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Certain subclasses of analytic and bi-univalent functions

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ABSTRACT

In the present paper, we introduce and investigate two interesting subclasses of normalized analytic and univalent functions in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

whose inverse has univalently analytic continuation to $\mathbb{U}$. Among other results, bounds for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ are found in our investigation.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by the following Taylor–Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

$\mathbb{C}$ being, as usual, the set of complex numbers. Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are *univalent* in $\mathbb{U}$ (for details, see [1]; see also the recent works [2,3]).

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\kappa)$ of *starlike functions of order* κ in $\mathbb{U}$ and the class $\mathcal{K}(\kappa)$ of *convex functions of order* κ in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*(\kappa) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \kappa \qquad (z \in \mathbb{U};\ 0 \leqq \kappa < 1) \right\} \tag{1.2}$$

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and

$$\mathcal{K}(\kappa) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \kappa \qquad (z \in \mathbb{U};\ 0 \leqq \kappa < 1) \right\}. \tag{1.3}$$

It readily follows from the definitions (1.2) and (1.3) that

$$f(z) \in \mathcal{K}(\kappa) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\kappa). \tag{1.4}$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots. \tag{1.5}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$.

Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by the Taylor–Maclaurin series expansion (1.1). Examples of functions in the class Σ are

$$\frac{z}{1-z}, \qquad -\log(1-z), \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ.

Lewin [4] investigated the bi-univalent function class Σ and showed that

$$|a_2| < 1.51.$$

Subsequently, Brannan and Clunie [5] conjectured that

$$|a_2| \leqq \sqrt{2}.$$

Netanyahu [6], on the other hand, showed that

$$\max_{f \in \Sigma} |a_2| = \frac{4}{3}.$$

The coefficient estimate problem for each of the following Taylor–Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1, 2\};\ \mathbb{N} := \{1, 2, 3, \ldots\})$$

is presumably still an open problem.

Brannan and Taha [7] (see also [8]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ (see [9]) of the univalent function class $\mathcal{S}$. Thus, following Brannan and Taha [7] (see also [8]), a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma[\alpha]$ $(0 < \alpha \leqq 1)$ of *strongly bi-starlike* functions of order α if each of the following conditions is satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \tag{1.6}$$

and

$$\left|\arg\left(\frac{zg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U};\ 0 < \alpha \leqq 1), \tag{1.7}$$

where g is the extension of f^{-1} to $\mathbb{U}$. The classes $\mathcal{S}^*_\Sigma(\kappa)$ and $\mathcal{K}_\Sigma(\kappa)$ of bi-starlike functions of order κ and bi-convex functions of order κ, corresponding (respectively) to the function classes $\mathcal{S}^*(\kappa)$ and $\mathcal{K}(\kappa)$ defined by (1.2) and (1.3), were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\kappa)$ and $\mathcal{K}_\Sigma(\kappa)$, they found *non-sharp* estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [7,8]).

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ.

2. Coefficient bounds for the function class $\mathcal{H}_\Sigma^\alpha$

Definition 1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma^\alpha$ $(0 < \alpha \leqq 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\big(f'(z)\big)\right| \leqq \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \tag{2.1}$$

and

$$\left|\arg\big(g'(w)\big)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U};\ 0 < \alpha \leqq 1), \tag{2.2}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots. \tag{2.3}$$

We first state and prove the following result.

Theorem 1. *Let $f(z)$ given by (1.1) be in the function class $\mathcal{H}_\Sigma^\alpha$. Then*

$$|a_2| \leqq \alpha\sqrt{\frac{2}{\alpha+2}} \qquad \textit{and} \qquad |a_3| \leqq \frac{\alpha(3\alpha+2)}{3}. \tag{2.4}$$

Proof. We can write the argument inequalities in (2.1) and (2.2) equivalently as follows:

$$f'(z) = [Q(z)]^\alpha \qquad \text{and} \qquad g'(w) = [L(w)]^\alpha, \tag{2.5}$$

respectively, where $Q(z)$ and $L(w)$ satisfy the following inequalities:

$$\Re\big(Q(z)\big) > 0 \quad (z \in \mathbb{U}) \qquad \text{and} \qquad \Re\big(L(w)\big) > 0 \quad (w \in \mathbb{U}). \tag{2.6}$$

Furthermore, the functions $Q(z)$ and $L(w)$ have the forms

$$Q(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.7}$$

and

$$L(w) = 1 + l_1 w + l_2 w^2 + \cdots, \tag{2.8}$$

respectively. Now, equating the coefficients of $f'(z)$ with $[Q(z)]^\alpha$ and the coefficients of $g'(w)$ with $[L(w)]^\alpha$, we get

$$2a_2 = \alpha c_1, \tag{2.9}$$

$$3a_3 = \alpha c_2 + \frac{\alpha(\alpha-1)}{2} c_1^2, \tag{2.10}$$

$$-2a_2 = \alpha l_1 \tag{2.11}$$

and

$$3(2a_2^2 - a_3) = \alpha l_2 + \frac{\alpha(\alpha-1)}{2} l_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$c_1 = -l_1 \qquad \text{and} \qquad 8a_2^2 = \alpha^2(c_1^2 + l_1^2). \tag{2.13}$$

