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A Unified Presentation of Some Classes of Meromorphically Multivalent Functions

H. M. SRIVASTAVA
Department of Mathematics and Statistics, University of Victoria
Victoria, British Columbia V8W 3P4, Canada
hmsri@uvvm.uvic.ca

H. M. HOSSEN AND M. K. AOUF
Department of Mathematics, Faculty of Science
University of Mansoura, Mansoura, Egypt
simfac@mum.mans.eun.eg

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Abstract—The authors introduce and investigate various properties of a general class

$$\mathcal{U}_k[p,\alpha,\beta,A,B]$$
$$(p,k\in\mathbb{N}:=\{1,2,3,\dots,\};\ 0\le\alpha<p;\ \beta\ge 0;$$
$$-1\le A<B\le 1;\ 0<B\le 1),$$

which unifies and extends several (known or new) subclasses of meromorphically multivalent functions. The properties and characteristics of this general class, which are presented here, include growth and distortion theorems; they also involve Hadamard products (or convolution) of functions belonging to the class $\mathcal{U}_k[p,\alpha,\beta,A,B]$.

Keywords—Meromorphic functions, p-valent functions, Hadamard product (or convolution), Analytic functions, Growth and distortion theorems, Cauchy-Schwarz inequality.

1. INTRODUCTION

Let $\sum_{p,k}$ denote the class of functions of the form

$$f(z)=\frac{1}{z^p}+\sum_{n=k}^{\infty}a_{n+p-1}z^{n+p-1},\qquad (p,k\in\mathbb{N}:=\{1,2,3,\dots,\}),\tag{1.1}$$

which are analytic and p-valent in the *punctured* unit disk

$$\mathcal{U}^*=\{z:z\in\mathbb{C}\text{ and }0<|z|<1\}.$$

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Many interesting families of analytic and multivalent functions were considered by earlier authors in *Geometric Function Theory* (cf., e.g., [1–3]). For a function $f(z)$ in $\sum_{p,k}$, and for fixed parameters A and B, with

$$-1 \leq A < B \leq 1, \qquad A + B \geq 0, \qquad \text{and} \qquad 0 < B \leq 1.$$

We say that $f(z)$ is a member of the class $\mathcal{Q}_k[p, \alpha, A, B]$ if and only if it satisfies the inequality

$$\left| \frac{(zf'(z)/f(z)) + p}{B(zf'(z)/f(z)) + [pB + (A - B)(p - \alpha)]} \right| < 1, \qquad (z \in \mathcal{U}^*;\ 0 \leq \alpha < p). \tag{1.2}$$

A function $f(z) \in \sum_{p,k}$ is said to belong to the class $\mathcal{R}_k[p, \alpha, A, B]$ if and only if

$$-\frac{zf'(z)}{p} \in \mathcal{Q}_k[p, \alpha, A, B]. \tag{1.3}$$

The classes $\mathcal{Q}_1[p, \alpha, A, B]$ and $\mathcal{Q}_1[p, 0, A, B]$ were introduced by Aouf [4] and Mogra [5], respectively. Some subclasses of $\sum_{p,k}$ when $k = p = 1$ were considered by (for example) Miller [6], Pommerenke [7], Clunie [8], and Royster [9]. Furthermore, several subclasses of $\sum_{p,k}$ when $k = 1$ were studied by (amongst others) Mogra [5,10], Aouf [4,11], and Uralegaddi and Ganigi [12].

Motivated essentially by many of these earlier works, we aim at investigating here various properties and characteristics of the above defined general class

$$\mathcal{U}_k[p, \alpha, \beta, A, B],$$
$$(p, k \in \mathbb{N};\ 0 \leq \alpha < p;\ \beta \geq 0;\ -1 \leq A < B \leq 1;\ 0 < B \leq 1)$$

of meromorphically p-valent functions in

$$\mathcal{U} := \mathcal{U}^* \cup \{0\} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

The following result can be proven fairly easily by appealing to the definition of the class $\mathcal{Q}_k[p, \alpha, A, B]$.

