ELSEVIER Applied Mathematics and Computation 141 (2003) 13–49

APPLIED MATHEMATICS AND COMPUTATION

www.elsevier.com/locate/amc

Certain classes of series associated with the Zeta and related functions

H.M. Srivastava

Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada

Abstract

A fascinatingly large number of seemingly independent solutions of the so-called *Basler problem* of evaluating the Riemann Zeta function $\zeta(s)$ when $s = 2$, which was of vital importance to Leonhard Euler (1707–1783) and the Bernoulli brothers (Jakob Bernoulli (1654–1705) and Johann Bernoulli (1667–1748)), have appeared in the mathematical literature ever since Euler first solved this problem in the year 1736. The main object of this two-part series of lectures is to present some recent developments on the evaluations and representations of $\zeta(s)$ when $s \in \mathbb{N} \setminus \{1\}$, $\mathbb{N}$ being the set of natural numbers. We emphasize upon several interesting classes of rapidly convergent series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ which have been developed in recent years. In two of many computationally useful special cases considered here, it is observed that $\zeta(3)$ can be represented by means of series which converge much more rapidly than that in Euler's celebrated formula as well as the series used recently by Roger Apéry (1916–1994) in his proof of the irrationality of $\zeta(3)$. Symbolic and numerical computations using *Mathematica* (Version 4.0) for Linux show, among other things, that only 50 terms of one of these series are capable of producing an accuracy of seven decimal places.

Keywords: Riemann Zeta function; Hurwitz (or generalized) Zeta function; Harmonic numbers; Dirichlet's *L*-functions; Series representations; Bernoulli numbers; Bernoulli polynomials; Generating functions; Euler numbers; Euler polynomials; Mellin transformation; Holomorphic function; Symbolic and numerical computations

E-mail address: harimsri@math.uvic.ca (H.M. Srivastava).

0096-3003/02/$ - see front matter © 2002 Elsevier Science Inc. All rights reserved.
doi:10.1016/S0096-3003(02)00318-1

1. Introduction, definitions, and motivation

The first part of this two-part series of lectures deals *chiefly* with the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s,a)$, which are defined (for $\Re(s) > 1$) by

$$\zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1) \end{cases} \tag{1.1}$$

and

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$
$$\left(\Re(s) > 1; \quad a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\right), \tag{1.2}$$

and (for $\Re(s) \leqq 1$; $s \neq 1$) by their meromorphic continuations (see, for details, [35]; see also [15] and [29]), so that (obviously)

$$\zeta(s,1) = \zeta(s) = (2^s - 1)^{-1} \quad \zeta\left(s, \frac{1}{2}\right) \quad \text{and} \quad \zeta(s,2) = \zeta(s) - 1. \tag{1.3}$$

More generally, we have the following relationships:

$$\zeta(s) = \frac{1}{m^s - 1} \sum_{j=1}^{m-1} \zeta\left(s, \frac{j}{m}\right) \quad (m \in \mathbb{N} \setminus \{1\}; \mathbb{N} := \{1,2,3,\ldots\}), \tag{1.4}$$

and

$$\zeta(s, ma) = \frac{1}{m^s} \sum_{j=0}^{m-1} \zeta\left(s, a + \frac{j}{m}\right) \quad (m \in \mathbb{N}). \tag{1.5}$$

We begin by considering the set $\mathscr{S}$ of all *non-trivial* integer kth powers, that is,

$$\mathscr{S} := \left\{n^k \colon n, k \in \mathbb{N} \setminus \{1\}\right\} = \{4, 8, 9, 16, 25, 27, 32, 36, \ldots\}. \tag{1.6}$$

An over two centuries old theorem of Christian Goldbach (1690–1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700–1782), was revived, not too long ago, as the following problem (see [22] and [24]):

$$\sum_{\omega \in \mathscr{S}} (\omega - 1)^{-1} = 1, \tag{1.7}$$

where the sum is extended over *all* elements ω of the set $\mathscr{S}$ (see also [1, p. 131]).

In terms of the Riemann Zeta function $\zeta(s)$ defined by (1.1), Goldbach's theorem (1.7) can easily be restated as

$$\sum_{k=2}^{\infty}\{\zeta(k)-1\}=1 \tag{1.8}$$

or, equivalently, as

$$\sum_{k=2}^{\infty}\mathscr{F}(\zeta(k))=1, \tag{1.9}$$

where, for convenience, $\mathscr{F}(x):=x-[x]$ denotes the *fractional* part of $x\in\mathbb{R}$. In fact, it is fairly easy to show also that

$$\sum_{k=2}^{\infty}(-1)^k\mathscr{F}(\zeta(k))=\frac{1}{2}, \tag{1.10}$$

$$\sum_{k=1}^{\infty}\mathscr{F}(\zeta(2k))=\frac{3}{4}, \tag{1.11}$$

and

$$\sum_{k=1}^{\infty}\mathscr{F}(\zeta(2k+1))=\frac{1}{4}. \tag{1.12}$$

Another remarkable *classical* result involving Riemann's ζ-function is the following elegant series representation for $\zeta(3)$:

$$\zeta(3)=-\frac{4\pi^2}{7}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}, \tag{1.13}$$

which was actually contained in a 1772 paper, entitled "*Exercitationes Analyticae*", by Leonhard Euler (1707–1783) (cf., e.g., [3, pp. 1084–1085]). In fact, this result of Euler was rediscovered (among others) by Ramaswami [21] (see also [23, p. 7, Eq. (2.23)]) and (more recently) by Ewell [9]. And, as pointed out by (for example) Chen and Srivastava [5, pp. 180–181], another series representation:

$$\zeta(3)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}, \tag{1.14}$$

which played a key rôle in the *celebrated* proof [2] of the irrationality of $\zeta(3)$ by Roger Apéry (1916–1994), was derived *independently* by (among others) Hjortnaes [14], Gosper [12], and Apéry [2].

Clearly, Euler's series in (1.13) converges faster than the defining series for $\zeta(3)$, but obviously not as fast as the series in (1.14). Such Zeta values as $\zeta(3)$, $\zeta(5)$, etc., are known to arise naturally in a wide variety of applications such as those in Elastostatics, Quantum Field Theory, etc., (see, for example, [19,20,36,41]). On the other hand, in the case of *even* integer arguments, we already have the following computationally useful relationship:

$$\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \tag{1.15}$$

with the *well-tabulated* Bernoulli numbers defined by the generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi), \tag{1.16}$$

so that

$$\begin{aligned}
&B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30},\\
&B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \quad B_{18} = \frac{43\,867}{798},\\
&B_{20} = -\frac{174\,611}{330}, \quad B_{22} = \frac{854\,513}{138}, \quad B_{24} = -\frac{236\,364\,091}{2730},\\
&B_{26} = \frac{8\,553\,103}{6}, \ldots, \quad B_{2n+1} = 0 \quad (n \in \mathbb{N}),
\end{aligned} \tag{1.17}$$

as well as the familiar recursion formula:

$$\zeta(2n) = \left(n + \frac{1}{2}\right)^{-1} \sum_{k=1}^{n-1} \zeta(2k)\zeta(2n-2k) \quad (n \in \mathbb{N} \setminus \{1\}). \tag{1.18}$$

Thus there is a need (as well as motivation) for expressing $\zeta(2n+1)$ as a rapidly converging series for all $n \in \mathbb{N}$. With this objective in view, we propose to present here a rather systematic investigation of the various interesting families of rapidly convergent series representations for the Riemann $\zeta(2n+1)$ $(n \in \mathbb{N})$. We also consider relevant connections of the results presented here with many other known series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$. In two of many computationally useful special cases considered here, it is observed that $\zeta(3)$ can be represented by means of series which converge much more rapidly than that in Euler's celebrated formula (1.13) as well as the series (1.14) used recently by Apéry [2] in his proof of the irrationality of $\zeta(3)$. Symbolic and numerical computations using *Mathematica* (Version 4.0) for Linux show, among other things, that only 50 terms of one of these series are capable of producing an accuracy of seven decimal places.

2. First group of series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$)

The various series identities considered in the preceding section, including (for example) Goldbach's theorem (1.7), are known to be derivable also from the following simple consequence of the binomial theorem *and* the definition (1.1):

$$\sum_{k=0}^{\infty} \frac{(s)_k}{k!} \zeta(s+k,a) t^k = \zeta(s,a-t) \quad (|t| < |a|), \tag{2.1}$$

which, for $a = 1$ and $t = \pm 1/m$, readily yields the series identity:

$$\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{m^{2k}} = \begin{cases} (2^s - 1)\zeta(s) - 2^{s-1} & (m = 2) \\ \frac{1}{2}\left[(m^s - 1)\zeta(s) - m^s - \sum_{j=2}^{m-2} \zeta(s, j/m)\right] & (m \in \mathbb{N} \setminus \{1,2\}), \end{cases} \tag{2.2}$$

$(\lambda)_n := \Gamma(\lambda+n)/\Gamma(\lambda)$ being the Pochhammer symbol (or the *shifted* factorial, since $(1)_n = n!$).

In terms of the familiar harmonic numbers

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \quad (n \in \mathbb{N}), \tag{2.3}$$

the following set of series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$) were proven recently by Srivastava [27] by appealing appropriately to the series identity (2.2) in its special cases when $m = 2$, 3, 4, and 6, and also to many other properties and characteristics of the Riemann Zeta function such as the familiar functional equation:

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s)\zeta(1-s) \tag{2.4}$$

or, equivalently,

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{1}{2}\pi s\right) \Gamma(s)\zeta(s), \tag{2.5}$$

the derivative formula:

$$\zeta'(-2n) = \lim_{\varepsilon \to 0} \frac{\zeta(-2n+\varepsilon)}{\varepsilon} = \frac{(-1)^n}{2(2\pi)^{2n}} (2n)!\, \zeta(2n+1) \quad (n \in \mathbb{N}) \tag{2.6}$$

with, of course,

$$\zeta(0) = -\frac{1}{2}; \qquad \zeta(-2n) = 0 \quad (n \in \mathbb{N}); \qquad \zeta'(0) = -\frac{1}{2}\log(2\pi), \tag{2.7}$$

and each of the following limit relationships:

$$\lim_{s\to-2n}\left\{\frac{\sin\left(\frac{1}{2}\pi s\right)}{s+2n}\right\}=(-1)^n\frac{\pi}{2}\quad(n\in\mathbb{N}) \tag{2.8}$$

and

$$\lim_{s\to-2n}\left\{\frac{\zeta(s+2k)}{s+2n}\right\}=\frac{(-1)^{n-k}}{2(2\pi)^{2(n-k)}}(2n-2k)!\,\zeta(2n-2k+1)$$
$$(k=1,\ldots,n-1;\ n\in\mathbb{N}\setminus\{1\}). \tag{2.9}$$

$$\zeta(2n+1)=(-1)^{n-1}\frac{(2\pi)^{2n}}{2^{2n+1}-1}\left[\frac{H_{2n}-\log\pi}{(2n)!}+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\frac{\zeta(2k+1)}{\pi^{2k}}\right.$$
$$\left.+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\frac{\zeta(2k)}{2^{2k}}\right]\quad(n\in\mathbb{N}); \tag{2.10}$$

$$\zeta(2n+1)=(-1)^{n-1}\frac{2(2\pi)^{2n}}{3^{2n+1}-1}\left[\frac{H_{2n}-\log\left(\frac{2}{3}\pi\right)}{(2n)!}+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}}\right.$$
$$\left.+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\frac{\zeta(2k)}{3^{2k}}\right]\quad(n\in\mathbb{N}); \tag{2.11}$$

$$\zeta(2n+1)=(-1)^{n-1}\frac{2(2\pi)^{2n}}{2^{4n+1}+2^{2n}-1}\left[\frac{H_{2n}-\log\left(\frac{1}{2}\pi\right)}{(2n)!}\right.$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\frac{\zeta(2k)}{4^{2k}}\right]\quad(n\in\mathbb{N}); \tag{2.12}$$

$$\zeta(2n+1)=(-1)^{n-1}\frac{2(2\pi)^{2n}}{3^{2n}(2^{2n}+1)+2^{2n}-1}\left[\frac{H_{2n}-\log\left(\frac{1}{3}\pi\right)}{(2n)!}\right.$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\frac{\zeta(2k+1)}{\left(\frac{1}{3}\pi\right)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\frac{\zeta(2k)}{6^{2k}}\right]$$
$$(n\in\mathbb{N}). \tag{2.13}$$

Here (*and elsewhere in this work*) an empty sum is to be interpreted (as usual) to be zero.

