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Inclusion properties of a subclass of analytic functions defined by an integral operator involving the Gauss hypergeometric function

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ABSTRACT

In the present paper, we introduce and investigate a new subclass of analytic functions in the open unit disk $\mathbb{U}$, which is defined by the convolution $(f_\mu)^{-1} * f(z)$, where

$$f_\mu(z) := (1-\mu)z\ {}_2F_1(a,b;c;z) + \mu z[z\ {}_2F_1(a,b;c;z)]' \quad (z \in \mathbb{U};\ \mu \geqq 0).$$

Several interesting properties including (for example) integral-preserving properties of this analytic function class are also considered.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also consider a class $\mathcal{M}$ of functions $\phi(z)$ which are analytic and univalent in $\mathbb{U}$ such that $\phi(\mathbb{U})$ is convex with

$$\phi(0) = 1 \quad \text{and} \quad \Re\{\phi(z)\} > 0 \quad (z \in \mathbb{U}).$$

We begin by recalling the principle of subordination between analytic functions.

Definition 1. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic (Schwarz) function $w(z)$ in $\mathbb{U}$, satisfying the following conditions:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)).$$

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We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if $g(z)$ is univalent in $\mathbb{U}$, then the subordination

$$f(z) \prec g(z) \quad (z \in \mathbb{U})$$

is equivalent to the following conditions:

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U})$$

(see, for details, [7,18]; see also [29]).

Definition 2. Each of the subclasses $\mathcal{S}^*(\phi),\ \mathcal{K}(\phi)$ and $\mathcal{C}(\phi,\psi)$ of the analytic function class $\mathcal{A}$ for $\phi,\psi \in \mathcal{M}$ is defined by using the above subordination principle (cf., e.g., [6,19]):

$$\mathcal{S}^*(\phi) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \mathbb{U};\ \phi \in \mathcal{M}) \right\}, \tag{1.2}$$

$$\mathcal{K}(\phi) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \quad (z \in \mathbb{U};\ \phi \in \mathcal{M}) \right\} \tag{1.3}$$

and

$$\mathcal{C}(\phi,\psi) := \left\{ f : f \in \mathcal{A},\ g \in \mathcal{S}^*(\phi) \quad \text{and} \quad \frac{zf'(z)}{g(z)} \prec \psi(z) \quad (z \in \mathbb{U};\ \phi,\psi \in \mathcal{M}) \right\}. \tag{1.4}$$

In particular, when

$$\phi(z) = \psi(z) = \frac{1+z}{1-z}$$

in the definitions (1.2) to (1.4), we have the familiar classes $\mathcal{S}^*,\ \mathcal{K}$ and $\mathcal{C}$ starlike, convex and close-to-convex function in $\mathbb{U}$, respectively. Furthermore, if we set

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leqq B < A \leqq 1)$$

in the definitions (1.2) and (1.3), we obtain the following function classes:

$$\mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right) = \mathcal{S}^*(A,B) \quad \text{and} \quad \mathcal{K}\left(\frac{1+Az}{1+Bz}\right) = \mathcal{K}(A,B). \tag{1.5}$$

Let $\mathcal{P}$ denote the class of functions of the form:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots,$$

which are analytic in $\mathbb{U}$ and satisfy the following inequality:

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

Denote by $D^\lambda : \mathcal{A} \to \mathcal{A}$ the Ruscheweyh derivative operator of order λ defined by the following Hadamard product (or convolution):

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \quad (\lambda > -1), \tag{1.6}$$

so that, obviously, we have

$$D^0 f(z) = f(z), \quad D^1 f(z) = zf'(z) \quad \text{and} \quad D^n f(z) = \frac{z\left[z^{n-1} f(z)\right]^{(n)}}{n!} \tag{1.7}$$

for

$$n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0,1,2,\ldots\} \quad (\mathbb{N} := \{1,2,3,\ldots\} = \mathbb{N}_0 \setminus \{0\}).$$

Recently, Noor et al. (see [22,23]) defined as integral operator $I_n : \mathcal{A} \to \mathcal{A}$, analogous to the Ruscheweyh derivative operator $D^\lambda f$, as follows.

Definition 3. Let the functions

$$f_n(z) = \frac{z}{(1-z)^{n+1}} \quad \text{and} \quad f_n^{(-1)}(z) \quad (n \in \mathbb{N}_0)$$

be defined such that

$$f_n(z) * f_n^{(-1)}(z) = \frac{z}{(1-z)^2} \quad (z \in \mathbb{U};\ n \in \mathbb{N}_0). \tag{1.8}$$

Then the integral operator $I_n : \mathcal{A} \to \mathcal{A}$ is defined by

$$I_n f(z) = f_n^{(-1)}(z) * f(z) = \left(\frac{z}{(1-z)^{n+1}}\right)^{-1} * f(z) \quad (f \in \mathcal{A}), \tag{1.9}$$

so that, clearly,

$$I^0 f(z) = zf'(z) \quad \text{and} \quad I_1 f(z) = f(z) \quad (f \in \mathcal{A})$$

The so-called Noor integral operator I_n of order n (see [3,16]) is an important operator which is used in defining several subclasses of analytic functions.

