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p-harmonic diffeomorphisms

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Introduction

In their paper [2], J.M.Coron and F.Hélein pointed out the following fact: if a harmonic map between surfaces satisfies a global hypothesis – namely being a C^1-diffeomorphism – then it is energy minimizing. Later Hélein [5, 6] proved a similar result for harmonic diffeomorphisms defined on a smooth domain of $\mathbf{R}^n$, $n \geq 3$ under the additional assumption that they are, roughly speaking, $SO(n)$-equivariant.

Coron-Hélein's result relies on an essentially two-dimensional technique while those of Hélein are based on one known as calibration or null Lagrangians. This latter technique had been used previously by F.H. Lin [8] in proving that the map $x/|x|$ from the n-dimensional ball to the $(n-1)$-dimensional sphere is energy minimizing.

In this paper, our aim is to establish in the framework of p-harmonic mappings results similar to those in [2] and [5, 6], again using calibration techniques. Let us mention extensions of the work of Lin [8] to the p-harmonic case: for the map $x/|x|$ first, by Avellaneda-Lin [7], and for a similar problem in complex projective space by Nichols [9].

1 Statement of results

In the sequel, Ω will be a smooth bounded domain in $\mathbf{R^n}$ endowed with the canonical metric c, and (M, h) a Riemannian manifold with boundary. The p-energy of a map u from Ω to M is

$$E_p(u) = \int_\Omega |du|^p(x)\mathrm{d}x^1 \wedge \ldots \mathrm{d}x^n,$$

where $|du(x)|$ is the Hilbert-Schmidt norm of du_x taken with respect to the metrics c and h. We also define $H^{1,p}(\Omega, M)$ to be the set of maps from Ω to M with finite p-energy.

A map u in $H^{1,p}(\Omega, M)$ is said to be p-harmonic if it is a critical point of the p-energy, i.e. if for any variation $(u_t)_t$ of u such that u_t and u agree on $\partial\Omega$ we have $\frac{d}{dt}E_p(u_t)_{|t=0} = 0$. Finally, if $\phi : \Omega \to M$ is a fixed map, we denote by $H^{1,p}_\phi(\Omega, M)$ the set of maps in $H^{1,p}(\Omega, M)$ agreeing with ϕ on $\partial\Omega$; the map ϕ will then be called p-minimizing if for all u in $H^{1,p}(\Omega, M)$, $E_p(u) \geq E_p(\phi)$. We are now able to state our first result, concerning 1-harmonic diffeomorphisms:

Theorem 1. *Let $\phi : (\Omega, c) \to (M, h)$ be a 1-harmonic C^1-diffeomorphism, with $\Omega \subset \mathbf{R}^2$, then ϕ is 1-minimizing. Moreover, ϕ is the only minimizer of E_1 in $H^{1,1}_\phi(\Omega, M)$.*

Our second result deals with equivariant diffeomorphisms defined on B^n, the unit ball in $\mathbf{R^n}$. A map $\phi : \mathsf{B}^n \to (M, h)$ is called $SO(n)$-equivariant if, for any $R \in SO(n)$ there exists an isometry γ of (M, h) such that $\phi \circ R = \gamma \circ \phi$.

Theorem 2. *Let $\phi : (\mathsf{B}^n, c) \to (M, h)$ be a p-harmonic, $SO(n)$-equivariant C^1-diffeomorphism with p an integer between 1 and $n-1$, then ϕ is p-minimizing. Moreover, ϕ is the only minimizer of E_p in $H^{1,p}_\phi(\mathsf{B}^n, M)$.*

The simplest example of a map satisfying the hypothesis in Theorems 1 and 2 is the identity map of of (Ω, c). Non trivial examples in the equivariant case have been constructed by A. Fardoun [4].

Before we proceed with the proofs, notice that if $u : (\Omega, c) \to (M, h)$, then $u \circ \phi^{-1} : (\Omega, c) \to (\Omega, \phi^* h)$ has same p-energy as u, so that what we really want to prove in Theorem 1 is that if the *identity map* $I : (\Omega, c) \to (\Omega, g)$ is 1-harmonic, with g a continuous metric, then I is the unique minimizer of E_1 in $H^{1,1}_I(\Omega, \Omega)$.