We choose to recall here the proof of (2.10) detailed by Srivastava [27]. Each of the other results (2.11) to (2.13) can be proven *mutatis mutandis*.

First of all, upon separating the first $n+1$ terms of the series occurring on the left-hand side of the case $m=2$ of the general result (2.2), if we transpose the terms for $k=0$ and $k=n$ to the right-hand side, we readily obtain the identity:

$$\sum_{k=1}^{n-1}\frac{(s)_{2k}}{(2k)!}2^{2(n-k)}\zeta(s+2k)+\sum_{k=1}^{\infty}\frac{(s)_{2n+2k}}{(2n+2k)!}\,\frac{\zeta(s+2n+2k)}{2^{2k}}$$
$$=2^{2n}(2^s-2)\zeta(s)-2^{s+2n-1}-\frac{(s)_{2n}}{(2n)!}\zeta(s+2n)\quad(n\in\mathbb{N}),\tag{2.14}$$

it being understood, as mentioned before, that an empty sum is to be interpreted as zero.

Now we apply the functional equation (2.4) in the first term on the right-hand side of (2.14) and divide both sides by $s+2n$. We thus find that

$$\sum_{k=1}^{n-1}\frac{(s)_{2k}}{(2k)!}2^{2(n-k)}\left\{\frac{\zeta(s+2k)}{s+2n}\right\}+\sum_{k=1}^{\infty}\frac{(s)_{2n}(s+2n+1)_{2k-1}}{(2n+2k)!}\,\frac{\zeta(s+2n+2k)}{2^{2k}}$$
$$=2^{s+2n}(2^s-2)\pi^{s-1}\Gamma(1-s)\zeta(1-s)\left\{\frac{\sin\left(\frac{1}{2}\pi s\right)}{s+2n}\right\}$$
$$-\left\{\frac{2^{s+2n-1}+\frac{(s)_{2n}}{(2n)!}\zeta(s+2n)}{s+2n}\right\}\quad(s\neq-2n;\ n\in\mathbb{N}).\tag{2.15}$$

Since

$$(-n)_k=(-1)^k\frac{n!}{(n-k)!}\quad(k=0,1,\ldots,n;\ n\in\mathbb{N}),$$

so that, obviously,

$$(-n)_n=(-1)^n\,n!\quad(n\in\mathbb{N}),\tag{2.16}$$

it is easily seen by *logarithmic differentiation* that

$$\frac{\mathrm{d}}{\mathrm{d}s}\{(s)_n\}=(s)_n\sum_{j=0}^{n-1}\frac{1}{s+j}\quad(n\in\mathbb{N}),\tag{2.17}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}s}\{(s)_{2n}\}\bigg|_{s=-2n}=-(2n)!\,H_{2n}\quad(n\in\mathbb{N}),\tag{2.18}$$

where H_n denotes the harmonic numbers defined by (2.3). We observe also that the limit formula (2.9) is needed in the first sum on the left-hand side of (2.15) only when this sum is non-zero (that is, only when $n\in\mathbb{N}\setminus\{1\}$). Furthermore, by l'Hôpital's rule once again, we have

$$\lim_{s\to -2n}\left\{\frac{2^{s+2n-1}+\frac{(s)_{2n}}{(2n)!}\zeta(s+2n)}{s+2n}\right\}$$
$$=\left[2^{s+2n-1}\log 2+\frac{\mathrm{d}}{\mathrm{d}s}\left\{(s)_{2n}\right\}\frac{\zeta(s+2n)}{(2n)!}+\frac{(s)_{2n}}{(2n)!}\zeta'(s+2n)\right]\Bigg|_{s=-2n}$$
$$=\frac{1}{2}(H_{2n}-\log\pi)\quad (n\in\mathbb{N}). \tag{2.19}$$

Finally, letting $s\to -2n$ in (2.15), and making use of the limit relationships (2.9) and (2.19), we obtain the *first* series representation for $\zeta(2n+1)$ $(n\in\mathbb{N})$ asserted by (2.10).

The series representation (2.10) is markedly different from each of the series representations for $\zeta(2n+1)$, which were given earlier by Zhang and Williams [42, p. 1590, Eq. (3.13)] and (subsequently) by Cvijović and Klinowski [6, p. 1265, Theorem A]. Since $\zeta(2k)\to 1$ as $k\to\infty$, the general term in the series representation (2.10) has the *order estimate*:

$$\mathrm{O}\left(2^{-2k}\, k^{-2n-1}\right)\quad (k\to\infty;\ n\in\mathbb{N}),$$

whereas the general term in each of these *earlier* series representations has the order estimate:

$$\mathrm{O}\left(2^{-2k}\, k^{-2n}\right)\quad (k\to\infty;\ n\in\mathbb{N}).$$

By suitably combining (2.10) and (2.12), it is fairly straightforward to obtain the series representation:

$$\zeta(2n+1)=(-1)^{n-1}\frac{2(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)}\left[\frac{\log 2}{(2n)!}+\sum_{k=1}^{n-1}\frac{(-1)^k(2^{2k}-1)}{(2n-2k)!}\frac{\zeta(2k+1)}{\pi^{2k}}-2\sum_{k=1}^{\infty}\frac{(2k-1)!(2^{2k}-1)}{(2n+2k)!}\frac{\zeta(2k)}{2^{4k}}\right]\quad (n\in\mathbb{N}). \tag{2.20}$$

Now, in terms of the Bernoulli numbers B_n and the Euler polynomials $E_n(x)$ defined by the generating functions (1.16) and

$$\frac{2\mathrm{e}^{xz}}{\mathrm{e}^{z}+1}=\sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!}\quad (|z|<\pi), \tag{2.21}$$

respectively, it is known that (cf., e.g., [18, p. 29])

$$E_n(0)=(-1)^n\, E_n(1)=\frac{2(1-2^{n+1})}{n+1}B_{n+1}\quad (n\in\mathbb{N}), \tag{2.22}$$

which, together with the identity (1.15), implies that

$$E_{2n-1}(0) = \frac{4(-1)^n}{(2\pi)^{2n}}(2n-1)!\left(2^{2n}-1\right)\zeta(2n) \quad (n \in \mathbb{N}). \tag{2.23}$$

By appealing to the relationship (2.23), the series representation (2.20) can immediately be put in the form:

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)}\Bigg[&\frac{\log 2}{(2n)!} \\ &+ \sum_{k=1}^{n-1}\frac{(-1)^k(2^{2k}-1)}{(2n-2k)!}\frac{\zeta(2k+1)}{\pi^{2k}} \\ &+ \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2n+2k)!}\left(\frac{\pi}{2}\right)^{2k}E_{2k-1}(0)\Bigg] \quad (n \in \mathbb{N}),\end{aligned} \tag{2.24}$$

which is a slightly modified (and corrected) version of a result proven, in a significantly different way, by Tsumura [37, p. 383, Theorem B].

Another interesting combination of our series representations (2.10) and (2.12) leads us to the following variant of Tsumura's result (2.20) or (2.24):

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\frac{\pi^{2n}}{2^{2n+1}-1}\Bigg[&\frac{H_{2n}-\log\left(\frac{1}{4}\pi\right)}{(2n)!} \\ &+ \sum_{k=1}^{n-1}\frac{(-1)^k(2^{2k+1}-1)}{(2n-2k)!}\frac{\zeta(2k+1)}{\pi^{2k}} \\ &- 4\sum_{k=1}^{\infty}\frac{(2k-1)!(2^{2k-1}-1)}{(2n+2k)!}\frac{\zeta(2k)}{2^{4k}}\Bigg] \quad (n \in \mathbb{N}),\end{aligned} \tag{2.25}$$

which is essentially the same as the *determinantal* expression for $\zeta(2n+1)$ derived recently by Ewell [10, p. 1010, Corollary 3] by employing an entirely different technique from ours.

Other similar combinations of the series representations (2.10) to (2.13) would yield the following interesting companions of Ewell's result (2.25):

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{(2^{2n+1}-1)(3^{2n}+1)}\Bigg[&\frac{H_{2n}-\log\left(\frac{1}{6}\pi\right)}{(2n)!} \\ &+ \sum_{k=1}^{n-1}\frac{(-1)^k(2^{2k+1}-1)}{(2n-2k)!}\frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}} \\ &- 4\sum_{k=1}^{\infty}\frac{(2k-1)!(2^{2k-1}-1)}{(2n+2k)!}\frac{\zeta(2k)}{6^{2k}}\Bigg] \quad (n \in \mathbb{N});\end{aligned} \tag{2.26}$$

$$\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{(2^{2n}+1)(3^{2n+1}-1)}\left[\frac{2H_{2n}-\log\left(\frac{\pi^2}{27}\right)}{(2n)!} + \sum_{k=1}^{n-1}\frac{(-1)^k(3^{2k+1}-1)}{(2n-2k)!}\frac{\zeta(2k+1)}{\pi^{2k}} - 6\sum_{k=1}^{\infty}\frac{(2k-1)!(3^{2k-1}-1)}{(2n+2k)!}\frac{\zeta(2k)}{6^{2k}}\right] \quad (n\in\mathbb{N}); \tag{2.27}$$

$$\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{3^{2n+2}-2^{2n+3}+1}\left[\frac{H_{2n}-\log\left(\frac{8\pi}{27}\right)}{(2n)!} + \sum_{k=1}^{n-1}\frac{(-1)^k(3^{2k+1}-2^{2k+1})}{(2n-2k)!}\frac{\zeta(2k+1)}{(2\pi)^{2k}} - 12\sum_{k=1}^{\infty}\frac{(2k-1)!(3^{2k-1}-2^{2k-1})}{(2n+2k)!}\frac{\zeta(2k)}{6^{2k}}\right] \quad (n\in\mathbb{N}); \tag{2.28}$$

$$\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{2^{4n+3}+2^{2n+2}-3^{2n+2}-1}\left[\frac{H_{2n}-\log\left(\frac{27\pi}{128}\right)}{(2n)!} + \sum_{k=1}^{n-1}\frac{(-1)^k(4^{2k+1}-3^{2k+1})}{(2n-2k)!}\frac{\zeta(2k+1)}{(2\pi)^{2k}} - 24\sum_{k=1}^{\infty}\frac{(2k-1)!(4^{2k-1}-3^{2k-1})}{(2n+2k)!}\frac{\zeta(2k)}{12^{2k}}\right] \quad (n\in\mathbb{N}); \tag{2.29}$$

$$\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{3^{2n+1}(2^{2n}+1)-2^{4n+2}+2^{2n}-1}\left[\frac{H_{2n}-\log\left(\frac{4\pi}{27}\right)}{(2n)!} + \sum_{k=1}^{n-1}\frac{(-1)^k(3^{2k+1}-2^{2k+1})}{(2n-2k)!}\frac{\zeta(2k+1)}{\pi^{2k}} - 12\sum_{k=1}^{\infty}\frac{(2k-1)!(3^{2k-1}-2^{2k-1})}{(2n+2k)!}\frac{\zeta(2k)}{12^{2k}}\right] \quad (n\in\mathbb{N}), \tag{2.30}$$

and many more (known or new) results.