For parameters

$$a, b \in \mathbb{C} \quad \text{and} \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}),$$

the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.10}$$

where $(v)_k$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$(v)_0 := 1 \quad \text{and} \quad (v)_k := \frac{\Gamma(v+k)}{\Gamma(v)} = v(v+1)\cdots(v+k-1) \quad (k \in \mathbb{N}). \tag{1.11}$$

The hypergeometric series in (1.10) converges absolutely for all $z \in \mathbb{U}$, so that it represents an analytic function in $\mathbb{U}$. In particular, the function $\varphi(a, c; z)$ given by

$$z\,{}_2F_1(1, a; c; z) =: \varphi(a, c; z) \quad (z \in \mathbb{U})$$

is the incomplete Beta function. Also, since

$$\varphi(a, 1; z) = \frac{z}{(1-z)^a} \quad (z \in \mathbb{U}),$$

the function $\varphi(2, 1; z)$ is precisely the Koebe function.

Many recent investigations in *geometric function theory* in Complex Analysis have made use of not only the familiar Gauss hypergeometric function ${}_2F_1(a, b; c; z)$, but also of its natural generalizations including (for example) the generalized hypergeometric function ${}_qF_s$ $(q, s \in \mathbb{N}_0)$ with q numerator and s denominator parameters:

$$\alpha_j \in \mathbb{C} \quad (j = 1, \ldots, q) \quad \text{and} \quad \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j = 1, \ldots, s),$$

defined by

$${}_qF_s(\alpha_1, \ldots, \alpha_q;\ \beta_1, \cdots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{q} (\alpha_j)_k}{\prod_{j=1}^{s} (\beta_j)_k} \frac{z^k}{k!}.$$

For example, we may cite the widely-investigated Dziok–Srivastava operator involving the generalized hypergeometric function ${}_qF_s$ $(q, s \in \mathbb{N}_0)$ (see, for details, [8–10]; see also [2,4,5,12–14,17,30] and the references cited in each of these earlier investigations).

Shukla and Shukla [28] studied the mapping properties of the function $f_\mu(a, b, c)(z)$ defined by

$$f_\mu(a, b, c)(z) := (1-\mu)z\,{}_2F_1(a, b; c; z) + \mu z[z\,{}_2F_1(a, b; c; z)]' \quad (z \in \mathbb{U};\ \mu \geqq 0). \tag{1.12}$$

On the other hand, Kim and Shon [15] introduced a linear operator $L_\mu : \mathcal{A} \to \mathcal{A}$ defined by

$$L_\mu(a, b, c)(f(z)) = f_\mu(a, b, c)(z) * f(z).$$

Here, in this paper, we define a function $(f_\mu)^{(-1)}$ by the means of the following Hadamard product (or convolution):

$$f_\mu(a, b, c)(z) * \left[f_\mu(a, b, c)(z)\right]^{(-1)} = \frac{z}{(1-z)^{\lambda+1}} \quad (\mu \geqq 0,\ \lambda > -1) \tag{1.13}$$

and introduce the linear operator $I_\mu^\lambda(a, b, c)$ by

$$I_\mu^\lambda(a,b,c)f(z) = [f_\mu(a,b,c)(z)]^{(-1)} * f(z). \tag{1.14}$$

Upon setting $\mu = 0$ in (1.13), we obtain the operator introduced earlier by Noor [21].

Since

$$\frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1} \quad (z \in \mathbb{U};\ \lambda \in \mathbb{C}), \tag{1.15}$$

by using (1.10) and (1.15) in (1.13), we get

$$\left(\sum_{k=0}^{\infty} \frac{(\mu k+1)(a)_k(b)_k}{(c)_k} \frac{z^{k+1}}{k!}\right) * [f_\mu(a,b,c)(z)]^{(-1)} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1}. \tag{1.16}$$

We thus obtain the following explicit representation for $[f_\mu(a,b,c)(z)]^{(-1)}$:

$$[f_\mu(a,b,c)(z)]^{(-1)} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k(c)_k}{(\mu k+1)(a)_k(b)_k} z^{k+1} \quad (z \in \mathbb{U}). \tag{1.17}$$

Eq. (1.14) now implies that

$$I_\mu^\lambda(a,b,c)f(z) = z + \sum_{k=1}^{\infty} \frac{(\lambda+1)_k(c)_k}{(\mu k+1)(a)_k(b)_k} a_{k+1} z^{k+1}. \tag{1.18}$$

In particular, we have

$$I_0^\lambda(a,\lambda+1,a)f(z) = f(z) \quad \text{and} \quad I_0^1(a,1,a)f(z) = zf'(z). \tag{1.19}$$

It can also be easily shown that

$$z\left[I_\mu^\lambda(a,b,c)f(z)\right]' = (\lambda+1)I_\mu^{\lambda+1}(a,b,c)f(z) - \lambda I_\mu^\lambda(a,b,c)f(z) \tag{1.20}$$

and

$$z\left[I_\mu^\lambda(a+1,b,c)f(z)\right]' = aI_\mu^\lambda(a,b,c)f(z) - (a-1)I_\mu^\lambda(a+1,b,c)f(z). \tag{1.21}$$