LEMMA 1. *Let a function $f(z)$ defined by (1.1) be in the class $\sum_{p,k}$. If*

$$\sum_{n=k}^{\infty} C(p, \alpha, A, B; n)\, |a_{n+p-1}| \leq D(p, \alpha, A, B) \tag{1.4}$$

$$(0 \leq \alpha < p;\ -1 \leq A < B \leq 1;\ 0 < B \leq 1),$$

where, for convenience,

$$C(p, \alpha, A, B; n) = (1 + B)(n - 1) + [2p + 2\alpha B + (B + A)(p - \alpha)], \qquad (n \geq k) \tag{1.5}$$

and

$$D(p, \alpha, A, B) = (B - A)(p - \alpha), \tag{1.6}$$

then $f(z) \in \mathcal{Q}_k[p, \alpha, A, B]$.

Next, by observing that

$$f(z) \in \mathcal{R}_k[p, \alpha, A, B] \iff -\frac{zf'(z)}{p} \in \mathcal{Q}_k[p, \alpha, A, B], \tag{1.7}$$

we arrive at Lemma 2.

LEMMA 2. *Let a function $f(z)$ defined by (1.1) be in the class $\sum_{p,k}$. If*

$$\sum_{n=k}^{\infty}\left(\frac{n+p-1}{p}\right)C(p,\alpha,A,B;n)\left|a_{n+p-1}\right| \leq D(p,\alpha,A,B) \tag{1.8}$$

$$(0 \leq \alpha < p;\ -1 \leq A < B \leq 1;\ 0 < B \leq 1),$$

where $C(p,\alpha,A,B,n)$ and $D(p,\alpha,A,B)$ are given by (1.5) and (1.6), respectively, then $f(z) \in \mathcal{R}_k[p,\alpha,A,B]$.

In view of Lemma 1 and Lemma 2, we define the subclasses $\mathcal{Q}_k^*[p,\alpha,A,B]$ of $\mathcal{Q}_k[p,\alpha,A,B]$ and $\mathcal{R}_k^*[p,\alpha,A,B]$ of $\mathcal{R}_k[p,\alpha,A,B]$ consisting of functions which, respectively, satisfy (1.4) and (1.8). Furthermore, we introduce and investigate the various properties and characteristics of the following general class $\mathcal{U}_k[p,\alpha,\beta,A,B]$ of functions $f(z) \in \sum_{p,k}$ which also satisfy the inequality

$$\sum_{n=k}^{\infty} C\left(p,\alpha,A,B;n\right)\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]\left|a_{n+p-1}\right| \leq D(p,\alpha,A,B), \tag{1.9}$$

$$(0 \leq \alpha < p;\ \beta \geq 0;\ -1 \leq A < B \leq 1;\ 0 < B \leq 1),$$

where $C(p,\alpha,A,B,n)$ and $D(p,\alpha,A,B)$ are given by (1.5) and (1.6), respectively. Clearly, we have

$$\mathcal{U}_k[p,\alpha,\beta,A,B] = (1-\beta)\mathcal{Q}_k^*[p,\alpha,A,B] + \beta\mathcal{R}_k^*[p,\alpha,A,B], \tag{1.10}$$

so that

$$\mathcal{U}_k[p,\alpha,0,A,B] = \mathcal{Q}_k^*[p,\alpha,A,B] \tag{1.11}$$

and

$$\mathcal{U}_k[p,\alpha,1,A,B] = \mathcal{R}_k^*[p,\alpha,A,B]. \tag{1.12}$$

2. GROWTH AND DISTORTION THEOREMS

THEOREM 1. *If a function $f(z)$ defined by (1.1) is in the class $\mathcal{U}_k[p,\alpha,\beta,A,B]$, then*

$$\begin{aligned}\frac{1}{|z|^p} - \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left((k+p-1)/p\right)\right]}|z|^{k+p-1} &\leq |f(z)| \\ \leq \frac{1}{|z|^p} + \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left((k+p-1)/p\right)\right]}|z|^{k+p-1}&,\end{aligned} \tag{2.1}$$

$$(\beta \geq 0;\ z \in \mathcal{U}^*)$$

and

$$\begin{aligned}\frac{p}{|z|^{p+1}} - \frac{(k+p-1)D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left((k+p-1)/p\right)\right]}|z|^{k+p-2} &\leq |f'(z)| \\ \leq \frac{p}{|z|^{p+1}} + \frac{(k+p-1)D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left((k+p-1)/p\right)\right]}|z|^{k+p-2}&,\end{aligned} \tag{2.2}$$

$$(\beta \geq 0;\ z \in \mathcal{U}^*).$$

The bounds in (2.1) and (2.2) are attained for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left((k+p-1)/p\right)\right]}z^{k+p-1}. \tag{2.3}$$

