!} = \frac{1}{f^k}\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\sum_{n=0}^{\infty}\mathcal{Y}_{n,\beta^f}\left(\frac{j}{f};k,a^f,b\right)\frac{(ft)^n}{n!}. \tag{1.32}$$

Upon comparing the coefficients of t^n from both sides of (1.32), we are led immediately to the assertion (1.31) of Theorem 1 (*cf.* [18, Theorem 1]).

Theorem 2 (Özden et al. [19, p. 2786, Theorem 7]). *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then*

$$\zeta_\beta(1-n,x;k,a,b) = (-1)^k\,\frac{(n-1)!}{(n+k-1)!}\,\mathcal{Y}_{n+k-1,\beta}(x;k,a,b). \tag{1.33}$$

In Section 2 of our paper, we make use of the *Umbral Calculus* convention in order to derive a recurrence relation for the unification $\mathcal{Y}_{n,\beta}(k,a,b)$ of the generalized Bernoulli, Euler and Genocchi numbers, which is defined by (1.28). Our systematic investigation of a unification of the partial zeta type functions $U_\beta(s,x;k,a,b;c)$ is carried out in Section 4. Finally, in Section 4, we present a number of new relationships involving a unified family of L-functions which are shown to enable us to compute residues of the related family of partial zeta type functions $U_\beta(s,x;k,a,b;c)$ at their poles.

2. A recurrence relation for $\mathcal{Y}_{n,\beta}(k,a,b)$

In this section, we derive a recurrence relation for the unification $\mathcal{Y}_\beta(k,a,b)$ of the Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi numbers. In fact, by using the *Umbral Calculus* convention, we get

$$\frac{2^{1-k}t^k}{\beta^b e^t - a^b} = e^{\mathcal{Y}_\beta(k,a,b)t},$$

which readily yields

$$2^{1-k}t^k = \beta^b \sum_{n=0}^{\infty} \left[\mathcal{Y}_\beta(k,a,b)+1\right]^n \frac{t^n}{n!} - a^b \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(k,a,b)\,\frac{t^n}{n!}, \tag{2.1}$$

where $\left[\mathcal{Y}_\beta(k,a,b)\right]^n$ is replaced *conventionally* by $\mathcal{Y}_{n,\beta}(k,a,b)$. From this last Eq. (2.1), we arrive at the following recurrence relation for the numbers $\mathcal{Y}_{n,\beta}(k,a,b)$.

Theorem 3. *The following recurrence relation holds true:*

$$\beta^b\left[\mathcal{Y}_\beta(k,a,b)+1\right]^n - a^b\mathcal{Y}_{n,\beta}(k,a,b) = \begin{cases} 2^{1-k} & (n=k) \\ 0 & (n \neq k), \end{cases} \tag{2.2}$$

where $\left[\mathcal{Y}_\beta(k,a,b)\right]^n$ *is replaced conventionally by* $\mathcal{Y}_{n,\beta}(k,a,b)$.

By applying Theorem 3, we can calculate all the numbers $\mathcal{Y}_{n,\beta}(k,a,b)$. For example, if we set $n=0$ in Theorem 3, we have

$$\mathcal{Y}_{0,\beta}(k,a,b) = \frac{2^{1-k}}{\beta^b - a^b} \quad \left(\beta^b \neq a^b\right). \tag{2.3}$$

Thus, by means of the number $\mathcal{Y}_{0,\beta}(k,a,b)$ given by (2.3), the other numbers $\mathcal{Y}_{n,\beta}(k,a,b)$ are easily calculated.

3. A unification of the family of partial zeta type functions

In this section, we define a new type of functions which unifies the family of partial zeta type functions. We derive many properties of this family of partial zeta type functions including (for example) its relationships with a unification of the Hurwitz type zeta functions and with a unification of the L-type functions.