In the present sequel to the aforementioned works, by using the operator $I_\mu^\lambda(a,b,c)$, we introduce and investigate the inclusion properties of each of the following interesting subclasses of analytic functions for

$$\phi,\psi \in \mathcal{M}, \quad \lambda > -1, \quad \phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leqq B < A \leqq 1) \quad \text{and} \quad \mu \geqq 0:$$

$$\mathcal{S}_\mu^\lambda(a,b,c)(\phi) := \left\{f : f \in \mathcal{A} \quad \text{and} \quad I_\mu^\lambda(a,b,c)f(z) \in \mathcal{S}^*(\phi)\right\}, \tag{1.22}$$

$$\mathcal{K}_\mu^\lambda(a,b,c)(\phi) := \left\{f : f \in \mathcal{A} \quad \text{and} \quad I_\mu^\lambda(a,b,c)f(z) \in \mathcal{K}(\phi)\right\} \tag{1.23}$$

and

$$\mathcal{C}_\mu^\lambda(a,b,c)(\phi,\psi) := \left\{f : f \in \mathcal{A} \quad \text{and} \quad \exists \quad g(z) \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi) \text{ such that } \frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)g(z)} \prec \psi(z) \quad (z \in \mathbb{U})\right\}. \tag{1.24}$$

It is easily seen from the definitions (1.22) and (1.23) that

$$f(z) \in \mathcal{K}(a,b,c)(\phi) \Longleftrightarrow zf'(z) \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi). \tag{1.25}$$

For the sake of convenience, we write

$$\mathcal{S}_\mu^\lambda(a,b,c)\left(\frac{1+Az}{1+Bz}\right) =: \mathcal{S}_\mu^\lambda(a,b,c,A,B) \quad (-1 \leqq B < A \leqq 1), \tag{1.26}$$

$$\mathcal{K}_\mu^\lambda(a,b,c)\left(\frac{1+Az}{1+Bz}\right) =: \mathcal{K}_\mu^\lambda(a,b,c,A,B) \quad (-1 \leqq B < A \leqq 1) \tag{1.27}$$

and

$$\mathcal{C}_\mu^\lambda(a,b,c)\left(\frac{1+Az}{1+Bz}, \frac{1+Az}{1+Bz}\right) =: \mathcal{C}_\mu^\lambda(a,b,c,A,B) \quad (-1 \leqq B < A \leqq 1). \tag{1.28}$$

The main objective of this paper is to investigate the inclusion properties of each of the above-defined function classes

$$\mathcal{S}_\mu^\lambda(a,b,c)(\phi), \quad \mathcal{K}_\mu^\lambda(a,b,c)(\phi) \quad \text{and} \quad \mathcal{C}_\mu^\lambda(a,b,c)(\phi,\psi).$$

Since

$$\mathcal{S}_0^\lambda(a,\lambda+1,a)\left(\frac{1+z}{1-z}\right) = \mathcal{S}^*, \quad \mathcal{K}_0^\lambda(a,\lambda+1,a)\left(\frac{1+z}{1-z}\right) = \mathcal{K} \tag{1.29}$$

and

$$\mathcal{C}_0^\lambda(a,\lambda+1,a)\left(\frac{1+z}{1-z},\frac{1+z}{1-z}\right) = \mathcal{C}, \tag{1.30}$$

the results presented in this paper can be suitably specialized to deduce the corresponding (known or new) results for the familiar function classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$.

2. Inclusion properties involving the operator $I_\mu^\lambda(a,b,c)$

The following lemmas will be required in our investigation.

Lemma 1 (see [20]). *Let the function $\phi(z)$ be convex univalent in $\mathbb{U}$. Suppose that the function $B(z)$ is analytic in $\mathbb{U}$* with

$$\Re\{B(z)\} \geqq E \quad (E \geqq 0).$$

If $g \in \mathcal{P}$ is analytic in $\mathbb{U}$, then

$$E^2z^2g''(z) + B(z)zg'(z) + g(z) \prec \phi(z) \quad (z \in \mathbb{U}) \tag{2.1}$$

implies that

$$g(z) \prec \phi(z) \quad (z \in \mathbb{U}).$$

Lemma 2 (see [26]). *Let $f \in \mathcal{K}$ and $g \in \mathcal{S}^*$. Then, for every analytic function Q in $\mathbb{U}$,*

$$\frac{(f * Qg)}{f * g}(\mathbb{U}) \subset \overline{\text{CO}}[Q(\mathbb{U})], \tag{2.2}$$

where $\overline{\text{CO}}[Q(\mathbb{U})]$ denotes the closed convex hull of $Q(\mathbb{U})$.

Lemma 3 (see [25]). *For complex numbers β and γ, let $\phi(z)$ be a convex univalent function in $\mathbb{U}$ with*

$$\phi(0) = 1 \quad \text{and} \quad \Re\{\beta\phi(z) + \gamma\} > 0 \quad (z \in \mathbb{U}).$$

Also let the function $q \in \mathcal{A}$ satisfy the following subordination condition:

$$q(z) \prec \phi(z) \quad (z \in \mathbb{U}).$$

If the function $p \in \mathcal{P}$ is analytic in $\mathbb{U}$, then

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \phi(z) \quad (z \in \mathbb{U}) \tag{2.3}$$

implies that

$$p(z) \prec \phi(z) \quad (z \in \mathbb{U}).$$

Lemma 4 (see [11]). *Let the parameters δ and η be complex numbers. Also let $\phi(z)$ be a convex univalent function in $\mathbb{U}$ with*

$$\phi(0) = 1 \quad \text{and} \quad \Re\{\delta\phi(z) + \eta\} > 0 \quad (z \in \mathbb{U}).$$

If the function $p \in \mathcal{P}$ is analytic in $\mathbb{U}$, then the following subordination condition:

$$p(z) + \frac{zp'(z)}{\delta p(z) + \eta} \prec \phi(z) \quad (z \in \mathbb{U}) \tag{2.4}$$

implies that

$$p(z) \prec \phi(z) \quad (z \in \mathbb{U}).$$

Our first main result is contained in Theorem 1 below.