Next, by setting $t = 1/m$ and differentiating both sides with respect to s, we find from the following obvious consequence of the series identity (2.1):

$$\sum_{k=0}^{\infty}\frac{(s)_{2k+1}}{(2k+1)!}\zeta(s+2k+1,a)t^{2k+1} = \frac{1}{2}[\zeta(s,a-t)-\zeta(s,a+t)] \quad (|t|<|a|) \tag{2.31}$$

that

$$\sum_{k=0}^{\infty}\frac{(s)_{2k+1}}{(2k+1)!m^{2k}}\left[\zeta'(s+2k+1,a)+\zeta(s+2k+1,a)\sum_{j=0}^{2k}\frac{1}{s+j}\right] = \frac{m}{2}\frac{\partial}{\partial s}\left\{\zeta\left(s,a-\frac{1}{m}\right)-\zeta\left(s,a+\frac{1}{m}\right)\right\} \quad (m\in\mathbb{N}\setminus\{1\}), \tag{2.32}$$

where we have made use of the derivative formula (2.17). In particular, when $m=2$, (2.32) immediately yields

$$\sum_{k=0}^{\infty}\frac{(s)_{2k+1}}{(2k+1)!2^{2k}}\left[\zeta'(s+2k+1,a)+\zeta(s+2k+1,a)\sum_{j=0}^{2k}\frac{1}{s+j}\right] = -\left(a-\frac{1}{2}\right)^{-s}\log\left(a-\frac{1}{2}\right). \tag{2.33}$$

By letting $s\to -2n-1$ $(n\in\mathbb{N})$ in the *further* special of this last identity (2.33) when $a=1$, Wilton [40, p. 92] obtained the following series representation for $\zeta(2n+1)$ (see also Hansen [13, p. 357, Entry (54.6.9)]):

$$\zeta(2n+1) = (-1)^{n-1}\pi^{2n}\left[\frac{H_{2n+1}-\log\pi}{(2n+1)!}+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{\pi^{2k}} + 2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{2^{2k}}\right] \quad (n\in\mathbb{N}), \tag{2.34}$$

which, in view of the identity:

$$\frac{(2k)!}{(2n+2k)!} = \frac{(2k-1)!}{(2n+2k-1)!} - 2n\frac{(2k-1)!}{(2n+2k)!} \quad (n\in\mathbb{N}), \tag{2.35}$$

would combine with the result (2.10) to yield the series representation:

$$\zeta(2n+1) = (-1)^n\frac{(2\pi)^{2n}}{n(2^{2n+1}-1)}\left[\sum_{k=1}^{n-1}\frac{(-1)^{k-1}k}{(2n-2k)!}\frac{\zeta(2k+1)}{\pi^{2k}} + \sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k)!}\frac{\zeta(2k)}{2^{2k}}\right] \quad (n\in\mathbb{N}). \tag{2.36}$$

The series representation (2.36) is precisely the aforementioned *main* result of Cvijović and Klinowski [6, p. 1265, Theorem A]. In fact, in view of the derivative formula (2.19), the series representation (2.35) is essentially the same as a result given earlier by Zhang and Williams [42, p. 1590, Eq. (3.13)] (see also [42, p. 1591, Eq. (3.16)] where an obviously more complicated (asymptotic) version of (2.36) was proven similarly).

Observing also that

$$\frac{(2k)!}{(2n+2k+1)!} = \frac{(2k-1)!}{(2n+2k)!} - (2n+1)\frac{(2k-1)!}{(2n+2k+1)!} \quad (n,k \in \mathbb{N}), \tag{2.37}$$

we obtain yet another series representation for $\zeta(2n+1)$ by applying (2.10) and (2.34):

$$\zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n-1)2^{2n}+1}\left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1}k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{\pi^{2k}} + \sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k+1)!}\frac{\zeta(2k)}{2^{2k}}\right] \quad (n \in \mathbb{N}), \tag{2.38}$$

which provides a *significantly* simpler (and *much more* rapidly convergent) version of the other *main* result of Cvijović and Klinowski [6, p. 1265, Theorem B]:

$$\zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n)!}\sum_{k=0}^{\infty} \Omega_{n,k}\frac{\zeta(2k)}{2^{2k}} \quad (n \in \mathbb{N}), \tag{2.39}$$

where the coefficients $\Omega_{n,k}$ are given explicitly as a finite sum of Bernoulli numbers [6, p. 1265, Theorem B(i)] (see, for details, [27, pp. 393–394]):

$$\Omega_{n,k} := \sum_{j=0}^{2n}\binom{2n}{j}\frac{B_{2n-j}}{(j+2k+1)(j+1)2^j} \quad (n \in \mathbb{N};\ k \in \mathbb{N}_0). \tag{2.40}$$

3. Another family of series representations

Starting once again from the identity (2.1) with (of course) $a = 1$, $t = \pm 1/m$, and s replaced by $s+1$, and applying (2.2), we find yet another class of series identities including, for example,

$$\sum_{k=1}^{\infty}\frac{(s+1)_{2k}}{(2k)!}\frac{\zeta(s+2k)}{2^{2k}} = (2^s - 2)\zeta(s), \tag{3.1}$$

and

$$\sum_{k=1}^{\infty} \frac{(s+1)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{m^{2k}} = \frac{1}{2m}\left[m(m^s-3)\zeta(s) + \left(m^{s+1}-1\right)\zeta(s+1) - 2\zeta\left(s+1,\frac{1}{m}\right) - \sum_{j=2}^{m-2}\left\{m\zeta\left(s,\frac{j}{m}\right) + \zeta\left(s+1,\frac{j}{m}\right)\right\}\right] \quad (m \in \mathbb{N} \setminus \{1,2\}). \tag{3.2}$$

It is the series identity (3.1) which was first applied by Zhang and Williams [42] (and, subsequently, by Cvijović and Klinowski [6]) in order to prove two (only seemingly different) versions of the series representation (2.36). Indeed, by appealing to (3.2) *with* $m=4$, we can derive the following much more rapidly convergent series representation for $\zeta(2n+1)$ (see [26, p. 9, Eq. (41)]):

$$\zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{n(2^{4n+1}+2^{2n}-1)}\left[\frac{4^{n-1}-1}{(2n)!}B_{2n}\log 2 - \frac{2^{2n-1}-1}{2(2n-1)!}\zeta'(1-2n) - \frac{4^{2n-1}}{(2n-1)!}\zeta'\left(1-2n,\frac{1}{4}\right) + \sum_{k=1}^{n-1}\frac{(-1)^{k-1}k}{(2n-2k)!}\frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} + \sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k)!}\frac{\zeta(2k)}{4^{2k}}\right] \quad (n \in \mathbb{N}), \tag{3.3}$$

where (*and in what follows*) a prime denotes the derivative of $\zeta(s)$ or $\zeta(s,a)$ with respect to s.

In view of the identities (2.35) and (2.37), the results (2.12) and (3.3) would lead us eventually to the following *additional* series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ (see [26, p. 10, Eqs. (42) and (43)]; see also [25]):

$$\zeta(2n+1) = (-1)^{n-1}\left(\frac{\pi}{2}\right)^{2n}\left[\frac{H_{2n+1}-\log\left(\frac{1}{2}\pi\right)}{(2n+1)!} + \frac{2(4^n-1)}{(2n+2)!}B_{2n+2}\log 2 - \frac{2^{2n+1}-1}{(2n+1)!}\zeta'(-2n-1) - \frac{2^{4n+3}}{(2n+1)!}\zeta'\left(-2n-1,\frac{1}{4}\right) + \sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} + 2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{4^{2k}}\right]$$

$$(n \in \mathbb{N}); \tag{3.4}$$

$$\zeta(2n+1) = (-1)^n \frac{4(2\pi)^{2n}}{n4^{2n+1} - 2^{2n} + 1} \left[\frac{2^{2n+1} - 1}{2(2n)!} \zeta'(-2n-1) + \frac{4^{2n+1}}{(2n)!} \zeta'\left(-2n-1, \frac{1}{4}\right) - \frac{(2n+1)(4^n - 1)}{(2n+2)!} B_{2n+2} \log 2 + \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k+1)!} \frac{\zeta(2k)}{4^{2k}} \right] \quad (n \in \mathbb{N}). \tag{3.5}$$

Explicit expressions for the derivatives $\zeta'(-2n \pm 1)$ and $\zeta'(-2n \pm 1, \frac{1}{4})$, occurring in the series representations (3.3) to (3.5), can be found and substituted into these results in order to represent $\zeta(2n+1)$ in terms of Bernoulli numbers and polynomials and various rapidly convergent series of ζ-functions (see, for details, [26, Section 3]).

Of the four seemingly analogous results (2.12), (3.3)–(3.5), the infinite series in (3.4) would obviously converge most rapidly, with its general term having the order estimate:

$$\mathrm{O}\left(k^{-2n-2}\ 4^{-2k}\right) \quad (k \to \infty;\ n \in \mathbb{N}).$$

We now turn to the work of Srivastava and Tsumura [31], who derived the following three *new* members of the class of the series representations (2.12) and (3.4):

$$\zeta(2n+1) = (-1)^{n-1} \left(\frac{2\pi}{3}\right)^{2n} \left[\frac{H_{2n+1} - \log\left(\frac{2}{3}\pi\right)}{(2n+1)!} + \frac{(3^{2n+2} - 1)\pi}{2\sqrt{3}(2n+2)!} B_{2n+2} + \frac{(-1)^{n-1}}{\sqrt{3}(2\pi)^{2n+1}} \zeta\left(2n+2, \frac{1}{3}\right) + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}} + 2\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{3^{2k}} \right] \quad (n \in \mathbb{N}), \tag{3.6}$$

$$\zeta(2n+1) = (-1)^{n-1} \left(\frac{\pi}{2}\right)^{2n} \left[\frac{H_{2n+1} - \log\left(\frac{1}{2}\pi\right)}{(2n+1)!} + \frac{2^{2n}(2^{2n+2} - 1)\pi}{(2n+2)!} B_{2n+2} + \frac{(-1)^{n-1}}{2(2\pi)^{2n+1}} \zeta\left(2n+2, \frac{1}{4}\right) + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} + 2\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{4^{2k}} \right] \quad (n \in \mathbb{N}), \tag{3.7}$$

and

$$\zeta(2n+1) = (-1)^{n-1}\left(\frac{\pi}{3}\right)^{2n}\left[\frac{H_{2n+1} - \log\left(\frac{1}{3}\pi\right)}{(2n+1)!} + \frac{2^{2n}(3^{2n+2}-1)\pi}{\sqrt{3}(2n+2)!}B_{2n+2} + \frac{(-1)^{n-1}}{2\sqrt{3}(2\pi)^{2n+1}}\left\{\zeta\left(2n+2,\frac{1}{3}\right) + \zeta\left(2n+2,\frac{1}{6}\right)\right\} + \sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{\left(\frac{1}{3}\pi\right)^{2k}} + 2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{6^{2k}}\right]$$

$$(n \in \mathbb{N}). \tag{3.8}$$

Indeed the general terms of the infinite series occurring in these three members [(3.6) to (3.8)] have the order estimates:

$$\mathrm{O}\left(k^{-2n-2}\, m^{-2k}\right) \quad (k \to \infty;\ n \in \mathbb{N};\ m = 3, 4, 6), \tag{3.9}$$

which exhibit the fact that *each* of the three series representations (3.6) to (3.8) converges more rapidly than Wilton's result (2.34) and two of them [cf. Eqs. (3.7) and (3.8)] at least as rapidly as Srivastava's result (3.4).