Definition 3. Let $s \in \mathbb{C}$ and let $d, F \in \mathbb{Z}$ $(0 < d < F)$. Suppose also that $\beta \in \mathbb{C}$ $(|\beta| < 1)$. A unification $U_{\beta|F}(s;k,a,b;d)$ of the family of partial zeta type functions is then defined by

$$U_{\beta|F}(s;k,a,b;d) := \left(-\frac{1}{2}\right)^{k-1} \sum_{\substack{n \equiv d (\mathrm{mod}\, F) \\ (n \in \mathbb{N})}}^{\infty} \frac{\beta^{b(n-1)}}{a^{bn} n^s}. \tag{3.1}$$

More generally, we define the family $U_{\beta|F}(s,x;k,a,b;d)$ of partial zeta type functions by

$$U_{\beta|F}(s,x;k,a,b;d) := \left(-\frac{1}{2}\right)^{k-1} \sum_{\substack{n \equiv d-1 (\mathrm{mod}\, F) \\ (n \in \mathbb{N}_0)}}^{\infty} \frac{\beta^{bn}}{a^{b(n+1)}(n+x)^s}. \tag{3.2}$$

Obviously, by comparing the definitions (3.1) and (3.2), we have

$$U_{\beta|F}(s,1;k,a,b;d) = U_{\beta|F}(s;k,a,b;d).$$

Remark 5. Upon setting $a=b=1$ in the definition (3.1), we have

$$U_{\beta|F}(s;k,1,1;d) = \left(-\frac{1}{2}\right)^{k-1} \sum_{\substack{n \equiv d (\mathrm{mod}\, F) \\ (n \in \mathbb{N})}}^{\infty} \frac{\beta^{(n-1)}}{n^s} \quad (\beta \in \mathbb{C}\ (|\beta|<1)), \tag{3.3}$$

which, for $k=1$ and $\beta \to 1$, yields the partial Riemann zeta type functions $\zeta_F(s;d)$ given by (*cf.* [40]; see also [25])

$$\zeta_F(s;d) := \sum_{\substack{n \equiv d (\mathrm{mod}\, F) \\ (n \in \mathbb{N})}}^{\infty} \frac{1}{n^s} \quad (\Re(s) > 1). \tag{3.4}$$

Moreover, if we let

$$k = 0 \quad \text{and} \quad \beta \to -\kappa \quad (\kappa^r = 1;\ r \in \mathbb{Z}),$$

we find from the definition (3.1) that (*cf.* [4,21,25,32])

$$U_{-\kappa|F}(s;0,1,1;d) = 2\sum_{\substack{n\,\equiv\,d(\mathrm{mod}\,F)\\(n\in\mathbb{N})}}^{\infty}\frac{(-\kappa)^n}{n^s}\quad(\kappa^r=1;\,r\in\mathbb{Z}). \tag{3.5}$$

Theorem 4. *Let the zeta type function* $\zeta_\beta(s,x;k,a,b)$ *be defined by* (1.18). *Then*

$$U_{\beta|F}(s,x;k,a,b;d) = \frac{\beta^{b(d-1)}}{F^s a^{bd}}\,\zeta_{\beta^F}\left(s,\frac{x+d-1}{F};k,a^F,b\right) \tag{3.6}$$

and

$$U_{\beta|F}(s;k,a,b;d) = \frac{\beta^{b(d-1)}}{F^s a^{bd}}\,\zeta_{\beta^F}\left(s,\frac{d}{F};k,a^F,b\right). \tag{3.7}$$

Proof. If we set $n = d + mF\ (m\in\mathbb{N}_0)$ in the definition (3.1), in terms of the zeta type function $\zeta_\beta(s,x;k,a,b)$ defined by (1.24), we find that

$$\begin{aligned}U_{\beta|F}(s,k;a,b;d) &= \left(-\frac{1}{2}\right)^{k-1}\sum_{m=0}^{\infty}\frac{\beta^{b(d+mF-1)}}{a^{b(d+mF)}(d+mF)^s}\\ &= \beta^{b(d-1)}a^{bd}\frac{1}{F^s}\left[\left(-\frac{1}{2}\right)^{k-1}\sum_{m=0}^{\infty}\frac{\beta^{bmF}}{a^{bmF}\left(m+\frac{d}{F}\right)^s}\right]\\ &= \beta^{b(d-1)}F^s a^{bd}\,\zeta_{\beta^F}\left(s,\frac{d}{F};k,a^F,b\right),\end{aligned}$$

which proves the *simpler* assertion (3.7) of Theorem 4.