Theorem 1. *Let the function $\phi(z)$ be convex univalent in $\mathbb{U}$ with*

$$\phi(0)=1 \quad \textit{and} \quad \Re\{\phi(z)\} \geqq 0 \quad (z\in\mathbb{U}).$$

Then

$$\mathcal{S}_{\mu}^{\lambda+1}(a,b,c)(\phi) \subset \mathcal{S}_{\mu}^{\lambda}(a,b,c)(\phi) \quad (\lambda>-1;\ \mu \geqq 0).$$

Proof. Let $f(z)\in\mathcal{S}_{\mu}^{\lambda+1}(a,b,c)(\phi)$ and suppose that

$$p(z)=\frac{z\left(I_{\mu}^{\lambda}(a,b,c)f(z)\right)'}{I_{\mu}^{\lambda}(a,b,c)f(z)} \quad (p(z)\in\mathcal{P}). \tag{2.5}$$

Then, by using (1.2) in (2.5) and differentiating the resulting equation, we get

$$\frac{z\left(I_{\mu}^{\lambda+1}(a,b,c)f(z)\right)'}{I_{\mu}^{\lambda+1}(a,b,c)f(z)}=p(z)+\frac{zp'(z)}{(\lambda+1)q(z)},$$

where

$$q(z)=\frac{I_{\mu}^{\lambda+1}(a,b,c)f(z)}{I_{\mu}^{\lambda}(a,b,c)f(z)}$$

and

$$q(z)\prec\phi(z) \quad (z\in\mathbb{U}).$$

Hence, by applying Lemma 3, we obtain

$$\frac{z\left(I_{\mu}^{\lambda}(a,b,c)f(z)\right)'}{I_{\mu}^{\lambda}(a,b,c)f(z)}\prec\phi(z) \quad (z\in\mathbb{U}),$$

which, in view of (1.22), yields

$$f(z)\in\mathcal{S}_{\mu}^{\lambda}(a,b,c)(\phi).$$

Our proof of Theorem 1 is thus completed. □

Theorem 2. *Let the function $\phi(z)$ be convex univalent in $\mathbb{U}$ with*

$$\phi(0)=1 \quad \text{and} \quad \Re\{\phi(z)\} \geqq 0 \quad (z\in\mathbb{U}).$$

Then

$$\mathcal{S}_{\mu}^{\lambda}(a,b,c)(\phi) \subset \mathcal{S}_{\mu}^{\lambda}(a+1,b,c)(\phi) \quad (\lambda>-1;\ \mu \geqq 0).$$

Proof. Applying the same technique as in the proof of Theorem 1, and using (1.21) in conjunction with Lemma 4, we obtain the result asserted by Theorem 2. □

Upon setting

$$\phi(z)=\frac{1+Az}{1+Bz} \quad (-1\leqq B<A\leqq 1)$$

in Theorems 1 and 2, we obtain the following result.

Corollary 1. *For $\lambda>-1$, $\mu\geqq 0$ and $\Re(a)>1$, the following inclusion properties hold true:*

$$\mathcal{S}_{\mu}^{\lambda+1}(a,b,c,A,B) \subset \mathcal{S}_{\mu}^{\lambda}(a,b,c,A,B)$$

and

$$\mathcal{S}_{\mu}^{\lambda}(a,b,c,A,B) \subset \mathcal{S}_{\mu}^{\lambda}(a+1,b,c,A,B).$$

If we set

$$\phi(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U})$$

in Theorems 1 and 2, we obtain the following result.

Corollary 2. *For* $\lambda > -1$, $\mu \geqq 0$ *and* $\Re(a) > 0$,

$$I_{\mu}^{\lambda+1}(a,b,c)f(z) \in \mathcal{S}^* \Rightarrow I_{\mu}^{\lambda}(a,b,c)f(z) \in \mathcal{S}^*.$$

Furthermore,

$$I_{\mu}^{\lambda}(a,b,c)f(z) \in \mathcal{S}^* \Rightarrow I_{\mu}^{\lambda}(a+1,b,c) \in \mathcal{S}^*.$$

Corollary 3. *For* $\lambda > -1$, $\mu \geqq 0$ *and* $\Re(a) > 0$,

$$\mathcal{K}_{\mu}^{\lambda+1}(a,b,c)(\phi) \subset \mathcal{K}_{\mu}^{\lambda}(a,b,c)(\phi)$$

and

$$\mathcal{K}_{\mu}^{\lambda}(a,b,c)(\phi) \subset \mathcal{K}_{\mu}^{\lambda}(a+1,b,c)(\phi).$$