4. A further class of series representations

In their aforecited work on the Ray–Singer torsion and topological field theories, Nash and O'Connor ([19] and [20]) obtained a number of remarkable integral expressions for $\zeta(3)$, including (for example) the following result [20, p. 1489 *et seq.*]:

$$\zeta(3) = \frac{2\pi^2}{7}\log 2 - \frac{8}{7}\int_0^{\pi/2} z^2 \cot z \,\mathrm{d}z. \tag{4.1}$$

Since [8, p. 51, Eq. 1.20 (3)]

$$z\cot z = -2\sum_{k=0}^{\infty}\zeta(2k)\left(\frac{z}{\pi}\right)^{2k} \quad (|z| < \pi), \tag{4.2}$$

the result (4.1) is obviously equivalent to the series representation (cf. [7, p. 202]; see also [5, p. 191, Eq. (3.19)]):

$$\zeta(3) = \frac{2\pi^2}{7}\left(\log 2 + \sum_{k=0}^{\infty}\frac{\zeta(2k)}{(k+1)2^{2k}}\right). \tag{4.3}$$

Moreover, upon integrating by parts, it is easily seen that

$$\int_0^{\pi/2} z^2 \cot z \,\mathrm{d}z = -2\int_0^{\pi/2} z \log\sin z \,\mathrm{d}z, \tag{4.4}$$

so that the result (4.1) is equivalent *also* to the integral representation:

$$\zeta(3) = \frac{2\pi^2}{7}\log 2 + \frac{16}{7}\int_0^{\pi/2} z \log\sin z \,\mathrm{d}z, \tag{4.5}$$

which was proven in the aforementioned 1772 paper by Euler (cf., e.g., [3, p. 1084]).

Next, since

$$\mathrm{i}\cot \mathrm{i}z = \coth z = \frac{2}{\mathrm{e}^{2z}-1} + 1 \quad \left(\mathrm{i} := \sqrt{-1}\right), \tag{4.6}$$

by replacing z in the known expansion (4.2) by $(1/2)\mathrm{i}\pi z$, it is easily seen that (cf., e.g., [17, p. 25]; see also [8, p. 51, Eq. 1.20 (1)])

$$\frac{\pi z}{\mathrm{e}^{\pi z}-1} + \frac{\pi z}{2} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\zeta(2k)}{2^{2k-1}} z^{2k} \quad (|z| < 2). \tag{4.7}$$

By setting $z = \mathrm{i}t$ in (4.7), multiplying both sides by t^{m-1} $(m \in \mathbb{N})$, and then integrating the resulting equation from $t = 0$ to τ $(0 < \tau < 2)$, Srivastava [28] derived the following series representations for $\zeta(2n+1)$ (see also [30]):

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{(2n)!(2^{2n+1}-1)}\left[\log 2 + \sum_{j=1}^{n-1}(-1)^j\binom{2n}{2j} \cdot \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1) + \sum_{k=0}^{\infty}\frac{\zeta(2k)}{(k+n)2^{2k}}\right] \quad (n \in \mathbb{N}) \tag{4.8}$$

and

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{(2n+1)!(2^{2n}-1)}\left[\log 2 + \sum_{j=1}^{n-1}(-1)^j\binom{2n+1}{2j} \cdot \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1) + \sum_{k=0}^{\infty}\frac{\zeta(2k)}{\left(k+n+\frac{1}{2}\right)2^{2k}}\right] \quad (n \in \mathbb{N}). \tag{4.9}$$

For $n=1$, (4.9) immediately reduces to the following series representation for $\zeta(3)$:

$$\zeta(3)=\frac{2\pi^2}{9}\left(\log 2+2\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+3)2^{2k}}\right), \tag{4.10}$$

which was proven *independently* by (among others) Glasser [11, p. 446, Eq. (12)], Zhang and Williams [42, p. 1585, Eq. (2.13)], and Dąbrowski [7, p. 206] (see also [5, p. 183, Eq. (2.15)]). And a special case of (4.8) when $n=1$ yields (cf. [7, p. 202]; see also [5, p. 191, Eq. (3.19)])

$$\zeta(3)=\frac{2\pi^2}{7}\left(\log 2+\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(k+1)2^{2k}}\right). \tag{4.11}$$

In view of the familiar sum:

$$\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+1)2^{2k}}=-\frac{1}{2}\log 2, \tag{4.12}$$

Euler's formula (1.13) is indeed a *simple* consequence of (4.11).

We remark in passing that an integral representation for $\zeta(2n+1)$, which is easily seen to be equivalent to the series representation (4.8), was given by Dąbrowski [7, p. 203, Eq. (16)], who [7, p. 206] mentioned the existence of (but did not fully state) the series representation (4.9) as well. The series representation (4.8) is derived also in a recent paper by Borwein et al. (cf. [4, p. 269, Eq. (57)]).

By suitably combining the series occurring in (4.3), (4.10), and (4.12), it is not difficult to derive several other series representations for $\zeta(3)$, which are analogous to Euler's formula (1.13). More generally, since

$$\begin{aligned}&\frac{\lambda k^2+\mu k+\nu}{(2k+2n-1)(2k+2n)(2k+2n+1)}\\&\quad=\frac{\mathscr{A}}{2k+2n-1}+\frac{\mathscr{B}}{2k+2n}+\frac{\mathscr{C}}{2k+2n+1},\end{aligned} \tag{4.13}$$

where, for convenience,

$$\mathscr{A}=\mathscr{A}_n(\lambda,\mu,\nu):=\frac{1}{2}\left[\lambda n^2-(\lambda+\mu)n+\frac{1}{4}(\lambda+2\mu+4\nu)\right], \tag{4.14}$$

$$\mathscr{B}=\mathscr{B}_n(\lambda,\mu,\nu):=-(\lambda n^2-\mu n+\nu), \tag{4.15}$$

and

$$\mathscr{C}=\mathscr{C}_n(\lambda,\mu,\nu):=\frac{1}{2}\left[\lambda n^2+(\lambda-\mu)n+\frac{1}{4}(\lambda-2\mu+4\nu)\right], \tag{4.16}$$

by applying (4.8) and (4.9), and another result (proven by Srivastava [28, p. 341, Eq. (3.17)]):

$$\sum_{j=1}^{n}(-1)^{j-1}\binom{2n+1}{2j}\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)$$
$$=\log 2+\sum_{k=0}^{\infty}\frac{\zeta(2k)}{\left(k+n+\frac{1}{2}\right)2^{2k}}\quad (n\in\mathbb{N}_0), \tag{4.17}$$

with n replaced by $n-1$, Srivastava [28] derived the following unification of a large number of known (or new) series representations for $\zeta(2n+1)$ $(n\in\mathbb{N})$, including (for example) Euler's formula (1.13):

$$\zeta(2n+1)=\frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)!\{(2^{2n+1}-1)\mathscr{B}+(2n+1)(2^{2n}-1)\mathscr{C}\}}$$
$$\cdot\left[\frac{1}{4}\lambda\log 2+\sum_{j=1}^{n-1}(-1)^{j}\binom{2n-1}{2j-2}\left\{2j(2j-1)\mathscr{A}\right.\right.$$
$$\left.+[\lambda(4n-1)-2\mu]nj+\lambda n\left(n+\frac{1}{2}\right)\right\}\frac{(2j-2)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)$$
$$\left.+\sum_{k=0}^{\infty}\frac{(\lambda k^2+\mu k+\nu)\zeta(2k)}{(2k+2n-1)(k+n)(2k+2n+1)2^{2k}}\right]$$
$$(n\in\mathbb{N};\ \lambda,\mu,\nu\in\mathbb{C}), \tag{4.18}$$

where $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are given by (4.14), (4.15), and (4.16), respectively.

Numerous other interesting series representations for $\zeta(2n+1)$, which are analogous to (4.8) and (4.9), were also given by Srivastava *et al.* [30]. For the sake of completeness, we choose to recall their results as follows:

$$\zeta(2n+1)=(-1)^{n-1}\frac{(2\pi)^{2n}}{(2n+1)!(3^{2n}-1)}\left[\log 3+4\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n+1)3^{2k}}\right.$$
$$+(2n+1)!\sum_{j=1}^{n-1}\frac{(-1)^{j}}{(2n-2j+1)!}\left(\frac{3^{2j}-1}{(2\pi)^{2j}}\right)\zeta(2j+1)$$
$$\left.-\frac{(2n+1)!}{\sqrt{3}}\sum_{j=1}^{n+1}\frac{(-1)^{j}}{(2n-2j+2)!}\frac{2\zeta\left(2j,\frac{1}{3}\right)-(3^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}}\right]$$
$$(n\in\mathbb{N}), \tag{4.19}$$

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{(2n)!(3^{2n+1}-1)}\left[\log 3 + 2\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(k+n)3^{2k}}\right.$$
$$+ (2n)!\sum_{j=1}^{n-1}\frac{(-1)^j}{(2n-2j)!}\left(\frac{3^{2j}-1}{(2\pi)^{2j}}\right)\zeta(2j+1)$$
$$\left. - \frac{(2n)!}{\sqrt{3}}\sum_{j=1}^{n}\frac{(-1)^j}{(2n-2j+1)!}\frac{2\zeta\left(2j,\frac{1}{3}\right)-(3^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}}\right]$$
$$(n \in \mathbb{N}), \tag{4.20}$$

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{(2n+1)!(2^{2n}-1)}\left[\log 2 + 4\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n+1)4^{2k}}\right.$$
$$+ (2n+1)!\sum_{j=1}^{n-1}\frac{(-1)^j}{(2n-2j+1)!}\left(\frac{2^{2j}-1}{(2\pi)^{2j}}\right)\zeta(2j+1)$$
$$\left. - (2n+1)!\sum_{j=1}^{n+1}\frac{(-1)^j}{(2n-2j+2)!}\frac{\zeta\left(2j,\frac{1}{4}\right)-2^{2j-1}(2^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}}\right]$$
$$(n \in \mathbb{N}), \tag{4.21}$$

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{(2n)!(2^{4n+1}+2^{2n}-1)}\left[\log 2 + 2\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(k+n)4^{2k}}\right.$$
$$+ (2n)!\sum_{j=1}^{n-1}\frac{(-1)^j}{(2n-2j)!}\left(\frac{2^{2j}-1}{(2\pi)^{2j}}\right)\zeta(2j+1)$$
$$\left. - (2n)!\sum_{j=1}^{n}\frac{(-1)^j}{(2n-2j+1)!}\frac{\zeta\left(2j,\frac{1}{4}\right)-2^{2j-1}(2^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}}\right]$$
$$(n \in \mathbb{N}), \tag{4.22}$$

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{(2^{2n}-1)(3^{2n}-1)}\left[-\frac{4}{(2n+1)!}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n+1)6^{2k}}\right.$$
$$+ \sum_{j=1}^{n-1}\frac{(-1)^j}{(2n-2j+1)!}\left(\frac{(2^{2j}-1)(3^{2j}-1)}{(2\pi)^{2j}}\right)\zeta(2j+1)$$
$$\left. + \frac{1}{\sqrt{3}}\sum_{j=1}^{n+1}\frac{(-1)^j}{(2n-2j+2)!}\frac{\zeta\left(2j,\frac{1}{3}\right)+\zeta\left(2j,\frac{1}{6}\right)-2^{2j-1}(3^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}}\right]$$
$$(n \in \mathbb{N}), \tag{4.23}$$

and

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2^{2n}+3^{2n}+6^{2n}-1}\left[\frac{2}{(2n)!}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(k+n)6^{2k}} - \sum_{j=1}^{n-1}\frac{(-1)^j}{(2n-2j)!}\left(\frac{(2^{2j}-1)(3^{2j}-1)}{(2\pi)^{2j}}\right)\zeta(2j+1) - \frac{1}{\sqrt{3}}\sum_{j=1}^{n}\frac{(-1)^j}{(2n-2j+1)!}\frac{\zeta\left(2j,\frac{1}{3}\right)+\zeta\left(2j,\frac{1}{6}\right)-2^{2j-1}(3^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}}\right] \quad (n\in\mathbb{N}). \tag{4.24}$$

It is not difficult to derive further series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ by appropriately combining two or more of the results (4.8), (4.9), (4.17) and (4.19)–(4.24). Thus we can arrive at several general results analogous (for example) to (4.18).