Similarly, by setting $n = d + mF - 1\ (m\in\mathbb{N}_0)$ in the definition (3.2), we can derive the *general* assertion (3.6) of Theorem 4. Our demonstration of Theorem 4 is thus completed. □

We next give an explicit formula for the family $U_{\beta|F}(s,x;k,a,b;d)$ of partial zeta type functions defined by (3.2) at negative integers. Indeed, upon setting $s = 1-n\ (n\in\mathbb{N})$ in the assertion (3.6) of Theorem 4, if we apply the known result (1.33) asserted by Theorem 2, we are led to Theorem 5 below.

Theorem 5. *Let* $n\in\mathbb{N}$ *and* $k\in\mathbb{N}_0$. *Then*

$$U_{\beta|F}(1-n,x;k,a,b;d) = (-1)^k\,\frac{(n-1)!}{(n+k-1)!}\,\frac{\beta^{b(d-1)}F^{n-1}}{a^{bd}}\cdot\mathcal{Y}_{n+k-1,\beta^F}\left(\frac{x+d-1}{F};k,a^F,b\right). \tag{3.8}$$

Remark 6. The importance of Theorem 5 lies in the fact that, by suitably specializing the parameters involved, we can make use of the partial zeta type function $U_{\beta|F}(s,x;k,a,b;d)$ defined by (3.2) in order to interpolate not only the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,b)$ generated by (1.4), but also various other polynomial systems which we have referred to in Section 1 (see, for example, [3,4,10,25], [32, p. 59 *et seq.*] and [33, p. 91 *et seq.*]).

Another important property of the general partial zeta type function $U_{\beta|F}(s,x;k,a,b;d)$ defined by (3.2) is given by the following theorem which provides its relationship with general Hurwitz–Lerch zeta function $\Phi(z,s,a)$ which is defined here by (1.14).

Theorem 6. *The following relationship holds true:*

$$U_{\beta|F}(s,x;k,a,b;d) = \left(-\frac{1}{2}\right)^{k-1}\frac{\beta^{b(d-1)}}{F^s a^{b(d+F)}}\,\Phi\left(\frac{\beta^{bF}}{a^{bF}},s,\frac{x+d-1}{F}\right). \tag{3.9}$$

Proof. The relationship (3.9) between the general partial zeta type function $U_{\beta|F}(s,x;k,a,b;d)$ defined by (3.2) and the general Hurwitz–Lerch zeta function $\Phi(z,s,a)$ defined here by (1.14) follows readily from (1.26) and (3.6). □

Remark 7. As already observed above (see Remark 3 *and* Ozden et al. [19, p. 2787, Remark 21], several interesting *multi-parameter* generalizations of the Hurwitz–Lerch zeta function $\Phi(z,s,a)$ were investigated by (for example) Garg et al. [7], Lin *et al.* [12] and Choi et al. [5]. Moreover, the interested reader should refer also to the works by (among others) Răducanu and Srivastava [20] and by Gupta et al. [8] for some recent applications of the general Hurwitz-Lerch zeta function $\Phi(z,s,a)$ in *Geometric Function Theory of Complex Analysis* and in *Probability Distribution Theory*, respectively (*cf.* [29,34,35]).