Also, from (2.10) and (2.12), we find that

$$6a_2^2 - \left(\alpha c_2 + \frac{\alpha(\alpha-1)}{2}\, c_1^2\right) = \alpha l_2 + \frac{\alpha(\alpha-1)}{2}\, l_1^2.$$

A rearrangement together with the second identity in (2.13) yields

$$6a_2^2 = \alpha(c_2 + l_2) + \frac{\alpha(\alpha-1)}{2}(l_1^2 + c_1^2) = \alpha(c_2 + l_2) + \alpha(\alpha-1)\,\frac{4a_2^2}{\alpha^2}.$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2}{2(\alpha+2)}(c_2 + l_2),$$

which, in conjunction with the following well-known inequalities (see [1, p. 41]):

$$|c_2| \leqq 2 \quad \text{and} \quad |l_2| \leqq 2,$$

gives us the desired estimate on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.12) from (2.10), we get

$$6a_3 - 6a_2^2 = \alpha c_2 + \frac{\alpha(\alpha-1)}{2}\, c_1^2 - \left(\alpha l_2 + \frac{\alpha(\alpha-1)}{2} l_1^2\right).$$

Upon substituting the value of a_2^2 from (2.13) and observing that

$$c_1^2 = l_1^2,$$

it follows that

$$a_3 = \frac{1}{4}\,\alpha^2 c_1^2 + \frac{1}{6}\,\alpha(c_2 - l_2).$$

The familiar inequalities (see [1, p. 41]):

$$|c_2| \leqq 2 \quad \text{and} \quad |l_2| \leqq 2$$

now yield

$$|a_3| \leqq \frac{1}{4}\,\alpha^2 \cdot 4 + \frac{1}{6}\,\alpha \cdot 4 = \frac{\alpha(3\alpha+2)}{3}.$$

This completes the proof of Theorem 1. □

3. Coefficient bounds for the function class $\mathcal{H}_\Sigma(\beta)$

Definition 2. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma(\beta)$ $(0 \leqq \beta < 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re\big(f'(z)\big) > \beta \qquad (z \in \mathbb{U};\, 0 \leqq \beta < 1) \tag{3.1}$$

and

$$\Re\big(g'(w)\big) > \beta \qquad (w \in \mathbb{U};\, 0 \leqq \beta < 1), \tag{3.2}$$

where the function g is defined by (2.3).

Theorem 2. *Let $f(z)$ given by (1.1) be in the function class $\mathcal{H}_\Sigma(\beta)$ $(0 \leqq \beta < 1)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{3}} \qquad \textit{and} \qquad |a_3| \leqq \frac{(1-\beta)(5-3\beta)}{3}. \tag{3.3}$$

Proof. First of all, the argument inequalities in (3.1) and (3.2) can easily be rewritten in their equivalent forms:

$$f'(z) = \beta + (1-\beta)Q(z) \qquad \text{and} \qquad g'(w) = \beta + (1-\beta)L(w), \tag{3.4}$$

respectively, where $Q(z)$ and $L(w)$ satisfy the following inequalities:

$$\Re\big(Q(z)\big) > 0 \quad (z \in \mathbb{U}) \qquad \text{and} \quad \Re\big(L(w)\big) > 0 \quad (w \in \mathbb{U}).$$

Moreover, the functions $Q(z)$ and $L(w)$ have the following forms:

$$Q(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{3.5}$$

and

$$L(w) = 1 + l_1 w + l_2 w^2 + \cdots. \tag{3.6}$$

As in the proof of Theorem 1, by suitably comparing coefficients, we get

$$2a_2 = (1-\beta)c_1, \tag{3.7}$$

$$3a_3 = (1-\beta)c_2, \tag{3.8}$$

$$-2a_2 = (1-\beta)l_1 \tag{3.9}$$

and

$$3(2a_2^2 - a_3) = (1-\beta)l_2. \tag{3.10}$$

Now, considering (3.7) and (3.9), we get

$$c_1 = -l_1 \quad \text{and} \quad 8a_2^2 = (1-\beta)^2(c_1^2 + l_1^2). \tag{3.11}$$

Also, from (3.8) and (3.10), we find that

$$6a_2^2 = 3a_3 + (1-\beta)l_2 = (1-\beta)c_2 + (1-\beta)l_2 = (1-\beta)(c_2 + l_2). \tag{3.12}$$

Therefore, we have

$$|a_2^2| \leqq \frac{(1-\beta)}{6}(|c_2| + |l_2|) = \frac{(1-\beta)}{6} \cdot 4 = \frac{2(1-\beta)}{3}.$$

This gives the bound on $|a_2|$ as asserted in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$6a_3 - 6a_2^2 = (1-\beta)(c_2 - l_2),$$

which, upon substitution of the value of a_2^2 from (3.11), yields

$$6a_3 = \frac{6}{8}\,(1-\beta)^2(c_1^2 + l_1^2) + (1-\beta)(c_2 - l_2).$$

This last equation, together with the well-known estimates:

$$|c_1| \leqq 2, \quad |l_1| \leqq 2, \quad |c_2| \leqq 2 \quad \text{and} \quad |l_2| \leqq 2,$$

lead us to the following inequality:

$$6|a_3| \leqq \frac{3}{4}(1-\beta)^2 \cdot 8 + (1-\beta) \cdot 4.$$

Therefore, we have

$$|a_3| \leqq \frac{(1-\beta)(5-3\beta)}{3}.$$

This completes the proof of Theorem 2. □

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