Proof. It is easily observed that

$$\begin{aligned} f(z) \in \mathcal{K}_{\mu}^{\lambda+1}(a,b,c)(\phi) &\Longleftrightarrow zf'(z) \in \mathcal{S}_{\mu}^{\lambda+1}(a,b,c)(\phi), \\ &\Rightarrow zf'(z) \in \mathcal{S}_{\mu}^{\lambda}(a,b,c)(\phi), \\ &\Longleftrightarrow I_{\mu}^{\lambda}(a,b,c)(zf'(z)) \in \mathcal{S}^*(\phi), \\ &\Longleftrightarrow z(I_{\mu}^{\lambda}(a,b,c)f(z))' \in \mathcal{S}^*(\phi), \\ &\Longleftrightarrow I_{\mu}^{\lambda}(a,b,c)f(z) \in \mathcal{K}(\phi), \\ &\Longleftrightarrow f(z) \in \mathcal{K}_{\mu}^{\lambda}(a,b,c)(\phi). \end{aligned}$$

The second assertion of Corollary 3 can be proved similarly. □

Theorem 3. *Let the function* $\phi(z)$ *be convex univalent in* $\mathbb{U}$ *with*

$$\phi(0) = 1 \quad \text{and} \quad \Re\{\phi(z)\} \geqq 0.$$

If $f(z) \in \mathcal{A}$ *satisfies the following condition:*

$$f(z) \in \mathcal{S}_{\mu}^{\lambda}(a,b,c)(\phi)$$

then

$$F(z) \in \mathcal{S}_{\mu}^{\lambda}(a,b,c)(\phi),$$

where the function $F(z)$ *is given by a one-parameter integral operator as follows:*

$$F(z) = \frac{\mathfrak{d}+1}{z^{\mathfrak{d}}} \int_0^z t^{\mathfrak{d}-1} f(z) \mathrm{d}t \quad (\mathfrak{d} > -1). \tag{2.6}$$

Proof. First of all, we find from the definition (2.6) that

$$z\left(I_{\mu}^{\lambda}(a,b,c)F(z)\right)' = (\mathfrak{d}+1)I_{\mu}^{\lambda}(a,b,c)f(z) - \mathfrak{d}I_{\mu}^{\lambda}(a,b,c)F(z). \tag{2.7}$$

Let

$$p(z) = \frac{z\left(I_{\mu}^{\lambda}(a,b,c)F(z)\right)'}{I_{\mu}^{\lambda}(a,b,c)F(z)} \quad (p \in \mathcal{P}).$$

Thus, by using (2.7), we get

$$p(z) + \mathfrak{d} = \frac{(\mathfrak{d}+1)I_{\mu}^{\lambda}(a,b,c)f(z)}{I_{\mu}^{\lambda}(a,b,c)F(z)}. \tag{2.8}$$

Differentiating both sides of (2.8) logarithmically, we obtain

$$p(z) + \frac{zp'(z)}{p(z) + \mathfrak{d}} = \frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)f(z)} \prec \phi(z) \quad (z \in \mathbb{U})$$

by means of the hypothesis of Theorem 3.

Finally, by applying Lemma 4, we have

$$\frac{z\left(I_\mu^\lambda(a,b,c)F(z)\right)'}{I_\mu^\lambda(a,b,c)F(z)} \prec \phi(z) \quad (z \in \mathbb{U}),$$

that is,

$$F(z) \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi),$$

as asserted by Theorem 3. □

In its special case when

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leqq B < A \leqq 1),$$

Theorem 3 yields the following result.

Corollary 4. *Let* $\lambda > -1$, $\mu \geqq 0$ *and* $\mathfrak{d} > -1$. *Also let the function F(z) be given by* (2.6). *If* $f(z) \in \mathcal{S}_\mu^\lambda(a,b,c,A,B)$, *then*

$$F(z) \in \mathcal{S}_\mu^\lambda(a,b,c,A,B).$$

Corollary 5. *Let* $\lambda > -1$, $\mu \geqq 0$ *and* $\mathfrak{d} > -1$. *Also let the function F(z) be given by* (2.6). *If* $f(z) \in \mathcal{K}_\mu^\lambda(a,b,c)(\phi)$, *then*

$$F(z) \in \mathcal{K}_\mu^\lambda(a,b,c)(\phi).$$

Proof. It is fairly easy to see that

$$\begin{aligned} f(z) \in \mathcal{K}_\mu^\lambda(a,b,c)(\phi) &\Longleftrightarrow zf'(z) \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi) \\ &\Longrightarrow z(F(z))' \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi) \\ &\Longleftrightarrow F(z) \in \mathcal{K}_\mu^\lambda(a,b,c)(\phi). \qquad \square \end{aligned}$$

Theorem 4. *Let* $f(z) \in \mathcal{A}$. *Then*

$$\mathcal{C}_\mu^{\lambda+1}(a,b,c,\phi,\psi) \subset \mathcal{C}_\mu^\lambda(a,b,c,\phi,\psi) \quad (\Re(a) > 0).$$

Proof. Let $f(z) \in \mathcal{C}_\mu^{\lambda+1}(a,b,c,\phi,\psi)$. Then, by definition, we have

$$\frac{z\left(I_\mu^{\lambda+1}(a,b,c,\phi,\psi)f(z)\right)'}{I_\mu^{\lambda+1}(a,b,c,\phi,\psi)g(z)} \prec \psi(z) \quad (z \in \mathbb{U})$$

for some $g(z) \in \mathcal{S}_\mu^{\lambda+1}(a,b,c)(\phi)$. Next, by setting

$$h(z) = \frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)g(z)} \tag{2.9}$$

and

$$H(z) = \frac{z\left(I_\mu^\lambda(a,b,c)g(z)\right)'}{I_\mu^\lambda(a,b,c)g(z)}, \tag{2.10}$$

we notice that

$$h(z) \in \mathcal{P} \quad \text{and} \quad H(z) \in \mathcal{P}.$$

Thus, by Theorem 2, $g(z) \in \mathcal{S}_{\mu}^{\lambda}(a,b,c)(\phi)$ and so $\Re\{H(z)\} > 0$. Moreover, (2.9) implies that

$$z\left(I_{\mu}^{\lambda}(a,b,c)f(z)\right)' = \left(I_{\mu}^{\lambda}(a,b,c)g(z)\right)h(z). \tag{2.11}$$