4. Relations involving a unified family of L-functions

In this section, we present various relationships between a unified family of L-functions and the partial zeta type function $U_{\beta|F}(s,x;k,a,b;d)$ defined by (3.2). We also show that of the function $U_{\beta|F}(s,x;k,a,b;d)$ is *meromorphic* in the complex s-plane and compute the residues at its *simple* poles at

$$s = 1, 2, \ldots, k.$$

Let χ be a Dirichlet character with conductor $f \in \mathbb{N}$. For $s, \beta \in \mathbb{C}$ $(|\beta| < 1)$, Özden and Simsek [18] considered the unified two-variable L-function $L_{\chi,\beta}(s,x;k,a,b)$ defined by

$$L_{\chi,\beta}(s,x;k,a,b) := \frac{1}{f^k}\left(-\frac{1}{2}\right)^{k-1}\sum_{n=0}^{\infty}\chi(n)\frac{\beta^{bn}}{a^{b(n+f)}(n+x)^s} \tag{4.1}$$

$$(\beta \in \mathbb{C}\ (|\beta| < 1);\ \Re(s) > 1),$$

with, as usual,

$$L_{\chi,\beta}(s;k,a,b) := L_{\chi,\beta}(s,1;k,a,b). \tag{4.2}$$

Indeed, by applying the elementary series identity:

$$\sum_{n=0}^{\infty}\Lambda(n) = \sum_{j=0}^{f-1}\sum_{n=0}^{\infty}\Lambda(fn+j) \quad (f \in \mathbb{N}), \tag{4.3}$$

it is not difficult to derive the following *alternative* form of the definition (4.1):

$$L_{\chi,\beta}(s,x;k,a,b) = \frac{1}{f^{s+k}}\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{jb}\zeta_{\beta^f}\left(s,\frac{x+j}{f};k,a^f,b\right), \tag{4.4}$$

in terms of the zeta type function $\zeta_\beta(s,x;k,a,b)$ defined by (1.24). The above formula (4.4) may be compared with a result proven by Özden and Simsek [[18] Theorem 2] for a Dirichlet character χ with conductor $f \in \mathbb{N}$.

Obviously, upon setting $\chi \equiv 1$ in the definition (4.1) or in the formula (4.4), we immediately obtain the zeta type function $\zeta_\beta(s,x;k,a,b)$ defined by (1.24) (*cf.* [2, p. 137 *et seq.*]; see also the recent works [9,38,42] dealing with several different aspects of the L-functions).

In terms of the generating function $\mathfrak{g}_\beta(x,t;k,a,b)$ occurring in (1.4), we have the following integral representation for the zeta type function $\zeta_\beta(s,x;k,a,b)$ defined by (1.24), which involves the Mellin transformation was given earlier by Ozden et al. (*cf.* [19, p. 2784, Eq. (4.1)]):

$$\zeta_\beta(s,x;k,a,b) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-k-1}\mathfrak{g}_\beta(x,-t;k,a,b)dt \quad (\min\{\Re(s),\Re(x)\} > 0), \tag{4.5}$$

where the *additional* constraint $\Re(x) > 0$ is required for the convergence of the infinite integral occurring on the right-hand side at its *upper* terminal. In fact, it was the integral representation (4.5) involving the Mellin transformation that led to the definition (1.24) by Ozden et al. [19, p. 2785, Definition 3]. Thus, upon substituting from (4.5) into the right-hand side of (4.4), we obtain the following formula involving the Mellin transformation (*cf.* [18, Eq. (2.7)]):

$$L_{\chi,\beta}(s,x;k,a,b) = \frac{1}{f^{s+k}\Gamma(s)}\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\cdot\int_0^\infty t^{s-k-1}\mathfrak{g}_{\beta^f}\left(\frac{x+j}{f},-t;k,a^f,b\right)dt \quad (\min\{\Re(s),\Re(x)\} > 0). \tag{4.6}$$

We now recall the following functional equation (*cf.* [18, Eq. (2.3)]):

$$\mathfrak{H}_{\chi,\beta}(x,t;k,a,b) = \frac{1}{f^k}\sum_{j=0}^{f-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\mathfrak{g}_{\beta^f}\left(\frac{x+j}{f},ft;k,a^f,b\right), \tag{4.7}$$

which involves the generating functions $\mathfrak{g}_\beta(x,t;k,a,b)$ and $\mathfrak{H}_{\chi,\beta}(x,t;k,a,b)$ occurring in (1.4) and (1.23), respectively. We multiply both sides of the functional Eq. (4.7) $\left(\textit{with}\ t \mapsto -\frac{t}{f}\right)$ by t^{s-k-1} and integrate each member with respect to t over the semi-infinite interval $(0,\infty)$. Then, by appealing once again to the Mellin transormation in (4.5), we obtain