As for Theorem 2, the additional hypothesis that ϕ be $SO(n)$-equivariant translates into $g = \phi^* h$ is an $SO(n)$-invariant metric on B^n. It suffices then to prove that given any continuous $SO(n)$-invariant metric g on B^n, if the identity map $I : (\mathsf{B}^n, c) \to (\mathsf{B}^n, g)$ is p-harmonic, with $1 \leq p \leq n-1$ an integer, then I is the unique minimizer of E_p in $H^{1,p}_I(\mathsf{B}^n, \mathsf{B}^n)$.

2 1-harmonic diffeomorphisms

Taking into account the above remarks, it suffices to study the case of the identity map $I : (\Omega, c) \to (\Omega, g)$, where g is a continuous riemannian metric on Ω such that I is p-harmonic.

We define the *stress energy* tensor S^p associated to the functional E_p at the map I by

$$S^p = |dI|^{p-2}\left(\frac{1}{p}|dI|^2 c - g\right) = T(g)^{p-2}\left(\frac{1}{p}T(g)^2 c - g\right),$$

where $T(g) = \sum_i g_{ii}$ is the trace of the metric g relative to the canonical metric c. It is then known (see [1] and [3]) that I is p-harmonic if and only if S^p is conserved, i.e. has null divergence. We now consider the n-form ω_p on $\Omega \times \Omega \subset \mathbf{R}^{2n}$ given in cartesian coordinates by

$$\omega_p(x,y) = \sum_{i,k} S^p_{ik}(y) \ \ \mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^{k-1} \wedge \mathrm{d}x^i \wedge \mathrm{d}y^{k+1} \wedge \ldots \wedge \mathrm{d}y^n .$$

We have

$$\mathrm{d}\omega_p(x,y) = -\sum_i (\delta S^p)_i \,(y) \ \ \mathrm{d}x^i \wedge \mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^n ,$$

where δS^p is the divergence of S^p. Therefore we have the

Lemma 1.1. *The identity map $I : (\Omega, c) \to (\Omega, g)$ is p-harmonic if and only if ω_p is a closed form*

Now, to any u mapping Ω to itself let us associate the map

$$\begin{aligned} \tilde{u} : \quad \Omega &\to \Omega \times \Omega \\ x &\to (x, u(x)) \end{aligned}$$

whose image is $\Gamma(u)$, the graph of u. A simple calculation shows that

$$\tilde{I}^* \omega_p = \left(\frac{n}{p} - 1 \right) |dI|^p \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n ,$$

so that

$$\int_{\Gamma(I)} \omega_p = \int_\Omega \tilde{I}^* \omega_p = \left(\frac{n}{p} - 1 \right) E_p(I).$$

But since ω_p is closed, for any u mapping Ω to itself and agreeing with the identity mapping I on the boundary of Ω, we have

$$\int_\Omega \tilde{u}^* \omega_p = \int_{\Gamma(u)} \omega_p = \int_{\Gamma(I)} \omega_p = \left(\frac{n}{p} - 1 \right) E_p(I).$$

In other words, the lagrangian $L_p(u, du)$ defined by

$$\tilde{u}^* \omega_p = L_p(u, du) \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$$

is a *null lagrangian*. Our Theorem 1 is now a consequence of the following lemma.

Lemma 1.2. *Let $n = 2$. For any $u \in H^{1,1}_I(\Omega, \Omega)$, we have the pointwise inequality*

$$|du|(x) \geq L_1(u, du)(x),$$

moreover, $|du| = L_1(u, du)$ holds a.e. in Ω if and only if $u = I$.