Differentiating both sides of (2.9), we get

$$\frac{z\left(z\left(I_{\mu}^{\lambda}(a,b,c)f(z)\right)'\right)'}{I_{\mu}^{\lambda}(a,b,c)g(z)} = H(z)h(z) + zh'(z) \tag{2.12}$$

which, in view of the identity (1.20), yields

$$\frac{z\left(I_{\mu}^{\lambda+1}(a,b,c)f(z)\right)'}{I_{\mu}^{\lambda+1}(a,b,c)g(z)} = \frac{I_{\mu}^{\lambda+1}(a,b,c)(zf'(z))}{I_{\mu}^{\lambda+1}(a,b,c)g(z)} = \frac{z\left(I_{\mu}^{\lambda}(a,b,c)(zf'(z))\right)' + \lambda I_{\mu}^{\lambda}(a,b,c)(zf'(z))}{z\left(I_{\mu}^{\lambda}(a,b,c)g(z)\right)' + \lambda I_{\mu}^{\lambda}(a,b,c)g(z)}$$

$$= \frac{\dfrac{z\left(I_{\mu}^{\lambda}(a,b,c)(zf'(z))\right)'}{I_{\mu}^{\lambda}(a,b,c)g(z)} + \dfrac{\lambda I_{\mu}^{\lambda}(a,b,c)(zf'(z))}{I_{\mu}^{\lambda}(a,b,c)g(z)}}{\dfrac{z\left(I_{\mu}^{\lambda}(a,b,c)g(z)\right)'}{I_{\mu}^{\lambda}(a,b,c)g(z)} + \lambda} = \frac{H(z)h(z) + zh'(z) + \lambda h(z)}{H(z) + \lambda} = h(z) + \frac{zh'(z)}{H(z) + \lambda} \prec \psi(z) \quad (z \in \mathbb{U}).$$

Now, by applying Lemma 1 for

$$E = 0 \quad \text{and} \quad B(z) = \frac{1}{H(z) + \lambda}$$

with

$$\Re\{B(z)\} = \frac{1}{|H(z) + \lambda|^2}\Re\{H(z) + \lambda\} > 0,$$

we get

$$h(z) \prec \psi(z) \quad (z \in \mathbb{U}),$$

which, by virtue of (2.9), implies that $f(z) \in \mathcal{C}_{\mu}^{\lambda}(a,b,c,\phi,\psi)$. □

Theorem 5. *Let $f \in \mathcal{A}$. Then*

$$\mathcal{C}_{\mu}^{\lambda}(a,b,c,\phi,\psi) \subset \mathcal{C}_{\mu}^{\lambda}(a+1,b,c,\phi,\psi) \quad (\Re(a) > 0).$$

Proof. By using arguments similar to those in the proof of Theorem 4, we get

$$h(z) + \frac{zh'(z)}{H(z) + a - 1} \prec \psi(z) \quad (z \in \mathbb{U})$$

for

$$h(z) = \frac{z\left(I_{\mu}^{\lambda}(a+1,b,c)f(z)\right)'}{I_{\mu}^{\lambda}(a+1,b,c)g(z)} \in \mathcal{P}$$

and

$$H(z) = \frac{z\left(I_{\mu}^{\lambda}(a+1,b,c)g(z)\right)'}{I_{\mu}^{\lambda}(a+1,b,c)g(z)} \in \mathcal{P}.$$

Now, by applying Lemma 1 for

$$E = 0 \quad \text{and} \quad B(z) = \frac{1}{H(z) + a - 1}$$

with

$$\Re(B(z)) = \frac{1}{|H(z) + a - 1|^2}\Re\{H(z) + a - 1\} > 0,$$

we obtain the required result. □

Theorem 6. *Let $\mathfrak{d} > -1$ and suppose that F(z) is given by (2.6). If $f(z) \in \mathcal{C}_\mu^\lambda(a,b,c,\phi,\psi)$, then*

$$F(z) \in \mathcal{C}_\mu^\lambda(a,b,c,\phi,\psi).$$

Proof. By employing the same technique as in proof of Theorem 4, we get

$$\frac{zh'(z)}{H(z)+\mathfrak{d}} + h(z) \prec \psi(z)$$

for

$$h(z) = \frac{z(I_\mu^\lambda(a,b,c)F(z))'}{I_\mu^\lambda(a,b,c)g(z)} \in \mathcal{P}$$

and

$$H(z) = \frac{z(I_\mu^\lambda(a,b,c)g(z))'}{I_\mu^\lambda(a,b,c)g(z)} \in \mathcal{P}.$$

Now, by applying Lemma 1 for

$$E = 0 \quad \text{and} \quad B = \frac{1}{H(z)+\mathfrak{d}}$$

with

$$\Re\{B(z)\} = \frac{1}{|H(z)+\mathfrak{d}|^2}\Re\{H(z)+\mathfrak{d}\} > 0,$$

we arrive at the result asserted by Theorem 6. □

3. Inclusion properties by convolution

In this section, we show that the function classes

$$\mathcal{S}_\mu^\lambda(a,b,c)(\phi), \quad \mathcal{K}_\mu^\lambda(a,b,c)(\phi) \quad \text{and} \quad \mathcal{C}_\mu^\lambda(a,b,c,\phi,\psi)$$

are invariant under convolution with convex functions.