5. Some useful deductions and consequences

For $\lambda = 0$, the series representation (4.18) simplifies to the form:

$$\zeta(2n+1) = \frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)!\left\{(2^{2n+1}-1)(\mu n - v) - (2^{2n}-1)\left(n+\frac{1}{2}\right)\left[\mu\left(n+\frac{1}{2}\right)-v\right]\right\}} \cdot \left[\sum_{j=1}^{n-1}(-1)^j\binom{2n-1}{2j-2}\left\{j(2j-1)\left[v-\mu\left(n-\frac{1}{2}\right)\right]-2\mu nj\right\} \cdot \frac{(2j-2)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1) + \sum_{k=0}^{\infty}\frac{(\mu k+v)\zeta(2k)}{(2k+2n-1)(k+n)(2k+2n+1)2^{2k}}\right] \quad (n\in\mathbb{N};\ \mu, v\in\mathbb{C}). \tag{5.1}$$

Furthermore, by setting

$$\lambda = \mu = 0 \quad \text{and} \quad v = 1$$

in (4.18) or (alternatively) by setting

$$\mu = 0 \quad \text{and} \quad v = 1$$

in (5.1), we immediately obtain the series representation:

$$\zeta(2n+1) = \frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)!\{2^{2n}(2n-3)-2n+1\}} \cdot \left[\sum_{j=1}^{n-1}(-1)^j \binom{2n-1}{2j-2} \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1) + 2\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n-1)(k+n)(2k+2n+1)2^{2k}}\right] \quad (n \in \mathbb{N}), \tag{5.2}$$

which, in the special case when $n = 1$, was given by Chen and Srivastava [5, p. 189, Eq. (2.45)].

Of the three representations (4.18), (5.1), and (5.2) for $\zeta(2n+1)$ $(n \in \mathbb{N})$, the infinite series in (5.2) converges most rapidly.

For various other suitable special values of the parameters λ, μ, and ν, we can easily deduce from (4.18) and (5.1) several known (or new) series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$. For example, if we set

$$\mu = 2 \quad \text{and} \quad \nu = 2n+1 \quad (n \in \mathbb{N})$$

in the series representation (5.1), we shall obtain

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{(2n)!(2^{2n+1}-1)}\left[\sum_{j=1}^{n-1}(-1)^j\binom{2n-1}{2j-1} \cdot \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1) - \sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n-1)(k+n)2^{2k}}\right] \quad (n \in \mathbb{N}), \tag{5.3}$$

which, in the special case when $n = 1$, immediately yields Euler's formula (1.13).

The following additional series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$, which are analogous to (5.3), can also be deduced similarly from (5.1):

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{(2n)!\{(2n-1)2^{2n}-2n\}}\left[2n\sum_{j=1}^{n-1}(-1)^j\binom{2n-1}{2j-2} \cdot \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1) - \sum_{k=0}^{\infty}\frac{\zeta(2k)}{(k+n)(2k+2n+1)2^{2k}}\right] \quad (n \in \mathbb{N}) \tag{5.4}$$

and

$$\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{(2n+1)!(2^{2n}-1)}\left[\sum_{j=1}^{n-1}(-1)^j\left(\frac{4nj-2j+1}{2j-1}\right)\cdot\binom{2n-1}{2j-2}\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1) - 4\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n-1)(2k+2n+1)2^{2k}}\right] \quad (n\in\mathbb{N}). \tag{5.5}$$

The special case of *each* of the last two series representations (5.4) and (5.5) when $n = 1$ was given by Zhang and Williams [42, p. 1586].

Next, with a view to *further* improving the rate of convergence in the reasonably rapidly convergent series representation (5.2), we observe that

$$\frac{1}{(2k+2n-1)(2k+2n)(2k+2n+1)(2k+2n+2)} = \frac{1}{6}\left(\frac{1}{2k+2n-1}-\frac{1}{2k+2n+2}\right)-\frac{1}{2}\,\frac{1}{(2k+2n)(2k+2n+1)}. \tag{5.6}$$

Thus, by applying the series representations (4.17) *with n* replaced by $n-1$, (4.8) *with n* replaced by $n+1$, and (5.4), we obtain

$$\zeta(2n+3) = \frac{2\pi^2\{2^{2n+2}+n(2n-3)(2^{2n}-1)-1\}}{(n+1)(2n+1)(2^{2n+3}-1)}\zeta(2n+1)+(-1)^{n-1}\cdot\frac{(2\pi)^{2n+2}}{(2n+2)!(2^{2n+3}-1)}\left[\sum_{j=1}^{n-1}(-1)^j\left\{\binom{2n-1}{2j}-\binom{2n+2}{2j}+6n\binom{2n-1}{2j-2}\right\}\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)+12\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n-1)(2k+2n)(2k+2n+1)(2k+2n+2)2^{2k}}\right]$$
$$(n\in\mathbb{N}), \tag{5.7}$$

where the series converges faster than that in (5.2).

In its special case when $n = 1$, (5.7) readily yields the following improved version of the series representation derivable from (5.2) for $n = 2$ (cf. [42, p. 1590, Eq. (3.14)]):

$$\zeta(5) = \frac{4\pi^2}{31}\zeta(3)+\frac{8\pi^4}{31}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+1)(2k+2)(2k+3)(2k+4)2^{2k}}, \tag{5.8}$$

in which $\zeta(3)$ can be replaced by its known value $-4\pi^2\zeta'(-2)$ given by (2.9) for $n = 1$.

Yet another rapidly convergent series representation for $\zeta(2n+3)$ $(n \in \mathbb{N})$, analogous to (5.7), can be derived by means of the identity:

$$\begin{aligned}&\frac{1}{(2k+2n)(2k+2n+1)(2k+2n+2)(2k+2n+3)}\\&\quad=\frac{1}{6}\left(\frac{1}{2k+2n}-\frac{1}{2k+2n+3}\right)-\frac{1}{2}\,\frac{1}{(2k+2n+1)(2k+2n+2)},\end{aligned}\tag{5.9}$$

together with our series representations (4.8) and (4.9) *with* n replaced by $n+1$, and (5.3) *with* n replaced by $n+1$. We thus obtain the series representation:

$$\begin{aligned}\zeta(2n+3) &= \frac{2\pi^2\left\{\frac{1}{3}(2n+1)(2n^2-4n+3)(2^{2n}-1)-2^{2n+1}+1\right\}}{(n+1)(2n+1)\{(2n-3)2^{2n+2}-2n\}}\zeta(2n+1)\\&\quad+(-1)^{n-1}\frac{(2\pi)^{2n+2}}{(2n+2)!\{(2n-3)2^{2n+2}-2n\}}\left[\sum_{j=1}^{n-1}(-1)^j\left\{\binom{2n}{2j}\right.\right.\\&\quad\left.-\binom{2n+3}{2j}+3\binom{2n+1}{2j-1}\right\}\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)\\&\quad\left.+12\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n)(2k+2n+1)(2k+2n+2)(2k+2n+3)2^{2k}}\right]\\&\quad(n\in\mathbb{N}),\end{aligned}\tag{5.10}$$

which, in the special case when $n = 1$, yields

$$\zeta(5)=\frac{2\pi^2}{27}\zeta(3)-\frac{4\pi^4}{9}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2)(2k+3)(2k+4)(2k+5)2^{2k}},\tag{5.11}$$

where the series obviously converges faster than that derivable from (5.2) for $n = 2$.

Lastly, by applying the identity:

$$\begin{aligned}&\frac{1}{2k(2k+2n-1)(2k+2n)(2k+2n+1)}\\&\quad=\frac{1}{2n(2n-1)(2n+1)}\,\frac{1}{2k}-\frac{1}{2(2n-1)}\,\frac{1}{2k+2n-1}+\frac{1}{2n}\,\frac{1}{2k+2n}\\&\qquad-\frac{1}{2(2n+1)}\,\frac{1}{2k+2n+1}\end{aligned}\tag{5.12}$$

in conjunction with the series representations (4.17) *with* n replaced by $n-1$, (4.8) and (4.9), and the known result (cf., e.g., [13, p. 356, Entry (54.5.3)]):

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} t^{2k} = \log\left[\pi t \csc(\pi t)\right] \tag{5.13}$$

with $t = 1/2$, we arrive at the following series representation for $\zeta(2n+1)$ $(n \in \mathbb{N})$:

$$\zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n-1)!\{n-1-(n-2)2^{2n}\}} \left[\frac{12n^2-1}{2n^2(4n^2-1)^2} - \frac{\log \pi}{n(4n^2-1)} - \sum_{j=2}^{n-1} (-1)^j \binom{2n-2}{2j-3} \frac{(2j-1)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) + \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+n)(2k+2n-1)(2k+2n+1)2^{2k}} \right] \quad (n \in \mathbb{N}), \tag{5.14}$$

where we have also applied the fact that $\zeta(0) = -1/2$.

For $n = 1$, (5.14) reduces immediately to Wilton's formula (cf. [40, p. 92] and [13, p. 357, Entry (54.5.9)]; see also [5, p. 181, Eq. (2.1)]):

$$\zeta(3) = \frac{\pi^2}{2} \left(\frac{11}{18} - \frac{1}{3} \log \pi + \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+1)(2k+1)(2k+3)2^{2k}} \right). \tag{5.15}$$

Furthermore, in its special case when $n = 2$, (5.14) would yield the following interesting companion of the series representations (5.8) and (5.11):

$$\zeta(5) = \frac{2\pi^4}{45} \left(\log \pi - \frac{47}{60} - 30 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+2)(2k+3)(2k+5)2^{2k}} \right), \tag{5.16}$$

which does *not* contain a term involving $\zeta(3)$ on the right-hand side.

By eliminating $\zeta(2n+3)$ between the results (5.7) and (5.10), we can obtain a series representation for $\zeta(2n+1)$ $(n \in \mathbb{N})$, which would converge as rapidly as the series in (5.14). We thus find that (cf. [28, pp. 348–349, Eq. (3.50)])

$$\zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!\Delta_n} \left[\sum_{j=1}^{n-1} (-1)^j \left(\left\{ (2n-3)2^{2n+2} - 2n \right\} \left\{ \binom{2n-1}{2j} - \binom{2n+2}{2j} + 6n \binom{2n-1}{2j-2} \right\} - (2^{2n+3}-1) \left\{ \binom{2n}{2j} - \binom{2n+3}{2j} + 3 \binom{2n+1}{2j-1} \right\} \right) \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) + 12 \sum_{k=0}^{\infty} \frac{(\xi_n k + \eta_n)\zeta(2k)}{(2k+2n-1)(2k+2n)(2k+2n+1)(2k+2n+2)(2k+2n+3)2^{2k}} \right] \quad (n \in \mathbb{N}), \tag{5.17}$$

where, for convenience,

$$\Delta_n := \left(2^{2n+3}-1\right)\left\{\frac{1}{3}(2n+1)\left(2n^2-4n+3\right)\left(2^{2n}-1\right)-2^{2n+1}+1\right\} - \left\{(2n-3)2^{2n+2}-2n\right\}\left\{2^{2n+2}+n(2n-3)\left(2^{2n}-1\right)-1\right\}, \quad (5.18)$$

$$\xi_n := 2\left\{(2n-5)2^{2n+2}-2n+1\right\}, \quad (5.19)$$

and

$$\eta_n := \left(4n^2-4n-7\right)2^{2n+2}-(2n+1)^2. \quad (5.20)$$

In its special case when $n=1$, (5.17) yields the following (*rather curious*) series representation:

$$\zeta(3) = -\frac{6\pi^2}{23}\sum_{k=0}^{\infty}\frac{(98k+121)\zeta(2k)}{(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)2^{2k}}, \quad (5.21)$$

where the series obviously converges much more rapidly than that in *each* of the *celebrated* results (1.13) and (1.14).