Proof. The stress energy tensor in the case $p = 1$, $n = 2$ is given by the 2×2 matrix

$$S^1 = T(g)^{-1/2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix},$$

and the form ω_1, written ω in the sequel, is given by

$$T(g)^{1/2}(y)\omega(x,y) = \mathrm{d}x^1 \wedge (g_{12}(y)\mathrm{d}y^1 + g_{22}(y)\mathrm{d}y^2) - \mathrm{d}x^2 \wedge (g_{11}(y)\mathrm{d}y^1 + g_{12}(y)\mathrm{d}y^2).$$

Therefore, for any $u = (u^1, u^2) \in H_I^{1,1}(\Omega, \Omega)$ and any $x \in \Omega$ we have, writing $y = u(x)$,

$$T(g)^{1/2}(y)\tilde{u}^*\omega(x) = \sum_{i,j \le 2} g_{ij}(y)\partial_i u^j(x)\mathrm{d}x^1 \wedge \mathrm{d}x^2,$$

hence the expression for the null lagrangian

$$T(g)^{1/2}(y)L(u,du)(x) = \sum_{i,j \le 2} g_{ij}(y)\partial_i u^j(x).$$

The right-hand side of this last equation happens to be the scalar product relative to the canonical metric c of the metric g at $y = u(x)$ with the form $T(X,Y) = c(du_x(X), Y)$. We can therefore assume that we are using a frame in which, $g(y)$ is a diagonal matrix:

$$g(y) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

with α, β two positive numbers. We now have

$$L(u,du)(x) = (\alpha + \beta)^{-1/2}(\alpha\ \partial_1 u^1(x) + \beta\ \partial_2 u^2(x)),$$

whereas the energy density is

$$|du|^2(x) = \alpha\left((\partial_1 u^1(x))^2 + (\partial_2 u^1(x))^2\right) + \beta\left((\partial_1 u^2(x))^2 + (\partial_2 u^2(x))^2\right),$$

so that, by a simple calculation,

$$|du|^2 = L(u,du)^2 + \frac{\alpha\beta}{\alpha+\beta}\left(\partial_1 u^1 - \partial_2 u^2\right)^2 + \alpha(\partial_2 u^1)^2 + \beta(\partial_1 u^2)^2.$$

It follows that $|du|(x) \ge L(u,du)(x)$ with equality holding if and only if du_x is a dilatation of $\mathbf{R}^2$. Lemma 2.2 follows at once by noting that if, for all $x \in \Omega$, du_x is a dilatation then u has to be a dilatation itself – in fact the identity map since the two agree on $\partial\Omega$.

3 Equivariant diffeomorphisms

We now consider the case where $\Omega = \mathsf{B}^n$, the unit ball in $\mathbf{R}^n$, and g is a $SO(n)$-invariant continuous metric on B^n – i.e. for any $R \in SO(n)$, $R^* g = g$. Finally, we assume that the identity map $I : (\mathsf{B}^n, c) \to (\mathsf{B}^n, g)$ is p-harmonic with p an integer strictly less than n.

The fact that g is $SO(n)$-equivariant implies that it can be written as,

$$g(x) = g_{\|}(r)\mathrm{d}r^2 + g_{\perp}(r)r^2\mathrm{d}\sigma^2,$$

where $\mathrm{d}\sigma^2$ is the canonical metric on the unit sphere in $\mathbf{R}^n$, $r = |x|$, and $g_{\|}$, $g_{\perp}$ are two continuous functions of r.

The stress-energy tensor S^p is now given by

$$S^p = T(g)^{\frac{p-2}{2}} \left[\left(\frac{1}{p} T(g) - g_{\|} \right) \mathrm{d}r^2 + \left(g_{\|} - g_{\perp} \right) r^2 \mathrm{d}\sigma^2 \right],$$

with $T(g) = g_{\|} + (n-1)g_{\perp}$. A straightforward calculation then gives

Lemma 1.3. *The identity map $I : (\mathsf{B}^n, c) \to (\mathsf{B}^n, g)$ is p-harmonic if and only if*

$$\left(T(g)^{\frac{p-2}{2}} \left(\frac{1}{p} T(g) - g_{\|} \right) \right)' = \frac{n-1}{r} T(g)^{\frac{p-2}{2}} \left(g_{\|} - g_{\perp} \right),$$

where $f'(r)$ is the derivative of f in the sense of distributions with respect to r.

