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Some applications of a subordination theorem for a class of analytic functions

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Abstract

By making use of a subordination theorem for analytic functions, we derive several subordination relationships between certain subclasses of analytic functions which are defined by means of the Sălăgean derivative operator. Some interesting corollaries and consequences of our results are also considered.

Keywords: Analytic functions; Univalent functions; Subordinating factor sequence; Sălăgean derivative operator; Hadamard product (or convolution)

1. Introduction, definitions and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ $(0 \leqq \alpha < 1)$ the class of *starlike functions of order* α and the class of *convex functions of order* α, respectively, where (see, for details, [2] and [4]; see also the references cited in *each* of these recent works)

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\quad 0 \leqq \alpha < 1)\right\}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U};\quad 0 \leqq \alpha < 1)\right\}.$$

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Clearly, we have

$$f(z) \in \mathcal{K}(\alpha) \iff zf'(z) \in \mathcal{S}^*(\alpha).$$

Sălăgean [5] introduced the following operator which is popularly known as the *Sălăgean derivative operator*:

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z)$$

and, in general,

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \ \ \mathbb{N} := \{1, 2, 3, \ldots\}).$$

We easily find from (1.1) that

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \quad (f \in \mathcal{A}; n \in \mathbb{N}_0).$$

Let $\mathcal{N}_{m,n}(\alpha, \beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\Re\left(\frac{D^m f(z)}{D^n f(z)}\right) > \beta \left|\frac{D^m f(z)}{D^n f(z)} - 1\right| + \alpha \quad (z \in \mathbb{U}; \ \ 0 \leqq \alpha < 1; \ \beta \geqq 0; \ m \in \mathbb{N}; \ n \in \mathbb{N}_0).$$

Also let $\mathcal{M}^s_{m,n}(\alpha, \beta)$ $(s \in \mathbb{N}_0)$ be the subclasses of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following condition:

$$f(z) \in \mathcal{M}^s_{m,n}(\alpha, \beta) \Longleftrightarrow D^s f(z) \in \mathcal{N}_{m,n}(\alpha, \beta).$$

For $s = 0$, it is easily verified that

$$\mathcal{M}^0_{m,n}(\alpha, \beta) \equiv \mathcal{N}_{m,n}(\alpha, \beta).$$

The function classes $\mathcal{N}_{m,n}(\alpha, \beta)$ and $\mathcal{M}^s_{m,n}(\alpha, \beta)$ were introduced by Eker and Owa [1], who gave the following coefficient inequalities associated with these function classes.

Theorem A (*Eker and Owa [1]*). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{j=2}^{\infty} \left\{\left|j^m - j^n - \alpha j^n\right| + \left(j^m + j^n - \alpha j^n\right) + 2\beta \left|j^m - j^n\right|\right\} \left|a_j\right| \leqq 2(1 - \alpha)$$
$$(0 \leqq \alpha < 1; \ \beta \geqq 0; \ m \in \mathbb{N}; \ n \in \mathbb{N}_0), \tag{1.2}$$

then $f(z) \in \mathcal{N}_{m,n}(\alpha, \beta)$.

Theorem B (*Eker and Owa [1]*). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{j=2}^{\infty} j^s \left\{\left|j^m - j^n - \alpha j^n\right| + \left(j^m + j^n - \alpha j^n\right) + 2\beta \left|j^m - j^n\right|\right\} \left|a_j\right| \leqq 2(1 - \alpha)$$
$$(0 \leqq \alpha < 1; \ \beta \geqq 0; \ m \in \mathbb{N}; \ n \in \mathbb{N}_0), \tag{1.3}$$

then $f(z) \in \mathcal{M}^s_{m,n}(\alpha, \beta)$.

In view of Theorems A and B, we now introduce the subclasses

$$\widehat{\mathcal{N}}_{m,n}(\alpha, \beta) \subset \mathcal{N}_{m,n}(\alpha, \beta) \quad \text{and} \quad \widehat{\mathcal{M}}^s_{m,n}(\alpha, \beta) \subset \mathcal{M}^s_{m,n}(\alpha, \beta),$$

which consist of functions $f(z) \in \mathcal{A}$ whose Taylor–Maclaurin coefficients satisfy the inequalities (1.2) and (1.3), respectively.

In this work, we prove several subordination relationships involving the function classes $\widehat{\mathcal{N}}_{m,n}(\alpha, \beta)$ and $\widehat{\mathcal{M}}^s_{m,n}(\alpha, \beta)$. In our proposed investigation of functions in these subclasses of the normalized analytic function class $\mathcal{A}$, we need the following definitions and results.