6. Inductive construction of series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$

Let $\mathbb{Z}$ and $\mathbb{Q}$ be the ring of integers and the field of rational numbers, respectively. We begin by defining the following sequences and rational functions. First we let

$$C_1(n,0) = \frac{2^{2n}H_{2n}}{(2^{2n+1}-1)(2n)!}, \qquad C_2(n,0) = -\frac{2^{2n}}{(2^{2n+1}-1)(2n)!};$$

$$D_k(n,0) = \frac{2^{2n}}{(2^{2n+1}-1)(2n-2k)!} \quad (1 \leqq k \leqq n-1);$$

$$f(x,n,0) = \frac{2^{2n}}{(2^{2n+1}-1)}\frac{1}{x+n} \quad (n \in \mathbb{N})$$

and

$$C_1(n,1) = \frac{H_{2n+1}}{(2n+1)!}, \qquad C_2(n,1) = -\frac{1}{(2n+1)!};$$

$$D_k(n,1) = \frac{1}{(2n-2k+1)!} \quad (1 \leqq k \leqq n-1);$$

$$f(x,n,1) = 2 \quad (n \in \mathbb{N}).$$

We note that $C_m(n,j)$, $D_k(n,j) \in \mathbb{Q}$, and $f(x,n,j) \in \mathbb{Q}(x)$. For $F(x)$, $G(x) \in \mathbb{Q}(x)$, we define

$$\deg\left\{\frac{F(x)}{G(x)}\right\} := \deg\{F(x)\} - \deg\{G(x)\}.$$

Then we readily see that $\deg\{f(x,n,j)\} = j-1$. Furthermore, we formally let

$$A(n,0) = 1 \quad \text{and} \quad A(n,1) = 1.$$

We recursively define these sequences and rational functions for $j \geqq 2$. We assume that we can define $A(n,l)$, $C_m(n,l)$, $D_k(n,l)$, and $f(x,n,l)$ for $1 \leqq l \leqq j-1$, $n \geqq 1$, $1 \leqq m \leqq 2$, and $1 \leqq k \leqq n-1$. Then we define

$$\begin{aligned}
A(n,j) &= A(n+1,j-1)D_n(n+1,j-2) - A(n+1,j-2)D_n(n+1,j-1);\\
C_m(n,j) &= A(n+1,j-1)C_m(n+1,j-2) - A(n+1,j-2)C_m(n+1,j-1);\\
D_k(n,j) &= A(n+1,j-1)D_k(n+1,j-2) - A(n+1,j-2)D_k(n+1,j-1);\\
f(x,n,j) &= (2x+2n+2j-1)(2x+2n+2j-2)A(n+1,j-1)\\
&\quad \cdot f(x,n+1,j-2) - A(n+1,j-2)f(x,n+1,j-1);\\
&\quad (m = 1,2; \quad 1 \leqq k \leqq n-1; \quad n,k \in \mathbb{N}).
\end{aligned}$$

By induction, we can easily derive Lemma 1 and Lemma 2 below.

Lemma 1. *$A(n,j)$, $C_m(n,j)$, $D_k(n,j) \in \mathbb{Q}$. Furthermore, $f(x,n,j) \in \mathbb{Q}(x)$ with*

$$\deg\{f(x,n,j)\} = j-1.$$

Lemma 2. *For $j \in \mathbb{N}_0$,*

$$\begin{aligned}
&(-1)^{n-1}A(n,j)\frac{\zeta(2n+1)}{\pi^{2n}}\\
&\quad = C_1(n,j) + C_2(n,j)\log\pi + \sum_{k=1}^{n-1}(-1)^k D_k(n,j)\frac{\zeta(2k+1)}{\pi^{2k}}\\
&\qquad + \sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+2j-1)!}f(k,n,j)\frac{\zeta(2k)}{2^{2k}} \quad (n \in \mathbb{N}).
\end{aligned} \tag{6.1}$$

Making use of Lemma 2, we immediately obtain the following general series representation (cf. [32,34]).

Theorem 1. *With the above notations, if $A(n,j) \neq 0$, then*

$$\begin{aligned}
\zeta(2n+1) = (-1)^{n-1}\frac{\pi^{2n}}{A(n,j)}\Bigg[&C_1(n,j) + C_2(n,j)\log\pi\\
&+ \sum_{k=1}^{n-1}(-1)^k D_k(n,j)\frac{\zeta(2k+1)}{\pi^{2k}} + \sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+2j-1)!}\\
&\cdot f(k,n,j)\frac{\zeta(2k)}{2^{2k}}\Bigg] \quad (n \in \mathbb{N};\ j \in \mathbb{N}_0).
\end{aligned} \tag{6.2}$$

Remark 1. Since $\deg\{f(x,n,j)\} = j-1$, we can easily observe that the general term of the infinite series in (6.2) has the order estimate:

$$\mathrm{O}\left(k^{-2n-j-1}\,2^{-2k}\right) \quad (k \to \infty;\ j \in \mathbb{N}_0).$$

By applying (6.2) in the special case of Theorem 1 when $j = 2$, we obtain

Proposition 1. *The following series representation holds true for* $\zeta(2n+1)$:

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\frac{6\pi^{2n}}{2^{2n+2}+1}\Bigg[&\frac{1}{(2n+3)!}\left\{(2n+3)2^{2n+2}H_{2n+2}\right.\\ &\left.-\left(2^{2n+3}-1\right)H_{2n+3} = \left[2^{2n+2}(2n+1)+1\right]\log\pi\right\}\\ &+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+3)!}\left\{2^{2n+2}(2n-2k+1)+1\right\}\frac{\zeta(2k+1)}{\pi^{2k}}\\ &+2\sum_{k=1}^{\infty}\frac{(2k-1)![2^{2n+2}(2n+2k+1)+1]}{(2n+2k+3)!}\frac{\zeta(2k)}{2^{2k}}\Bigg] \quad (n \in \mathbb{N}).\end{aligned} \tag{6.3}$$

Remark 2. Since the general term of the infinite series in (6.3) has the order estimate:

$$\mathrm{O}\left(k^{-2n-3}\,2^{-2k}\right) \quad (k \to \infty),$$

the infinite series in (6.3) converges more rapidly than that in Wilton's result (2.34).

Example 1. In its special case when $n = 1$, (6.3) readily yields

$$\begin{aligned}\zeta(3) = -\frac{6}{17}\pi^2\Bigg[&\frac{1}{120}(80H_4 - 31H_5 - 49\log\pi)\\ &+2\sum_{k=1}^{\infty}\frac{32k+49}{(2k)(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)}\frac{\zeta(2k)}{2^{2k}}\Bigg].\end{aligned} \tag{6.4}$$

Corresponding to (2.36) and (2.38), we now define the following sequences and rational functions in the same manner as we have already detailed above. We first let

$$\mathscr{A}(n,0)=1;$$
$$\mathscr{D}_k(n,0)=\frac{2^{2n}}{n(2^{2n+1}-1)}\frac{k}{(2n-2k)!}\quad(1\leqq k\leqq n-1);$$
$$g(x,n,0)=\frac{2^{2n-1}}{n(2^{2n+1}-1)}\frac{1}{x+n};\quad(n\in\mathbb{N})$$

and

$$\mathscr{A}(n,1)=1;$$
$$\mathscr{D}_k(n,1)=\frac{2^{2n+1}}{(2n-1)2^{2n}+1}\frac{k}{(2n-2k+1)!}\quad(1\leqq k\leqq n-1);$$
$$g(x,n,1)=\frac{2^{2n+1}}{(2n-1)2^{2n}+1};\quad(n\in\mathbb{N}).$$

By induction, for $j\in\mathbb{N}\setminus\{1\}$, we also let

$$\begin{aligned}&\mathscr{A}(n,j)=\mathscr{A}(n+1,j-1)\mathscr{D}_n(n+1,j-2)-\mathscr{A}(n+1,j-2)\mathscr{D}_n(n+1,j-1);\\&\mathscr{D}_k(n,j)=\mathscr{A}(n+1,j-1)\mathscr{D}_k(n+1,j-2)-\mathscr{A}(n+1,j-2)\mathscr{D}_k(n+1,j-1);\\&g(x,n,j)=(2x+2n+2j-1)(2x+2n+2j-2)\mathscr{A}(n+1,j-1)\\&\qquad\cdot g(x,n+1,j-2)-\mathscr{A}(n+1,j-2)g(x,n+1,j-1);\\&\quad(1\leqq k\leqq n-1;\ n,k\in\mathbb{N}).\end{aligned}$$

Thus, by induction, we can obtain Lemma 3 below.

Lemma 3. *$\mathscr{A}(n,j),\mathscr{D}_k(n,j)\in\mathbb{Q}$ and $g(x,n,j)\in\mathbb{Q}(x)$ with* $\deg\{g(x,n,j)\}=j-1$.

Next, by using the above notations, we obtain the following general series representation (cf. [32,34]).

Theorem 2. *With the above notations, if $\mathscr{A}(n,j)\neq 0$, then*

$$\zeta(2n+1)=(-1)^n\frac{\pi^{2n}}{\mathscr{A}(n,j)}\left[\sum_{k=1}^{n-1}(-1)^{k-1}\mathscr{D}_k(n,j)\frac{\zeta(2k+1)}{\pi^{2k}}+\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k+2j-1)!}g(k,n,j)\frac{\zeta(2k)}{2^{2k}}\right]$$
$$(n\in\mathbb{N};\ j\in\mathbb{N}_0).\tag{6.5}$$

Remark 3. Since $\deg\{g(x,n,j)\}=j-1$, it is easily seen that the general term of the infinite series in (6.5) has the order estimate:

$$\mathrm{O}(k^{-2n-j}\ 2^{-2k}) \quad (k \to \infty;\ j \in \mathbb{N}_0).$$

Finally, we have

Proposition 2. *The following series representation holds true for* $\zeta(2n+1)$:

$$\begin{aligned}\zeta(2n+1) = (-1)^n \frac{6\pi^{2n}}{n\{2^{2n+2}(2n-1)+(2n+5)\}} \Bigg[\sum_{k=1}^{n-1} \frac{(-1)^{k-1}k}{(2n-2k+3)!} \\ \cdot \left\{2^{2n+2}(4n^2-4kn+4n-2k-1)+(4n-2k+5)\right\} \frac{\zeta(2k+1)}{\pi^{2k}} \\ + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k+3)!} \{2[2^{2n+2}(2n+1)+1]k + 2^{2n+2}(4n^2+4n-1) \\ + 4n+5\} \frac{\zeta(2k)}{2^{2k}} \Bigg] \quad (n \in \mathbb{N}). \end{aligned} \tag{6.6}$$

Example 2. By applying (6.6) in the special cases when $n = 1$ and 2, we obtain Srivastava's result (5.21) and

$$\begin{aligned}\zeta(5) = \frac{\pi^4}{67} \Bigg[\frac{281}{40} \frac{\zeta(3)}{\pi^2} \\ + 3 \sum_{k=0}^{\infty} \frac{214k+495}{(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)(2k+6)(2k+7)} \frac{\zeta(2kt)}{2^{2k}} \Bigg], \end{aligned} \tag{6.7}$$

respectively.