The proof of Theorem 2 will follow – as in the previous section – from the existence of an n-form ω on $\mathsf{B}^n \times \mathsf{B}^n$ that “calibrates” the graph of the identity map – e.g. a form such that

i) $\mathrm{d}\omega = 0$
ii) $\int_{\Gamma(I)} \omega = E_p(I)$
iii) $\int_{\Gamma(u)} \omega \leq E_p(u)$, for all $u \in H_I^{1,p}(\mathsf{B}^n, \mathsf{B}^n)$,

equality holding in iii) if and only if $u = I$.

In search of ω

The cartesian coordinates on $\mathsf{B}^n \times \mathsf{B}^n$ are still $(x^1, \ldots, x^n, y^1, \ldots, y^n)$. For property iii) above to hold, the form ω should certainly be homogeneous of degree p in the $\mathrm{d}y^i$ ’s. It is also very reasonable – considering the symetries of the problem – to require that ω be invariant by the action of $SO(n)$ on $\mathsf{B}^n \times \mathsf{B}^n$. These two simple requirements leave in fact little choice.

Let us introduce the following notations: for any multi-index $I = \{i_1, \ldots, i_k\}$ subset of $\{1, \ldots, n\}$, with $i_1 < i_2 < \cdots < i_n$, write

$$\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \ldots \wedge \mathrm{d}x^{i_k}.$$

For any integer $1 \leq s \leq k$, and $J = \{j_1 < j_2 < \cdots < j_s\} \subset I$, let $(\mathrm{d}x^I)_J$ be the k-form on $\mathsf{B}^n \times \mathsf{B}^n$ obtained from $\mathrm{d}x^I$ by replacing the factors $\mathrm{d}x^{j_1},\ldots,\mathrm{d}x^{j_s}$ there by $\mathrm{d}y^{j_1},\ldots,\mathrm{d}y^{j_s}$, respectively. The sum of the $(\mathrm{d}x^I)_J$'s where J runs over the set of all s-subindices of I – e.g. $J \in \{\{j_1 < \cdots < j_s\} \subset I\}$— is denoted by $(\mathrm{d}x^I)_s$. For instance, we have

$$(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3)_1 = \mathrm{d}y^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \mathrm{d}x^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}x^3 + \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^3.$$

Now for a general k-form on B^n, say $\tau = \sum_{I \in I_k} \tau_I(x)\mathrm{d}x^I$, where $I_k = \{\{i_1, \ldots, i_k\};\ 1 \leq i_1 < \cdots < i_k \leq n\}$ — and for any $s \leq k$ – we define a new k-form on $\mathsf{B}^n \times \mathsf{B}^n$ given by

$$(\tau)_s(x, y) = \sum_{I \in I_k} \tau_I(y)(\mathrm{d}x^I)_s,$$

while we set $(\tau)_0 = \sum_{I \in I_k} \tau_I(y)\mathrm{d}x^I$. The form $(\tau)_s$ is then of degree s in the $\mathrm{d}y^i$'s and of degree $k - s$ in the $\mathrm{d}x^i$'s.

Set $\sigma = *(r\mathrm{d}r) = *\sum_i x^i \mathrm{d}x^i$, where $*$ is the hodge operator with respect to the canonical metric on B^n. We will look for an ω of the form

$$\omega(x, y) = \alpha(\rho)\rho\mathrm{d}\rho \wedge (\sigma)_{p-1} + \beta(\rho)(r\mathrm{d}r)_0 \wedge (\sigma)_p,$$

where $\rho = |y|$ and therefore $\rho\mathrm{d}\rho = \sum_i y^i \mathrm{d}y^i$.