PROOF. Noting that

$$\sum_{n=k}^{\infty}\left|a_{n+p-1}\right| \leq \frac{D\left(p,\alpha,A,B\right)}{C\left(p,\alpha,A,B;k\right)\left[1-\beta+\beta\left((k+p-1)/p\right)\right]} \tag{2.4}$$

for $f(z) \in \mathcal{U}_k[p,\alpha,\beta,A,B]$, we have

$$\begin{aligned} |f(z)| &\geq \frac{1}{|z|^p} - |z|^{k+p-1} \sum_{n=k}^{\infty} |a_{n+p-1}| \\ &\geq \frac{1}{|z|^p} - \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\,[1-\beta+\beta\,((k+p-1)/p)]} |z|^{k+p-1}, \\ &\qquad (\beta \geq 0; z \in \mathcal{U}^*) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} |f(z)| &\leq \frac{1}{|z|^p} + |z|^{k+p-1} \sum_{n=k}^{\infty} |a_{n+p-1}| \\ &\leq \frac{1}{|z|^p} + \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\,[1-\beta+\beta\,((k+p-1)/p)]} |z|^{k+p-1}, \\ &\qquad (\beta \geq 0;\ z \in \mathcal{U}^*). \end{aligned} \tag{2.6}$$

We also observe that

$$\begin{aligned} &\frac{C(p,\alpha,A,B;k)\,[1-\beta+\beta\,((k+p-1)/p)]}{k+p-1} \sum_{n=k}^{\infty} (n+p-1)\,|a_{n+p-1}| \\ &\leq \sum_{n=k}^{\infty} C(p,\alpha,A,B;n) \left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right] |a_{n+p-1}| \leq D(p,\alpha,A,B), \qquad (\beta \geq 0), \end{aligned} \tag{2.7}$$

which readily yields the following distortion inequalities:

$$\begin{aligned} |f'(z)| &\geq \frac{p}{|z|^{p+1}} - |z|^{k+p-2} \sum_{n=k}^{\infty} (n+p-1)\,|a_{n+p-1}| \\ &\geq \frac{p}{|z|^{p+1}} - \frac{(k+p-1)D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\,[1-\beta+\beta\,((k+p-1)/p)]} |z|^{k+p-2}, \\ &\qquad (\beta \geq 0;\ z \in \mathcal{U}^*) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} |f'(z)| &\leq \frac{p}{|z|^{p+1}} + |z|^{k+p-2} \sum_{n=k}^{\infty} (n+p-1)\,|a_{n+p-1}| \\ &\leq \frac{p}{|z|^{p+1}} + \frac{(k+p-1)D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\,[1-\beta+\beta\,((k+p-1)/p)]} |z|^{k+p-2}, \\ &\qquad (\beta \geq 0;\ z \in \mathcal{U}^*). \end{aligned} \tag{2.9}$$

Now it is easy to see that the bounds in (2.1) and (2.2) are attained for the function $f(z)$ given by (2.3).

Taking $\beta = 0$ in Theorem 1, we have the following.

COROLLARY 1. *If a function $f(z)$ defined by (1.1) is in the class $\mathcal{Q}_k^*[p,\alpha,A,B]$, then*

$$\begin{aligned} &\frac{1}{|z|^p} - \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)} |z|^{k+p-1} \leq |f(z)| \\ &\leq \frac{1}{|z|^p} + \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)} |z|^{k+p-1}, \qquad (z \in \mathcal{U}^*) \end{aligned} \tag{2.10}$$

and

$$\frac{p}{|z|^{p+1}} - \frac{(k+p-1)D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}|z|^{k+p-2} \le |f'(z)|$$
$$\le \frac{p}{|z|^{p+1}} + \frac{(k+p-1)D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}|z|^{k+p-2}, \qquad (z \in \mathcal{U}^*). \tag{2.11}$$

The bounds in (2.10) and (2.11) are attained for the function

$$f(z) = \frac{1}{z^p} + \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)} z^{k+p-1}. \tag{2.12}$$

Letting $\beta = 1$ in Theorem 1, we have the following.