$$\int_0^\infty t^{s-k-1}\mathfrak{H}_{\chi,\beta}\left(x,-\frac{t}{f};k,a,b\right)dt = \frac{\Gamma(s)}{f^k}\sum_{j=0}^{F-1}\chi(j)\left(\frac{\beta}{a}\right)^{jb}\zeta_{\beta^f}\left(s,\frac{x+j}{f};k,a^f,b\right) \tag{4.8}$$

$$(\min\{\Re(s),\Re(x)\} > 0).$$

A relationship between the functions $L_{\chi,\beta}(s,x;k,a,b)$ and $U_{\beta|F}(s,x;k,a,b;d)$ is provided by Theorem 7 below.

Theorem 7. *Let $s \in \mathbb{C}$. Also let χ be a Dirichlet character with conductor $f \in \mathbb{N}$ and suppose that F is any multiple of f. Then*

$$L_{\chi,\beta}(s,x;k,a,b) = \frac{a^{bd}}{\beta^{b(d-1)} f^k} \sum_{j=0}^{F-1} \chi(j) \left(\frac{\beta}{a}\right)^{jb} U_{\beta^F|F}\left(s, \frac{x+j+d-1}{f}; k, a^F, b; d\right). \tag{4.9}$$

Proof. By using (3.6) and (4.8), we easily arrive at the desired result (4.9) asserted by Theorem 7. □

As asserted by Theorem 8 below, the unified L-function $L_{\chi,\beta}(s,x;k,a,b)$ can be used to interpolate the unified Bernoulli–Euler-Genocchi polynomials $\mathcal{Y}_{n,\chi,\beta}(x;k,a,b)$ defined by the generating function (1.23) attached to the Dirichlet character χ with conductor $f \in \mathbb{N}$.

Theorem 8. *Let $n \in \mathbb{N}$. Then*

$$L_{\chi,\beta}(1-n,x;k,a,b) = \frac{(-1)^k}{f} \frac{(n-1)!}{(n+k-1)!} \mathcal{Y}_{n+k-1,\chi,\beta}(x;k,a,b). \tag{4.10}$$

Proof. By substituting from (3.8) into (4.9), one can easily obtain the desired result (4.10) asserted by Theorem 8. □

Remark 8. A different proof of the assertion (4.10) of Theorem 8 was given by Özden and Simsek [18], Theorem 3]; it was based essentially upon the formula (4.4).

Next, upon substituting from (1.29) into (3.8), we get

$$U_{\beta|F}(1-n,x;k,a,b;d) = (-1)^k \frac{(n-1)!}{(n+k-1)!} \frac{\beta^{b(d-1)} F^{n-k}}{a^{bd}} \cdot \sum_{j=0}^{n+k-1} \binom{n+k-1}{j} \left(\frac{x+d-1}{F}\right)^{n-j+k-1} \mathcal{Y}_{j,\beta^F}(k,a^F,b), \tag{4.11}$$

which, upon setting $n \mapsto n-k+1$, yields

$$U_{\beta|F}(k-n,x;k,a,b;d) = (-1)^k \frac{(n-k)!}{n!} \frac{\beta^{b(d-1)} F^{n-k}}{a^{bd}} \cdot \sum_{j=0}^{n} \binom{n}{j} \left(\frac{x+d-1}{F}\right)^{n-j} \mathcal{Y}_{j,\beta^F}(k,a^F,b). \tag{4.12}$$

Since

$$\frac{(n-k)!}{n!} = \frac{1}{(n-k+1)(n-k+2)\cdots(n-1)n} \qquad (0 \leqq k \leqq n), \tag{4.13}$$

it is not difficult to derive the primitive (original) corresponding to this last formula (4.12), which is asserted by Theorem 9 below.