It is clear that such an ω has the required degree in the $\mathrm{d}y^i$'s and also the $SO(n)$-invariance. Besides, as a direct calculation shows,

$$\begin{aligned} \mathrm{d}(\sigma)_s &= (s + 1)(\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n)_{s+1} \\ &= \frac{s + 1}{\rho^2} \left(\rho\mathrm{d}\rho \wedge (\sigma)_s + (r\mathrm{d}r)_0 \wedge (\sigma)_{s+1}\right), \end{aligned}$$

and

$$\mathrm{d}(r\mathrm{d}r)_0 \wedge (\sigma)_s = \frac{1}{\rho^2}(r\mathrm{d}r)_0 \wedge \rho\mathrm{d}\rho \wedge (\sigma)_s,$$

so that

$$\mathrm{d}\omega = \frac{1}{\rho^2} \left(p(\alpha - \beta) - \rho\beta'\right) (r\mathrm{d}r)_0 \wedge \rho\mathrm{d}\rho \wedge (\sigma)_p.$$

Therefore, ω is closed if and only if

$$\rho\beta' = p(\alpha - \beta).$$

Lemma 3.1 prompts us to choose as α and β the functions:

$$\beta = T(g)^{\frac{p-2}{2}} \left(\frac{1}{p}T(g) - g_{\|}\right),$$

$$\alpha = \frac{n - 1}{p} T(g)^{\frac{p-2}{2}} \left(g_{\|} - g_{\perp}\right) + \beta = \frac{n - p}{p} T(g)^{\frac{p-2}{2}} g_{\|}.$$

With this choice, ω is closed if and only if the identity map $I : (\mathsf{B}^n, c) \to (\mathsf{B}^n, g)$ is p-harmonic. Moreover,

$$\tilde{I}^*\omega = \frac{1}{p} C^p_{n-1} T(g)^{\frac{p}{2}} \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n = \frac{1}{p} C^p_{n-1} |dI|^p \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n,$$

so that

$$\int_{\Gamma(I)} \omega = \int_{\mathsf{B}^n} \tilde{I}^*\omega = \frac{1}{p} C^p_{n-1} E_p(I).$$

Things are beginning to look good, but we still have to check property iii) above (up to the factor $\frac{1}{p} C^p_{n-1}$). Namely, let $L(u, du)$ be the lagrangian defined for any $u \in H^{1,p}_I(\mathsf{B}^n, \mathsf{B}^n)$ by

$$\tilde{u}^*\omega = L(u, du)\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n,$$

then we have the

Lemma 1.4. *For all $u \in H^{1,p}_I(\mathsf{B}^n, \mathsf{B}^n)$,*

$$L(u, du) \leq \frac{C^p_{n-1}}{p} |du|^p.$$

Moreover, equality holds a.e. if and only if $u = I$ in B^n.

Proof. Because of the $SO(n)$-invariance of ω and $|du|^p$, it suffices to check the inequality at a point $x = (r, 0, \ldots, 0)$ where the covectors $\mathrm{d}r$ and $\mathrm{d}x^1$ coincide. Then we have

$$L(u, du) = \sum_{I \in I^1_p} \alpha(|u(x)|) M_I + \sum_{I \in I^0_p} \beta(|u(x)|) M_I,$$

where I^1_p is the set of ordered p-indices $i_1 < \ldots < i_p$ such that $i_1 = 1$, I^0_p is the set of all other ordered p-indices, and where M_I – with $I = \{i_1 < \ldots < i_p\}$ – is the principal p-minor formed from the lines and columns of indices $i_1, \ldots, i_p$ of the jacobian matrix du_x, e.g. $M_I = \left|\partial_{i_l} u^{i_k}(x)\right|_{1 \leq k, l \leq p}$.

Hadamard's inequality now tells us that a determinant is bounded from above by the product of the euclidean norms of its columns, so that

$$M_I \leq \prod_{l=1}^{p} \left(\sum_{k \leq p} \left(\partial_{i_l} u^{i_k}(x)\right)^2 \right)^{1/2} \leq |\partial_{i_1} u(x)| \cdots |\partial_{i_p} u(x)| = C^I.$$

On the other hand, we claim that $\alpha(\rho)$ and $\beta(\rho)$ are positive: this is clear for α and we have from the closure condition for ω

$$(\rho^p \beta)' = \rho^{p-1}(\rho\beta' + p\beta) = p\rho^{p-1}\alpha > 0.$$

and therefore $\beta \geq 0$. Then we can deduce the inequality

$$L(u, du) \leq \sum_{I \in I_p^1} \alpha(|u(x)|)C^I + \sum_{I \in I_p^0} \beta(|u(x)|)C^I,$$

while on the other hand we have

$$|du|^p(x) = \left(g_{\|}(|u(x)|)|\partial_1 u(x)|^2 + g_{\perp}(|u(x)|) \sum_{i \geq 2} |\partial_i u(x)|^2 \right)^{p/2}.$$