Theorem 7. *Let a > 0, b > 0 and $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$. Suppose also that $\phi, \psi \in \mathcal{M}$ and $g \in \mathcal{K}$. Then*

(i) $f \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi) \Rightarrow g * f \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi)$;
(ii) $f \in \mathcal{K}_\mu^\lambda(a,b,c)(\phi) \Rightarrow g * f \in \mathcal{K}_\mu^\lambda(a,b,c)(\phi)$

and

(iii) $f \in \mathcal{C}_\mu^\lambda(a,b,c,\phi,\psi) \Rightarrow g * f \in \mathcal{C}_\mu^\lambda(a,b,c,\phi,\psi)$.

Proof. We consider the following three cases:

(i) Let $f \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi)$. Then

$$\frac{z\left(I_\mu^\lambda(a,b,c)f\right)'}{I_\mu^\lambda(a,b,c)f} \prec \phi(w(z)) \quad (z \in \mathbb{U}),$$

which yields

$$\frac{z\left(I_\mu^\lambda(a,b,c)(g*f)(z)\right)'}{I_\mu^\lambda(a,b,c)(g*f)(z)} = \frac{g(z) * z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{g(z) * I_\mu^\lambda(a,b,c)f(z)} = \frac{g(z) * \phi(w)I_\mu^\lambda(a,b,c)f(z)}{g(z) * I_\mu^\lambda(a,b,c)f(z)}. \tag{3.1}$$

Thus, by using Lemma 2, we conclude that

$$\frac{\left\{g * \phi(w)I_\mu^\lambda(a,b,c)f\right\}}{\left\{g * I_\mu^\lambda(a,b,c)f\right\}}(\mathbb{U}) \subset \overline{\text{CO}}[\phi(\mathbb{U})] \subset \phi(\mathbb{U}),$$

since ϕ is convex univalent and $I_\mu^\lambda(a,b,c)f \in \mathcal{S}^*(\phi)$. By the definition of subordination, we see that the function quotient in (3.1) is subordinate to $\phi(z)$ in $\mathbb{U}$, and so we have

$$g * f \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi).$$

(ii) Suppose that $f \in \mathcal{K}_\mu^\lambda(a,b,c)(\phi)$. Then, by (1.23), $zf'(z) \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi)$. Hence, by means of (i), we have

$$g * zf'(z) \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi).$$

We notice also that

$$g(z) * zf'(z) = z(g*f)'(z).$$

Thus, by applying (1.23) again, we get

$$g * f \in \mathcal{K}_\mu^\lambda(a,b,c)(\phi).$$

(iii) Let $f \in \mathcal{C}_\mu^\lambda(a,b,c,\phi,\psi)$. Then there exists a function $q \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi)$ such that

$$\frac{z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{I_\mu^\lambda(a,b,c)q(z)} \prec \psi(z) \quad (z \in \mathbb{U}). \tag{3.2}$$

Therefore, we get

$$z\left(I_\mu^\lambda(a,b,c)f(z)\right)' = \psi(w(z))I_\mu^\lambda(a,b,c)q(z) \quad (z \in \mathbb{U}), \tag{3.3}$$

where $w(z)$ is an analytic function in $\mathbb{U}$ with

$$|w(z)| < 1 \quad (z \in \mathbb{U}) \quad \text{and} \quad w(0) = 0.$$

Now, since $I_\mu^\lambda(a,b,c)q \in \mathcal{S}^*(\phi)$, we have

$$\frac{z\left(I_\mu^\lambda(a,b,c)(g*f)(z)\right)'}{g * I_\mu^\lambda(a,b,c)q} = \frac{g(z) * z\left(I_\mu^\lambda(a,b,c)f(z)\right)'}{g(z) * I_\mu^\lambda(a,b,c)q(z)} = \frac{g(z) * \psi(w(z))I_\mu^\lambda(a,b,c)q(z)}{g(z) * I_\mu^\lambda(a,b,c)q(z)} \prec \psi(z) \quad (z \in \mathbb{U}). \tag{3.4}$$

Thus the assertion (iii) of Theorem 7 is proved. We complete the proof of Theorem 7. □

We next investigate the functions $\omega_1(z)$ and $\omega_2(z)$ defined by (see [24,27])

$$\omega_1(z) := \sum_{k=1}^{\infty}\left(\frac{\mathfrak{c}+1}{\mathfrak{c}+k}\right)z^k \quad (\Re(\mathfrak{c}) \geqq 0;\ z \in \mathbb{U}) \tag{3.5}$$

and

$$\omega_2(z) := \frac{1}{1-\kappa}\log\left(\frac{1-\kappa z}{1-z}\right) \quad (\log 1 := 0;\ |\kappa| \leqq 1\ (\kappa \neq 1);\ z \in \mathbb{U}), \tag{3.6}$$

respectively. Then it is known from the earlier works [1,27] that the functions $\omega_1(z)$ and $\omega_2(z)$ are convex univalent in $\mathbb{U}$. Therefore, we have the following immediate consequences of Theorem 7.