Example 3. By applying (2.38) in the special case when $n = 2$, we get

$$\zeta(5) = \frac{32}{49}\pi^4 \left[\frac{\zeta(3)}{6\pi^2} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)} \frac{\zeta(2k)}{2^{2k}} \right], \tag{6.8}$$

which, when subtracted from (6.7), yields

$$\begin{aligned}\zeta(3) = -\frac{120}{1573}\pi^2 \sum_{k=0}^{\infty} \frac{8576k^2+24\,286k+17\,283}{(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)(2k+6)(2k+7)} \\ \cdot \frac{\zeta(2k)}{2^{2k}}. \end{aligned} \tag{6.9}$$

The series in (6.9) has the same rate of convergence as that in (6.4), but (6.9) obviously does not contain any finite part analogous to that occurring on the right-hand side of (6.4).

Remark 4. If we apply the above procedure in conjunction with the known results given earlier by Srivastava and Tsumura [31, p. 331, Lemma 5; p. 332, Theorem 4] instead of (2.36) and (2.38), we can get several other series representations for $\zeta(2n+1)$ with their general terms having the order estimate: $\mathrm{O}(k^{-2n-m}\ l^{-2k})$ $(k \to \infty)$, where $l = 3, 4, 6$ and m is an arbitrary natural number. But these formulas would contain terms involving such values of Hurwitz's Zeta function $\zeta(s,a)$ as, for example, $\zeta(2n+m+1, \frac{1}{3})$, $\zeta(2n+m+1, \frac{1}{6})$, and so on. As a concrete instance, we can thus get

$$\begin{aligned}\zeta(3) = \frac{\pi^2}{399}\Bigg[&\frac{1}{60}\left(270H_4 - 29H_5 - 241\log\frac{\pi}{3}\right) - \frac{42\,224\sqrt{3}}{135}B_6\pi \\ &+ \frac{29\sqrt{3}}{96\pi^5}\left\{\zeta\left(6,\frac{1}{3}\right) + \zeta\left(6,\frac{1}{6}\right)\right\} \\ &+ 4\sum_{k=1}^{\infty}\frac{108k+241}{(2kt)(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)}\,\frac{\zeta(2k)}{6^{2k}}\Bigg]. \end{aligned} \tag{6.10}$$

The series in (6.10) converges more rapidly than those in the earlier results of Srivastava and Tsumura [31, p. 332, Theorem 4].

7. Rapidly convergent series representations for the Dirichlet functions $L(2n,\chi)$ and $L(2n+1,\chi)$

For a non-trivial primitive Dirichlet character χ of modulus q, let $L(s,\chi)$ denote the Dirichlet L-function defined (for $\Re(s) > 1$) by

$$L(s,\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \quad (\Re(s) > 1), \tag{7.1}$$

and (for $\Re(s) \leqq 1$) by its analytic continuations (see, e.g., [38, Chapter 4]). Then, in terms of the familiar generalized Bernoulli numbers $B_{n,\chi}$ defined by means of the generating function:

$$\sum_{k=1}^{q}\frac{\chi(k)te^{kt}}{e^{qt}-1} = \sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!} \quad \left(|t| < \frac{2\pi}{q}\right), \tag{7.2}$$

it is fairly well known that

$$L(2n+1,\chi) = \frac{(-1)^n \mathrm{i}\tau(\chi)}{2(2n+1)!}\left(\frac{2\pi}{q}\right)^{2n+1} B_{2n+1,\overline{\chi}} \quad (n \in \mathbb{N}_0) \tag{7.3}$$

and

$$L(2n,\chi) = \frac{(-1)^{n-1}\tau(\chi)}{2(2n)!}\left(\frac{2\pi}{q}\right)^{2n} B_{2n,\overline{\chi}} \quad (n \in \mathbb{N}) \tag{7.4}$$

for χ with $\chi(-1) = -1$ and 1, respectively; here $\mathrm{i} := \sqrt{-1}$ and $\tau(\chi)$ is the Gauss sum defined by

$$\tau(\chi) := \sum_{k=1}^{q} \chi(k) \exp\left(\frac{2k\pi \mathrm{i}}{q}\right). \tag{7.5}$$

But no such simple (and useful) representations exist for $L(2n,\chi)$ and $L(2n+1,\chi)$ for χ with $\chi(-1) = -1$ and 1, respectively.

Recently, by making use of the Mellin transformation technique, Katsurada [16] proved the following series representations (see [16, p. 82, Theorem 3]): *Let* $u \in \mathbb{R}$ *with* $|u| \leqq 1$. *If* $\chi(-1) = 1$ *and* $\chi \neq 1$, *then*

$$\begin{aligned} nL(2n+1,\chi) &- n\sum_{l=1}^{\infty} \frac{\chi(l)}{l^{2n+1}} \cos\left(\frac{2l\pi u}{q}\right) - \frac{\pi u}{q}\sum_{l=1}^{\infty} \frac{\chi(l)}{l^{2n}} \sin\left(\frac{2l\pi u}{q}\right) \\ &= (-1)^n \left(\frac{2\pi u}{q}\right)^{2n} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1}k}{(2n-2k)!} \frac{L(2k+1,\chi)}{(2\pi u/q)^{2k}} \right. \\ &\qquad \left. + \frac{\tau(\chi)}{q}\sum_{k=1}^{\infty} \frac{(2k)!}{(2n+2k)!} L(2k,\overline{\chi}) u^{2k}\right] \quad (n \in \mathbb{N}). \end{aligned} \tag{7.6}$$

Furthermore, *if* $\chi(-1) = -1$, *then*

$$\begin{aligned} L(2n,\chi) &- \sum_{l=1}^{\infty} \frac{\chi(l)}{l^{2n}} \cos\left(\frac{2l\pi u}{q}\right) \\ &= (-1)^n \left(\frac{2\pi u}{q}\right)^{2n-1} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{(2n-2k)!} \frac{L(2k,\chi)}{(2\pi u/q)^{2k-1}} \right. \\ &\qquad \left. + \frac{2\tau(\chi)\mathrm{i}}{q}\sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} L(2k+1,\overline{\chi}) u^{2k+1}\right] \quad (n \in \mathbb{N}). \end{aligned} \tag{7.7}$$

In this section, we present two (presumably new) members of the class of the series representations (7.6) and (7.7) (see [32,33]). The infinite series occurring in these two members ([see Eqs. (7.8) and (7.9)] below) converge remarkably faster than those in (7.6) and (7.7).

Throughout this section, we assume that χ is a non-trivial primitive Dirichlet character of modulus q. We define the sequence $\{\beta_{n,\chi}(x)\}_{n=0}^{\infty}$ by means of the generating function [cf. Eq. (7.2)]:

$$F(x,t,\chi) := \sum_{k=1}^{q} \frac{\chi(k)tx^{q-k}\mathrm{e}^{kt}}{\mathrm{e}^{qt}-x^{q}} = \sum_{n=0}^{\infty} \beta_{n,\chi}(x)\frac{t^n}{n!} \qquad \left(|t| < \frac{2\pi}{q}\right)$$
$$(1 \leqq x \leqq 1+c;\ c>0), \tag{7.8}$$

so that, clearly,

$$\beta_{n,\chi}(1) = B_{n,\chi} \quad (n \in \mathbb{N}_0). \tag{7.9}$$

Since $\Sigma_{k=1}^{q}\chi(k) = 0$ and since the zeros of $\mathrm{e}^{qt} - x^q$ are given by

$$t = \frac{2n\pi\mathrm{i}}{q} + \log x \quad (n \in \mathbb{Z}), \tag{7.10}$$

the radius of convergence of the series in (7.8) is at least $2\pi/q$. Hence, by the Cauchy–Hadamard theorem for absolute convergence (cf., e.g., [39, p. 30]), we have

Lemma 4. *Let the sequence* $\{\beta_{n,\chi}(x)\}_{n=0}^{\infty}$ *be defined by* (7.8). *Then there exists some non-negative real number* κ *such that*

$$\liminf_{n\to\infty}\left(\frac{|\beta_{n,\chi}(x)|}{n!}\right)^{-1/n} = \frac{2\pi}{q} + \kappa \quad (\kappa \geqq 0). \tag{7.11}$$

We now consider the following Dirichlet series [cf. Eq. (7.1)]:

$$\omega(s,x,\chi) := \sum_{n=1}^{\infty} \frac{x^{-n}\chi(n)}{n^s} \quad (1 \leqq x \leqq 1+c;\ c>0), \tag{7.12}$$

so that, clearly,

$$\omega(s,1,\chi) = L(s,\chi). \tag{7.13}$$

In case $1 < x \leqq 1+c$ $(c>0)$, we can see that the function $\omega(s,x,\chi)$ is holomorphic on the whole complex s-plane.

It is not difficult to prove Lemmas 5 and 6 below (cf. [33]).

Lemma 5. *Let* $\beta_{n,\chi}(x)$ *and* $\omega(s,x,\chi)$ *be defined by* (7.8) *and* (7.12), *respectively. Then*

$$\omega(1-n,x,\chi) = -\frac{\beta_{n,\chi}(x)}{n} \quad (n \in \mathbb{N}). \tag{7.14}$$

Lemma 6. *Let* $n \in \mathbb{N}$ *and* $|\theta| < 2\pi/q$ $(\theta \in \mathbb{R})$.

(1) *If* $\chi(-1)=1$ *and* $\chi\neq 1$, *then*

$$\sum_{k=1}^{\infty}\frac{x^{-k}\chi(k)}{k^{2n+2}}\sin(k\theta)=\sum_{k=0}^{n}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\omega(2n-2k+1,x,\chi)-\sum_{k=n+1}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\frac{\beta_{2k-2n,\chi}(x)}{2k-2n}\quad (1<x\leqq 1+c;\ c>0). \tag{7.15}$$

(2) *If* $\chi(-1)=-1$, *then*

$$\sum_{k=1}^{\infty}\frac{x^{-k}\chi(k)}{k^{2n+1}}\sin(k\theta)=\sum_{k=0}^{n-1}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\omega(2n-2k,x,\chi)-\sum_{k=n}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\cdot\frac{\beta_{2k-2n+1,\chi}(x)}{2k-2n+1}\quad (1<x\leqq 1+c;\ c>0). \tag{7.16}$$

By applying Lemma 6, we obtain the following result (cf. [32,33]).

Proposition 3. *Let* $u\in\mathbb{R}$ *and* $|u|\leqq 1$.
(1) *If* $\chi(-1)=1$ *and* $\chi\neq 1$, *then*

$$L(2n+1,\chi)-\frac{q}{2\pi u}\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n+2}}\sin\left(\frac{2l\pi u}{q}\right)=(-1)^{n+1}\left(\frac{2\pi u}{q}\right)^{2n}\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{L(2k+1,\chi)}{(2\pi u/q)^{2k}}+\frac{2\tau(\chi)}{q}\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}L(2k,\overline{\chi})u^{2k}\right]\quad (n\in\mathbb{N}). \tag{7.17}$$

(2) *If* $\chi(-1)=-1$, *then*

$$L(2n,\chi)-\frac{q}{2\pi u}\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n+1}}\sin\left(\frac{2l\pi u}{q}\right)=(-1)^{n+1}\left(\frac{2\pi u}{q}\right)^{2n-1}\left[\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n+2k+1)!}\frac{L(2k,\chi)}{(2\pi u/q)^{2k-1}}-\frac{2\tau(\chi)i}{q}\cdot\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k+1)!}L(2k+1,\overline{\chi})u^{2k+1}\right]\quad (n\in\mathbb{N}). \tag{7.18}$$

Remark 5. The infinite series occurring on the right-hand sides of (7.17) and (7.18) *obviously* converge more rapidly than the corresponding ones in (7.6) and (7.7), respectively.