Let $\theta = T(g)^{-1}(x) g_{\|}(|u(x)|)$, so that $0 < \theta < 1$. For $1 \leq i \leq n$, write $\lambda_i = T(g)^{1/2}(x)|\partial_i u(x)|$, and if $I = \{i_1, \ldots, i_p\}$, $\lambda^I = \lambda_{i_1} \ldots \lambda_{i_p}$. The following holds:

$$L(u, du)(x) \leq f(\lambda_1, \ldots, \lambda_n),$$

with

$$f(\lambda_1, \ldots, \lambda_n) = \frac{n-p}{p} \theta \sum_{I \in I_p^1} \lambda^I + (\frac{1}{p} - \theta) \sum_{I \in I_p^0} \lambda^I,$$

and

$$\frac{C_{n-1}^p}{p} |du|^p(x) = g(\lambda_1, \ldots, \lambda_n),$$

with

$$g(\lambda_1, \ldots, \lambda_n) = \frac{C_{n-1}^p}{p} \left(\theta \lambda_1^2 + \frac{1-\theta}{n-1} \sum_{i=2}^n \lambda_i^2 \right)^{p/2}.$$

It remains to compare two functions f and g, both homogeneous of degree p, hoping that g is always greater than f. To actually prove this, we look for the maximum of f under the constraint g = Constant. It is not difficult to see that this maximum must be achieved at a point of the form $(\lambda, \mu, \ldots, \mu) \in \mathbf{R^n}$, with λ, μ two positive numbers. Hence, it suffices to check the inequality at such points, and furthermore – because f and g are homogeneous – we can assume $\mu = 1$. To sum up, we are left with proving that $f(\lambda, 1, \ldots, 1) \leq g(\lambda, 1, \ldots, 1)$, which after minor simplifications amounts to

$$1 + p\theta(\lambda - 1) \leq (1 + \theta(\lambda^2 - 1))^{p/2},$$

λ being a positive number. This last inequality is clear by noting that the right-hand side is a convex function of λ when $p \geq 1$, and that the linear term of its Taylor expansion at $\lambda = 1$ is precisely the left-hand side. Moreover, one sees that equality holds if and only if $\lambda = \mu = 1$.

We now have proved the following inequality at x:

$$L(u, du)(x) \leq \frac{C_{n-1}^p}{p} |du|^p(x),$$

where equality holds if and only if $|\partial_1 u|(x) = \cdots = |\partial_n u|(x) = c$ and for all $I \in I_p$, $M_I = C^I = c^p$. These two conditions are in fact equivalent to $du(x)$ being a dilatation of $\mathbf{R^n}$. Indeed, take $1 \leq i \neq j \leq n$, and $I = \{i_1 < \cdots < i_p\}$ such that $i \in I$ and $j \notin I$, then

$$c^p = M_I \leq \prod_{l=1}^{p} \left(\sum_{k \leq p} \left(\partial_{i_l} u^{i_k}(x)\right)^2 \right)^{1/2} \leq \left(\sum_{k \leq p} \left(\partial_i u^{i_k}(x)\right)^2 \right)^{1/2} c^{p-1} \leq$$

$$\leq \left(c^2 - (\partial_i u^j(x))^2\right)^{1/2} c^{p-1}.$$

Therefore $M_I = c^p$ for all I implies $\partial_i u^j(x) = 0$ forall $i \neq j$. Hence, the matrix du_x is diagonal and for any $1 \leq i \leq n$ we have $|\partial_i u^i(x)| = c$. In fact, since the p-minors M_I are all positive, the $\partial_i u^i(x)$'s must all have the same sign, so that $du(x) = \pm c(x)I$, for all x. As in Lemma 2.2, this implies that $u = I$.

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