Definition 1 (*Hadamard Product or Convolution*). Given two functions f and g in the class $\mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{j=2}^{\infty} b_j z^j, \tag{1.4}$$

the *Hadamard product* (or *convolution*) $f * g$ is defined (as usual) by

$$(f * g)(z) := z + \sum_{j=2}^{\infty} a_j b_j z^j =: (g * f)(z) \quad (z \in \mathbb{U}).$$

Definition 2 (*Subordination Principle*). For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is *subordinate* to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \implies f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence [3, p. 4]:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 3 (*Subordinating Factor Sequence*). A sequence $\{b_j\}_{j=1}^{\infty}$ of complex numbers is said to be a *Subordinating Factor Sequence* if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by

$$\sum_{j=1}^{\infty} a_j b_j z^j \prec f(z) \quad (z \in \mathbb{U};\ a_1 = 1). \tag{1.5}$$

Theorem C (*Wilf [6]*). *The sequence $\{b_j\}_{j=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left(1 + 2\sum_{j=1}^{\infty} b_j z^j\right) > 0 \quad (z \in \mathbb{U}). \tag{1.6}$$

2. Subordination result for the function class $\widehat{\mathcal{N}}_{m,n}(\alpha, \beta)$

Theorem 1. *Let the function $f(z)$ defined by (1.1) be in the class $\widehat{\mathcal{N}}_{m,n}(\alpha, \beta)$. Suppose also that*

$$\mathcal{K} := \mathcal{K}(0)$$

denotes the familiar class of functions $f(z) \in \mathcal{A}$, which are univalent and convex in $\mathbb{U}$. Then

$$\Omega_{m,n}(\alpha, \beta) \cdot (f * g)(z) \prec g(z) \quad (z \in \mathbb{U};\ \ 0 \leqq \alpha < 1;\ \beta \geqq 0;\ \ m \in \mathbb{N};\ n \in \mathbb{N}_0;\ g \in \mathcal{K}) \tag{2.1}$$

and

$$\Re(f(z)) > -\frac{2(1-\alpha) + |2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|} \quad (z \in \mathbb{U}), \tag{2.2}$$

where, for convenience,

$$\Omega_{m,n}(\alpha,\beta) := \frac{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}{2[2(1-\alpha) + |2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|]}. \tag{2.3}$$

The constant factor $\Omega_{m,n}(\alpha,\beta)$ in the subordination result (2.1) cannot be replaced by a larger one.

Proof. Let $f(z) \in \widehat{\mathcal{N}}_{m,n}(\alpha,\beta)$ and suppose that

$$g(z) = z + \sum_{j=2}^{\infty} c_j z^j \in \mathcal{K} := \mathcal{K}(0).$$

Then, for $f \in \mathcal{A}$ given by (1.1), we have

$$\Omega_{m,n}(\alpha,\beta) \cdot (f * g)(z) = \Omega_{m,n}(\alpha,\beta) \cdot \left(z + \sum_{j=2}^{\infty} a_j c_j z^j \right), \tag{2.4}$$

where $\Omega_{m,n}(\alpha,\beta)$ is defined by (2.3). Thus, by Definition 3, the subordination result (2.1) will hold true if the sequence

$$\left\{ \Omega_{m,n}(\alpha,\beta) \cdot a_j \right\}_{j=1}^{\infty} \tag{2.5}$$

is a subordinating factor sequence, with (of course) $a_1 = 1$. In view of Theorem C, this is equivalent to the following inequality:

$$\Re\left(1 + \sum_{j=1}^{\infty} \frac{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}{2(1-\alpha) + |2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|} a_j z^j \right) > 0 \quad (z \in \mathbb{U}). \tag{2.6}$$

Now, since

$$\left| j^m - j^n - \alpha j^n \right| + \left(j^m + j^n - \alpha j^n \right) + 2\beta \left| j^m - j^n \right|$$

is an increasing function of $j \in \mathbb{N}$, we have

$$\begin{aligned}
&\Re\left(1 + \sum_{j=1}^{\infty} \frac{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}{2(1-\alpha) + |2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}\ a_j z^j \right) \\
&\quad = \Re\Bigg(1 + \frac{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}{2(1-\alpha) + |2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}\ a_1 z \\
&\qquad + \frac{1}{2(1-\alpha) + |2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|} \\
&\qquad \cdot \sum_{j=2}^{\infty} \left[|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n| \right] a_j z^j \Bigg) \\
&\quad \geqq 1 - \frac{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}{2(1-\alpha) + |2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}\ r \\
&\qquad - \frac{1}{2(1-\alpha) + |2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|} \\
&\qquad \cdot \sum_{j=2}^{\infty} \left[\left| j^m - j^n - \alpha j^n \right| + \left(j^m + j^n - \alpha j^n \right) + 2\beta|j^m - j^n| \right] |a_j| r^j \\
&\quad > 1 - \frac{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}{2(1-\alpha) + |2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}\ r \\
&\qquad - \frac{2(1-\alpha)}{2(1-\alpha) + |2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}\ r \\
&\quad = 1 - r > 0 \quad (|z| = r < 1),
\end{aligned}$$