8. Symbolic and numerical computations using Mathematica (Version 4.0)

In this concluding section, we choose first to summarize below the results of our symbolic and numerical computations with the series in (5.21) using *Mathematica* (Version 4.0) for Linux:

$$\mathrm{In}[1] := (98k+121)\mathrm{Zeta}[2k]/((2k+1)(2k+2)(2k+3)(2k+4) \cdot (2k+5)2^{\urcorner}(2k))$$

$$\mathrm{Out}[1] = \frac{(121+98k)\mathrm{Zeta}[2k]}{2^{2k}(1+2k)(2+2k)(3+2k)(4+2k)(5+2k)}$$

$$\mathrm{In}[2] := \mathrm{Sum}[\%, \{k, 1, \mathrm{Infinity}\}]//\mathrm{Simplify}$$

$$\mathrm{Out}[2] = \frac{121}{240} - \frac{23\mathrm{Zeta}[3]}{6\mathrm{Pi}^2}$$

$$\mathrm{In}[3] := \mathrm{N}[\%]$$

$$\mathrm{Out}[3] = 0.0372903$$

$$\mathrm{In}[4] := \mathrm{Sum}[\mathrm{N}[\%1]//\mathrm{Evaluate}, \quad \{k, 1, 50\}]$$

$$\mathrm{Out}[4] = 0.0372903$$

$$\mathrm{In}[5] := \mathrm{N}\ \mathrm{Sum}[\%1//\mathrm{Evaluate}, \quad \{k, 1, \mathrm{Infinity}\}]$$

$$\mathrm{Out}[5] = 0.0372903$$

Remark 6. Since $\zeta(0) = -1/2$, Out[2] evidently validates the series representation (5.21) *symbolically*. Furthermore, our *numerical* computations in Out[3], Out[4], and Out[5], together, exhibit the fact that only 50 terms ($k = 1$ to 50) of the series in (5.21) can produce an accuracy of seven decimal places.

Our symbolic and numerical computations with the series in (6.9) using *Mathematica* (Version 4.0) for Linux lead to the following table:

Number of terms	Precision of computation
4	6
10	11
20	18
50	38
98	69

In fact, since the general term of the series in (6.9) has the order estimate:

$$\mathrm{O}(2^{-2k}\ k^{-5}) \quad (k \to \infty),$$

for getting p *exact* digits, we must have

$$2^{-2k}\ k^{-5} < 10^{-p}.$$

Solving this inequality *symbolically*, we find that

$$k \cong \frac{5}{\log 4}\text{ProductLog}\left(\frac{10^{p/5}\log 4}{5}\right),$$

where the function ProductLog (also known as Lambert's function) is the solution of the equation:

$$x\mathrm{e}^{x} = a.$$

Here are some relevant details about our symbolic and numerical computations with the series in (6.9) using *Mathematica* (Version 4.0) for Linux.

$$\text{In}[1] := \text{expr} = (8576k\,\hat{}\,2 + 24286k + 17283)\text{Zeta}[2k]/((2k+1)(2k+2)$$
$$\cdot\ (2k+3)(2k+4)(2k+5)(2k+6)(2k+7)2\,\hat{}\,(2k))$$

$$\text{Out}[1] = \frac{(17283 + 24286k + 8576k^2)\text{Zeta}[2k]}{2^{2k}(1+2k)(2+2k)(3+2k)(4+2k)(5+2k)(6+2k)(7+2k)}$$

$$\text{In}[2] := \text{Sum}[\text{expr}, \{k, 0, \text{infinity}\}]//\text{Simplify}$$

$$\text{Out}[2] = -\frac{1573}{120\text{Pi}^2}\text{Zeta}[3]$$

$$\text{In}[3] := \text{N}[-1573/(120\text{Pi}\,\hat{}\,2)\text{Zeta}[3], 50]$$
$$-\ \text{Sum}[\text{expr}, \{k, 0, 50\}]$$

$$\text{Out}[3] = 4.00751120011 \cdot 10^{-38}$$

$$\text{In}[4] := \text{N}[-1573/(120\text{Pi}\,\hat{}\,2)\text{Zeta}[3], 100]$$
$$-\ \text{Sum}[\text{expr}, \{k, 0, 50\}]$$

$$\text{Out}[4] = 4.0075112001\langle\text{skip}\rangle 3481 \cdot 10^{-38}$$

Thus the result does not change appreciably when we increase the precision of computation of the symbolic result from 50 to 100. This is expected, because of the following numerical computation of the last term for $k = 50$:

$$\text{In}[5] := \text{N}[\text{expr}/.k \to 50, 50]$$

$$\text{Out}[5] = 1.3608530374922376861443887454551514233575702860179 \cdot 10^{-37}$$

Acknowledgements

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

References

[1] V.S. Adamchik, H.M. Srivastava, Some series of the Zeta and related functions, Analysis 18 (1998) 131–144.

[2] R. Apéry, Irrationalité de ζ (2) et ζ (3), in Journées Arithmétiques de Luminy (Colloq. Internat. CNRS, Centre Univ. Luminy, Luminy, 1978), Astérisque 61, Soc. Math. France, Paris, 1979, pp. 11–13.

[3] R. Ayoub, Euler and the Zeta function, Amer. Math. Monthly 81 (1974) 1067–1086.

[4] J.M. Borwein, D.M. Bradley, R.E. Crandall, Computational strategies for the Riemann Zeta function, J. Comput. Appl. Math. 121 (2000) 247–296.

[5] M.-P. Chen, H.M. Srivastava, Some families of series representations for the Riemann $\zeta(3)$, Resultate Math. 33 (1998) 179–197.

[6] D. Cvijović, J. Klinowski, New rapidly convergent series representations for $\zeta(2n+1)$, Proc. Amer. Math. Soc. 125 (1997) 1263–1271.

[7] A. Dąbrowski, A note on the values of the Riemann Zeta function at positive odd integers, Nieuw Arch. Wisk. 4 (14) (1996) 199–207.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, vol. 1, McGraw-Hill Book Company, New York, Toronto, and London, 1953.

[9] J.A. Ewell, A new series representation for $\zeta(3)$, Amer. Math. Monthly 97 (1990) 219–220.

[10] J.A. Ewell, On the Zeta function values $\zeta(2k+1)$, $k = 1, 2, \ldots$, Rocky Mountain J. Math. 25 (1995) 1003–1012.

[11] M.L. Glasser, Some integrals of the arctangent function, Math. Comput. 22 (1968) 445–447.

[12] R.W. Gosper Jr., A calculus of series rearrangements, in: J.F. Traub (Ed.), Algorithms and Complexity: New Directions and Recent Results, Academic Press, New York, London, and Toronto, 1976, pp. 121–151.

[13] E.R. Hansen, A Table of Series and Products, Prentice-Hall, Englewood Cliffs, NJ, 1975.

[14] M.M. Hjortnaes, OverfØring av rekken $\Sigma_{k=1}^{\infty}(1/k^3)$ til et bestemt integral, Proceedings of the Twelfth Scandanavian Mathematical Congress (Lund, 10–15 August 1953) Scandanavian Mathematical Society, Lund, 1954, pp. 211–213.

[15] A. Ivić, The Riemann Zeta-Function, John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1985.

[16] M. Katsurada, Rapidly convergent series representations for $\zeta(2n+1)$ and their χ-analogue, Acta Arith. 40 (1999) 79–89.

[17] N. Koblitz, p-Adic Numbers, p-Adic Analysis, and Zeta-Functions, Graduate Texts in Mathematics, vol. 58, Springer-Verlag, New York, Heidelberg, and Berlin, 1977.

[18] W. Magnus, F. Oberhettinger, R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Third Enlarged Edition, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtingung der Anwendungsgebiete, Bd. 52, Springer-Verlag, New York, 1966.

[19] C. Nash, D. O'Connor, Ray–Singer torsion, topological field theories and the Riemann Zeta function at $s = 3$, in: H. Osborn (Ed.), Low-Dimensional Topology and Quantum Field Theory, Proceedings of a NATO Advanced Research Workshop held at the Isaac Newton Institute at Cambridge, UK, 6–12 September 1992, Plenum Press, New York and London, 1993, pp. 279–288.

[20] C. Nash, D. O'Connor, Determinants of Laplacians, the Ray–Singer torsion on lens spaces and the Riemann Zeta function, J. Math. Phys. 36 (1995) 1462–1505.

[21] V. Ramaswami, Notes on Riemann's ζ-function, J. London Math. Soc. 9 (1934) 165–169.

[22] J.D. Shallit, K. Zikan, A theorem of Goldbach, Amer. Math. Monthly 93 (1986) 402–403.

[23] H.M. Srivastava, A unified presentation of certain classes of series of the Riemann Zeta function, Riv. Mat. Univ. Parma 4 (14) (1988) 1–23.

[24] H.M. Srivastava, Sums of certain series of the Riemann Zeta function, J. Math. Anal. Appl. 134 (1988) 129–140.

[25] H.M. Srivastava, Certain families of rapidly convergent series representations for $\zeta(2n+1)$, Math. Sci. Res. Hot-Line 1 (6) (1997) 1–6 (Research Announcement).

[26] H.M. Srivastava, Further series representations for $\zeta(2n+1)$, Appl. Math. Comput. 97 (1998) 1–15.

[27] H.M. Srivastava, Some rapidly converging series for $\zeta(2n+1)$, Proc. Amer. Math. Soc. 127 (1999) 385–396.

[28] H.M. Srivastava, Some simple algorithms for the evaluations and representations of the Riemann Zeta function at positive integer arguments, J. Math. Anal. Appl. 246 (2000) 331–351.

[29] H.M. Srivastava, J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston, and London, 2001.

[30] H.M. Srivastava, M.L. Glasser, V.S. Adamchik, Some definite integrals associated with the Riemann Zeta function, Z. Anal. Anwendungen 19 (2000) 831–846.

[31] H.M. Srivastava, H. Tsumura, A certain class of rapidly convergent series representations for $\zeta(2n+1)$, J. Comput. Appl. Math. 118 (2000) 323–335.

[32] H.M. Srivastava, H. Tsumura, New rapidly convergent series representations for $\zeta(2n+1)$, $L(2n,\chi)$ and $L(2n+1,\chi)$, Math. Sci. Res. Hot-Line 4 (7) (2000) 17–24 (Research Announcement).

[33] H.M. Srivastava, H. Tsumura, Certain classes of rapidly convergent series representations for $L(2n,\chi)$ and $L(2n+1,\chi)$, Acta Arith. 100 (2001) 195–201.

[34] H.M. Srivastava, H. Tsumura, Inductive construction of rapidly convergent series representations for $\zeta(2n+1)$ (Preprint 2001).

[35] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function, Oxford University (Clarendon) Press, Oxford and London, 1951; second ed., revised by D.R. Heath-Brown, 1986.

[36] F.G. Tricomi, Sulla somma delle inverse delle terze e quinte potenze dei numeri naturali, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 8 (47) (1969) 16–18.

[37] H. Tsumura, On evaluation of the Dirichlet series at positive integers by q-calculation, J. Number Theory 48 (1994) 383–391.

[38] L.C. Washington, Introduction to Cyclotomic Fields, Second ed., Springer-Verlag, New York, Berlin, and Heidelberg, 1997.

[39] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions, Fourth ed., Cambridge University Press, Cambridge, London, and New York, 1927.

[40] J.R. Wilton, A proof of Burnside's formula for $\log\Gamma(x+1)$ and certain allied properties of Riemann's ζ-function, Messenger Math. 52 (1922–1923) 90–93.

[41] E. Witten, On quantum gauge theories in two dimensions, Comm. Math. Phys. 141 (1991) 153–209.

[42] N.-Y. Zhang, K.S. Williams, Some series representations of $\zeta(2n+1)$, Rocky Mountain J. Math. 23 (1993) 1581–1592.