COROLLARY 2. *If a function $f(z)$ defined by (1.1) is in the class $\mathcal{R}_k^*[p,\alpha,A,B]$, then*

$$\frac{1}{|z|^p} - \frac{pD(p,\alpha,A,B)}{(k+p-1)C(p,\alpha,A,B;k)}|z|^{k+p-1} \le |f(z)|$$
$$\le \frac{1}{|z|^p} + \frac{pD(p,\alpha,A,B)}{(k+p-1)C(p,\alpha,A,B;k)}|z|^{k+p-1}, \qquad (z \in \mathcal{U}^*) \tag{2.13}$$

and

$$\frac{p}{|z|^{p+1}} - \frac{pD(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}|z|^{k+p-2} \le |f'(z)|$$
$$\le \frac{p}{|z|^{p+1}} + \frac{pD(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}|z|^{k+p-2}, \qquad (z \in \mathcal{U}^*). \tag{2.14}$$

The bounds in (2.13) and (2.14) are attained for the function

$$f(z) = \frac{1}{z^p} + \frac{pD(p,\alpha,A,B)}{(k+p-1)C(p,\alpha,A,B;k)} z^{k+p-1}. \tag{2.15}$$

3. CONVOLUTION PROPERTIES

For functions

$$f_j(z) = \frac{1}{z^p} + \sum_{n=k}^{\infty} a_{n+p-1,j}\, z^{n+p-1}, \qquad (j = 1,2) \tag{3.1}$$

belonging to the class $\sum_{p,k}$, we denote by $(f_1 * f_2)(z)$ the convolution (or Hadamard product) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) := \frac{1}{z^p} + \sum_{n=k}^{\infty} a_{n+p-1,1}\, a_{n+p-1,2}\, z^{n+p-1}. \tag{3.2}$$

THEOREM 2. *Let the functions $f_j(z)$ $(j = 1,2)$ defined by (3.1) be in the class $\mathcal{U}_k[p,\alpha,\beta,A,B]$. Then*

$$(f_1 * f_2)(z) \in \mathcal{U}_k[p,\gamma,\beta,A,B],$$

where

$$\gamma = p - \frac{(B-A)(1+B)(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2 [1-\beta+\beta((k+p-1)/p)] + \{D(p,\alpha,A,B)\}^2}. \tag{3.3}$$

The result is sharp for the functions

$$f_j(z) = \frac{1}{z^p} + \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)[1-\beta+\beta((k+p-1)/p)]} z^{k+p-1}, \qquad (j = 1,2). \tag{3.4}$$

PROOF. In order to prove Theorem 2, we must find the largest γ such that

$$\sum_{n=k}^{\infty} \frac{C(p,\gamma,A,B;n)\,[1-\beta+\beta\,((n+p-1)/p)]}{D(p,\gamma,A,B)} |a_{n+p-1,1}|\,|a_{n+p-1,2}| \le 1 \tag{3.5}$$

for $f_j(z) \in \mathcal{U}_k[p,\gamma,\beta,A,B]\,(j=1,2)$. Since $f_j(z) \in \mathcal{U}_k[p,\alpha,\beta,A,B]\,(j=1,2)$, we readily see that

$$\sum_{n=k}^{\infty} \frac{C(p,\alpha,A,B;n)\,[1-\beta+\beta\,((n+p-1)/p)]}{D(p,\alpha,A,B)} |a_{n+p-1,j}| \le 1, \qquad (j=1,2). \tag{3.6}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=k}^{\infty} \frac{C(p,\alpha,A,B;n)\,[1-\beta+\beta\,((n+p-1)/p)]}{D(p,\alpha,A,B)} \sqrt{|a_{n+p-1,1}|\,|a_{n+p-1,2}|} \le 1. \tag{3.7}$$

This implies that we need only show that

$$\begin{aligned} &\frac{C(p,\gamma,A,B;n)}{p-\gamma} |a_{n+p-1,1}|\,|a_{n+p-1,2}| \\ &\le \frac{C(p,\alpha,A,B;n)}{p-\alpha} \sqrt{|a_{n+p-1,1}|\,|a_{n+p-1,2}|}, \qquad (n \ge k) \end{aligned} \tag{3.8}$$

or equivalently, that

$$\sqrt{|a_{n+p-1,1}|\,|a_{n+p-1,2}|} \le \frac{(p-\gamma)C(p,\alpha,A,B;n)}{(p-\alpha)C(p,\gamma,A,B;n)}, \qquad (n \ge k). \tag{3.9}$$