where we have also made use of the assertion (1.2) of Theorem A. This evidently proves the inequality (2.6), and hence also the subordination result (2.1) asserted by Theorem 1. The inequality (2.2) asserted by Theorem 1 would follow from (2.1) upon setting

$$g(z) = \frac{z}{1-z} = \sum_{j=1}^{\infty} z^j \in \mathcal{K} := \mathcal{K}(0).$$

Finally, we consider the function $q(z)$ given by

$$q(z) = z - \frac{2(1-\alpha)}{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|}\, z^2$$
$$(m \in \mathbb{N};\ n \in \mathbb{N}_0;\ 0 \leqq \alpha < 1;\ \beta \geqq 0), \tag{2.7}$$

which is a member of the function class $\widehat{\mathcal{N}}_{m,n}(\alpha,\beta)$. Then, by using (2.1), we have

$$\Omega_{m,n}(\alpha,\beta) \cdot q(z) \prec \frac{z}{1-z} \quad (z \in \mathbb{U}),$$

where $\Omega_{m,n}(\alpha,\beta)$ is defined (as before) by (2.3). Moreover, it can easily be verified for the function $q(z)$ given by (2.7) that

$$\min_{z\in\mathbb{U}} \left\{\Re\left(\Omega_{m,n}(\alpha,\beta) \cdot q(z)\right)\right\} = -\frac{1}{2},$$

which evidently completes the proof of Theorem 1. □

Upon setting $\beta = 0$, $n = 0$ and $m = 1$ in Theorem 1, we get the following consequence.

Corollary 1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}^*(\alpha)$ and suppose that $g(z) \in \mathcal{K}$. Then*

$$\left(\frac{2-\alpha}{2(3-2\alpha)}\right) \cdot (f * g)(z) \prec g(z) \quad (z \in \mathbb{U}) \tag{2.8}$$

and

$$\Re(f(z)) > -\frac{3-2\alpha}{2-\alpha} \quad (z \in \mathbb{U}).$$

The constant factor

$$\frac{2-\alpha}{2(3-2\alpha)}$$

in the subordination result (2.8) cannot be replaced by a larger one.

By taking $\beta = 0$, $n = 1$ and $m = 2$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{K}(\alpha)$ and suppose that $g(z) \in \mathcal{K}$. Then*

$$\left(\frac{2-\alpha}{5-3\alpha}\right) \cdot (f * g)(z) \prec g(z) \quad (z \in \mathbb{U}) \tag{2.9}$$

and

$$\Re(f(z)) > -\frac{5-3\alpha}{2(2-\alpha)} \quad (z \in \mathbb{U}).$$

The constant factor

$$\frac{2-\alpha}{5-3\alpha}$$

in the subordination result (2.9) cannot be replaced by a larger one.

3. Subordination result for the function class $\widehat{\mathcal{M}}^s_{m,n}(\alpha, \beta)$

The proof of the following subordination result is much akin to that of Theorem 1 of the preceding section. We, therefore, choose to omit the details involved.

Theorem 2. *Let the function $f(z)$ defined by (1.1) be in the class $\widehat{\mathcal{M}}^s_{m,n}(\alpha, \beta)$. Then*

$$\Lambda^s_{m,n}(\alpha, \beta) \cdot (f * g)(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \left(0 \leqq \alpha < 1; \quad \beta \geqq 0; \quad m \in \mathbb{N}; \quad n \in \mathbb{N}_0; \quad g(z) \in \mathcal{K}\right) \tag{3.1}$$

and

$$\Re(f(z)) > -\frac{(1-\alpha) + 2^{s-1}\{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|\}}{2^{s-1}\{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|\}} \quad (z \in \mathbb{U}), \tag{3.2}$$

where, for convenience,

$$\Lambda^s_{m,n}(\alpha, \beta) := \frac{2^{s-2}\{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|\}}{(1-\alpha) + 2^{s-1}\{|2^m - 2^n - \alpha 2^n| + (2^m + 2^n - \alpha 2^n) + 2\beta|2^m - 2^n|\}}. \tag{3.3}$$

The constant factor $\Lambda^s_{m,n}(\alpha, \beta)$ in the subordination result (3.1) cannot be replaced by a larger one.

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