Theorem 9. *Let*

$$s,\beta \in \mathbb{C} \quad (|\beta| < 1), \quad d \in \mathbb{N}, \quad b \in \mathbb{R}, \quad k \in \mathbb{N}_0 \quad \text{and} \quad a \in \mathbb{R}.$$

Then

$$U_{\beta|F}(s,x;k,a,b;d) = \frac{\beta^{b(d-1)}}{(s-1)(s-2)\cdots(s-k+1)(s-k)a^{bd} F^s} \cdot \sum_{j=0}^{\infty} \binom{k-s}{j} \left(\frac{x+d-1}{F}\right)^{k-s-j} \mathcal{Y}_{j,\beta^F}(k,a^F,b), \tag{4.14}$$

provided that each member of (4.14) exists.

Remark 9. Obviously, since

$$\binom{k-s}{0} = 1 \quad \text{and} \quad \binom{k-s}{j} = \frac{k-s}{j} \binom{k-s-1}{j-1} \qquad (j \in \mathbb{N}),$$

the only singularities of the *meromorphic* function $U_{\beta|F}(s,x;k,a,b;d)$ in the complex s-plane are *simple* poles at

$$s = 1, 2, \cdots, k.$$

The residues of this meromorphic function $U_{\beta|F}(s,x;k,a,b;d)$ at the simple poles at $s=1$, $s=2$, $s=k-1$ and $s=k$ are given, respectively, by

$$\text{Res}_{s=1}\{U_{\beta|F}(s,x;k,a,b;d)\} = \frac{(-1)^{k-1}\beta^{b(d-1)}}{a^{bd}F \cdot (k-1)!} \cdot \sum_{j=0}^{k-1} \binom{k-1}{j} \left(\frac{x+d-1}{F}\right)^{k-j-1} \mathcal{Y}_{j,\beta^F}(k,a^F,b), \tag{4.15}$$

$$\mathrm{Res}_{s=2}\left\{U_{\beta|F}(s,x;k,a,b;d)\right\} = \frac{(-1)^k\,\beta^{b(d-1)}}{a^{bd}\,F^2\cdot(k-2)!}\cdot\sum_{j=0}^{k-2}\binom{k-2}{j}\left(\frac{x+d-1}{F}\right)^{k-j-2}\mathcal{Y}_{j,\beta^F}\left(k,a^F,b\right), \tag{4.16}$$

$$\begin{aligned}\mathrm{Res}_{s=k-1}\left\{U_{\beta|F}(s,x;k,a,b;d)\right\} &= -\frac{\beta^{b(d-1)}}{a^{bd}\,F^{k-1}\cdot(k-2)!}\sum_{j=0}^{1}\binom{1}{j}\left(\frac{x+d-1}{F}\right)^{1-j}\mathcal{Y}_{j,\beta^F}\left(k,a^F,b\right)\\ &= \frac{\beta^{b(d-1)}}{a^{bd}\,F^{k-1}\cdot(k-2)!}\left[\frac{2^{1-k}}{a^{bF}-\beta^{bF}}\left(\frac{x+d-1}{F}\right)-\mathcal{Y}_{1,\beta^F}\left(k,a^F,b\right)\right]\left(\beta^{bF}\neq a^{bF}\right)\end{aligned} \tag{4.17}$$

and

$$\mathrm{Res}_{s=k}\left\{U_{\beta|F}(s,x;k,a,b;d)\right\} = \frac{2^{1-k}\,\beta^{b(d-1)}}{a^{bd}\,F^k\left(\beta^{bF}-a^{bF}\right)\cdot(k-1)!}\quad\left(\beta^{bF}\neq a^{bF}\right), \tag{4.18}$$

where, in the very last step in both (4.17) and (4.18), we have made use of the formula (2.3). In fact, the residues of the meromorphic function $U_{\beta|F}(s,x;k,a,b;d)$ at its simple poles at

$$s = 1,2,\ldots,k$$

can also be found by using the *Euler-Maclaurin Summation Formula* which would provide us with the meromorphic continuation of this function on $\mathbb{C}$.

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