Hence, by the inequality (3.7), it is sufficient to prove that

$$\frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;n)\,[1-\beta+\beta\,((n+p-1)/p)]} \le \frac{(p-\gamma)C(p,\alpha,A,B;n)}{(p-\alpha)C(p,\gamma,A,B;n)}, \qquad (n \ge k). \tag{3.10}$$

It follows from (3.10) that

$$\gamma \le p - \frac{(B-A)\,(1+B)\,(n+p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;n)\}^2\,[1-\beta+\beta\,((n+p-1)/p)] + \{D(p,\alpha,A,B)\}^2}, \qquad (n \ge k). \tag{3.11}$$

Now, defining the function $\varphi(n)$ by

$$\varphi(n) := p - \frac{(B-A)\,(1+B)\,(n+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;n)\}^2\,[1-\beta+\beta\,((n+p-1)/p)] + \{D(p,\alpha,A,B)\}^2}, \qquad (n \ge k). \tag{3.12}$$

We see that $\varphi(n)$ is an increasing function of n. Therefore, we conclude that

$$\gamma \le \varphi(k) = p - \frac{(B-A)\,(1+B)\,(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2\,[1-\beta+\beta\,((k+p-1)/p)] + \{D(p,\alpha,A,B)\}^2}, \tag{3.13}$$

which evidently completes the proof of Theorem 2.

Letting $\beta = 0$ in Theorem 2, we arrive at Corollary 3.

COROLLARY 3. *Let the functions $f_j(z)\,(j=1,2)$ defined by (3.1) be in the class $\mathcal{Q}_k^*[p,\alpha,A,B]$. Then*

$$(f_1 * f_2)(z) \in \mathcal{Q}_k^*[p,\gamma,A,B],$$

where

$$\gamma = p - \frac{(B-A)\,(1+B)\,(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2 + \{D(p,\alpha,A,B)\}^2}. \tag{3.14}$$

The result is sharp for the functions

$$f_j(z) = \frac{1}{z^p} + \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)} z^{k+p-1}, \qquad (j=1,2). \tag{3.15}$$

Putting $\beta = 1$ in Theorem 2, we have Corollary 4.

COROLLARY 4. *Let the functions $f_j(z)\,(j=1,2)$ defined by (3.1) be in the class $\mathcal{R}_k^*[p,\alpha,A,B]$. Then*

$$(f_1 * f_2)(z) \in \mathcal{R}_k^*[p,\gamma,A,B],$$

where

$$\gamma = p - \frac{p\,(B-A)\,(1+B)\,(k+2p-1)(p-\alpha)^2}{(k+p-1)\,\{C(p,\alpha,A,B;k)\}^2 + p\,\{D(p,\alpha,A,B)\}^2}. \tag{3.16}$$

The result is sharp for the functions

$$f_j(z) = \frac{1}{z^p} + \frac{pD(p,\alpha,A,B)}{(k+p-1)C(p,\alpha,A,B;k)} z^{k+p-1}, \qquad (j=1,2). \tag{3.17}$$

Finally, we prove the following.

THEOREM 3. *Let the functions $f_j(z)\,(j=1,2)$ defined by (3.1) be in the class $\mathcal{U}_k[p,\alpha,\beta,A,B]$. Then the function $h(z)$ defined by*

$$h(z) := \frac{1}{z^p} + \sum_{n=k}^{\infty} \left(a_{n+p-1,1}^2 + a_{n+p-1,2}^2\right) z^{n+p-1} \tag{3.18}$$

belongs to the class $\mathcal{U}_k[p,\gamma,\beta,A,B]$, where

$$\gamma = p - \frac{2\,(B-A)\,(1+B)\,(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2\,[1-\beta+\beta\,((k+p-1)/p)] + 2\,\{D(p,\alpha,A,B)\}^2}. \tag{3.19}$$

The result is sharp for the functions $f_j(z)\,(j=1,2)$ given by (3.4).