Corollary 6. *Let $a > 0$, $b > 0$ and $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$. Suppose that $\phi, \psi \in \mathcal{M}$. Also let the functions $\omega_1(z)$ and $\omega_2(z)$ be defined by* (3.5) and (3.6), *respectively. Then*

(i) $f \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi) \Rightarrow \omega_j * f \in \mathcal{S}_\mu^\lambda(a,b,c)(\phi)\ (j=1,2)$;
(ii) $f \in \mathcal{K}_\mu^\lambda(a,b,c)(\phi) \Rightarrow \omega_j * f \in \mathcal{K}_\mu^\lambda(a,b,c)(\phi)\ (j=1,2)$

and

(iii) $f \in \mathcal{C}_\mu^\lambda(a,b,c,\phi,\psi) \Rightarrow \omega_j * f \in \mathcal{C}_\mu^\lambda(a,b,c,\phi,\psi)\ (j=1,2)$.

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References

[1] R.W. Barnard, Ch. Kellogg, Applications of convolution operators to problems in univalent function theory, Michigan Math. J. 27 (1980) 81–94.
[2] B.C. Carlson, D.B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal. 15 (1984) 737–745.
[3] N.E. Cho, The Noor integral operator and strongly close-to-convex functions, J. Math. Anal. Appl. 283 (2003) 202–212.
[4] N.E. Cho, Inclusion properties for certain subclasses of analytic functions defined by a linear operator, Abstr. Appl. Anal. 2008 (2008) 1–8 (Article ID 246876).
[5] N.E. Cho, T.H. Kim, Multiplier transformations and strongly close-to-convex functions, Bull. Korean Math. Soc. 40 (2003) 399–410.
[6] J.H. Choi, M. Saigo, H.M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl. 276 (2002) 432–445.
[7] P.L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
[8] J. Dziok, H.M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. 103 (1999) 1–13.
[9] J. Dziok, H.M. Srivastava, Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function, Adv. Stud. Contemp. Math. 5 (2002) 115–125.
[10] J. Dziok, H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transforms Spec. Funct. 14 (2003) 7–18.
[11] P. Eenigenburg, S.S. Miller, P.T. Mocanu , M.O. Reade, On a Briot-Bouquet differential subordination, in: General Inequalities, vol. 3, International Series of Numerical Mathematics, 64 (1983) 339–348, Birkhäuser Verlag, Basel, 1983; see also Rev. Roumaine Math. Pures Appl. 29 (1984) 567–573.
[12] R.M. Goel, B.S. Mehrok, On the coefficients of a subclass of starlike functions, Indian J. Pure Appl. Math. 12 (1981) 634–647.
[13] Yu. E. Hohlov, Operators and operations in the class of univalent functions, Izv. Vysŝ. Uĉebn. Zaved. Matematika 197 (10) (1978) 83–89.
[14] W. Janowski, Some external problems for certain families of analytic functions, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys. 21 (1973) 17–25.
[15] J.A. Kim, K.H. Shon, Mapping properties for convolutions involving hypergeometric functions, Internat. J. Math. Math. Sci. 17 (2003) 1083–1091.
[16] J.-L. Liu, The Noor integral operator and strongly close-to-convex functions, J. Math. Anal. Appl. 261 (2001) 441–447.
[17] J.-L. Liu, H.M. Srivastava, A class of multivalently analytic functions associated with the Dziok–Srivastava operator, Integral Transforms Spec. Funct. 20 (2009) 401–417.
[18] S.S. Miller, P.T. Mocanu, Differential Subordination: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker Incorporated, New York and Basel, 2000.
[19] W. Ma, D. Minda, An internal geometric characterization of strongly starlike functions, Ann. Univ. Mariae Curie-Skłodowska. Sect. A 45 (1991) 89–97.
[20] S.S. Miller, P.T. Mocanu, Differential subordinations and inequalities in the complex plane, J. Differ. Equations 67 (1987) 199–211.
[21] K.I. Noor, Integral operators defined by convolution with hypergeometric functions, Appl. Math. Comput. 182 (2006) 1872–1881.
[22] K.I. Noor, On new classes of integral operators, J. Nat. Geom. 16 (1999) 71–80.
[23] K.I. Noor, M.A. Noor, On integral operators, J. Math. Anal. Appl. 283 (1999) 341–352.
[24] S. Owa, H.M. Srivastava, Some applications of the generalized Libera integral operator, Proc. Japan Acad. Ser. A Math. Sci. 62 (1986) 125–128.
[25] K.S. Padmanabhan, R. Parvatham, Some applications of differential subordination, Bull. Austral. Math. Soc. 32 (1985) 321–330.
[26] St. Ruscheweyh, T. Sheil-Small, Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture, Comment. Math. Helv. 48 (1973) 119–135.
[27] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975) 109–115.
[28] N. Shukla, P. Shukla, Mapping properties of analytic function defined by hypergeometric function. II, Soochow J. Math. 25 (1999) 29–36.
[29] H.M. Srivastava, S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.
[30] H.M. Srivastava, D.-G. Yang, N-E. Xu, Subordinations for multivalent analytic functions associated with the Dziok–Srivastava operator, Integral Transforms Spec. Funct. 20 (2009) 581–606.