PROOF. Noting that

$$\begin{aligned} &\sum_{n=k}^{\infty} \frac{\{C(p,\alpha,A,B;n)\}^2\,[1-\beta+\beta\,(n+p-1/p)]^2}{\{D(p,\alpha,A,B)\}^2} |a_{n+p-1,j}|^2 \\ &\le \left(\sum_{n=k}^{\infty} \frac{C(p,\alpha,A,B;n)\,[1-\beta+\beta\,((n+p-1)/p)]}{D(p,\alpha,A,B)} |a_{n+p-1,j}|\right)^2 \le 1, \qquad (j=1,2) \end{aligned} \tag{3.20}$$

for $f_j(z) \in \mathcal{U}_k\,[p,\alpha,\beta,A,B]$ $(j=1,2)$, we have

$$\sum_{n=k}^{\infty} \frac{\{C(p,\alpha,A,B;n)\}^2\,[1-\beta+\beta\,((n+p-1)/p)]^2}{2\,\{D(p,\alpha,A,B)\}^2} \left|a_{n+p-1,1}^2 + a_{n+p-1,2}^2\right| \le 1. \tag{3.21}$$

Therefore, we have to find the largest γ such that

$$\frac{C\,(p,\gamma,A,B;n)}{p-\gamma} \le \frac{\{C(p,\alpha,A,B;n)\}^2\,[1-\beta+\beta\,((n+p-1)/p)]}{2\,(B-A)\,(p-\alpha)^2}, \qquad (n \ge k), \tag{3.22}$$

that is, that

$$\gamma \le p - \frac{2\,(B-A)\,(1+B)\,(n+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;n)\}^2\,[1-\beta+\beta\,((n+p-1)/p)] + 2\,\{D(p,\alpha,A,B)\}^2}, \qquad (n \ge k). \tag{3.23}$$

Now, defining a function $\psi(n)$ by

$$\psi(n) := p - \frac{2\,(B-A)\,(1+B)\,(n+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;n)\}^2\,[1-\beta+\beta\,((n+p-1)/p)] + 2\,\{D(p,\alpha,A,B)\}^2}, \qquad (n \ge k). \tag{3.24}$$

We observe that $\psi(n)$ is an increasing function of n. Thus, we conclude that

$$\gamma \le \psi(k) = p - \frac{2\,(B-A)\,(1+B)\,(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2\,[1-\beta+\beta\,((k+p-1)/p)] + 2\,\{D(p,\alpha,A,B)\}^2}, \tag{3.25}$$

which completes the proof of Theorem 3.

By setting $\beta = 0$, Theorem 3 leads us to Corollary 5.

COROLLARY 5. *Let the functions $f_j(z)\,(j=1,2)$ defined by (3.1) be in the class $\mathcal{Q}_k^*[p,\alpha,A,B]$. Then the function $h(z)$ defined by (3.18) belongs to the class $\mathcal{Q}_k^*[p,\gamma,A,B]$, where*

$$\gamma = p - \frac{2\,(B-A)\,(1+B)\,(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2 + 2\,\{D(p,\alpha,A,B)\}^2}. \tag{3.26}$$

The result is sharp for the functions $f_j(z)\,(j=1,2)$ given by (3.15).

Letting $\beta = 1$ in Theorem 3, we have the following.

COROLLARY 6. *Let the functions $f_j(z)(j=1,2)$ defined by (3.1) be in the class $\mathcal{R}_k^*[p,\alpha,A,B]$. Then the function $h(z)$ defined by (3.18) belongs to the class $\mathcal{R}_k^*[p,\gamma,A,B]$, where*

$$\gamma = p - \frac{2p\,(B-A)\,(1+B)\,(k+2p-1)(p-\alpha)^2}{(k+p-1)\,\{C(p,\alpha,A,B;k)\}^2 + 2p\,\{D(p,\alpha,A,B)\}^2}. \tag{3.27}$$

The result is sharp for the functions $f_j(z)\,(j=1,2)$ given by (3.17).

Many of our results in this paper, (especially Corollaries 1 to 6) would simplify considerably when we set

$$A = -1 \qquad \text{and} \qquad B = 1.$$

The details involved in the derivation of these and other special cases of our results may be left as an exercise for the interested reader.

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