Relation Between Constants Connected with the Zeta Function

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In this paper, we shall give a relation between the Stieltjes' Constants, the derivatives of the zeta function at 0 and a new set of constants closely related to the zeta function. The Stieltjes' Constants, denoted γ_n are defined as the coefficients in the Laurent expansion of $\zeta(s)$ about $s = 1$:

$$\zeta(1+s) = \frac{1}{s} + \sum_{n=0}^{\infty} (-1)^n \gamma_n \frac{s^n}{n!}.$$

Clearly, $\gamma_0 = \gamma$, where γ is Euler's Constant. We shall denote the derivatives of the zeta function at zero by g_n, so

$$\zeta(s) = \sum_{n=0}^{\infty} g_n \frac{s^n}{n!}$$

Clearly, $g_0 = -1/2$ and $g_1 = -\log(2\pi)/2$. The new set of constants that we will link to γ_n and g_n will be denoted P_n and are defined for $n > 1$ by

$$P_n = \sum_{\rho} \rho^{-n}$$

where the sum is over the nontrivial zeros of the zeta function. It turns out that if we let A_n, B_n and C_n be the collection of γ_k, g_k and P_k for $1 \leq k \leq n$ then given any one of A_n, B_n or C_n the others can be determined.

The two important formula which will be used in this paper are:

- The product for the zeta function

$$\zeta(s) = \frac{e^{cs}}{2(s-1)\Gamma(s/2+1)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \tag{1}$$

where the product is over the nontrivial zeros and the constant c is given by $\log(2\pi) - 1 - \gamma/2$

- The formal identity for exponentiation of a power series

$$\exp\left[b_0 + \sum_{n=1}^{\infty} \frac{b_n}{n} x^n\right] = e^{b_0}\left[1 + \sum_{n=1}^{\infty} [b_1, -b_2, \ldots, (-1)^{n+1} b_n] \frac{x^n}{n!}\right] \qquad (2)$$

where the symbol $[a_1, a_2, \ldots, a_n]$ is given by the $n \times n$ determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \ldots & a_n \\ (n-1) & a_1 & a_2 & a_3 & \ldots & a_{n-1} \\ 0 & (n-2) & a_1 & a_2 & \ldots & a_{n-2} \\ 0 & 0 & (n-3) & a_1 & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & a_1 \end{vmatrix}$$

Clearly, we have

$$\log\left((s-1)\zeta(s)\right) = -\log 2 + cs - \log\left(\Gamma(s/2+1)\right) + \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right) + \frac{s}{\rho}.$$

Now, it is well known that

$$\log\left(\Gamma(x+1)\right) = -\gamma x + \sum_{n=2}^{\infty} (-1)^n \zeta(n) \frac{x^n}{n}$$

so with a little work, it can be shown that

$$\log\left((s-1)\zeta(s)\right) = -\log 2 + \left(\log(2\pi) - 1\right)s + \sum_{n=2}^{\infty} \left((-1)^{n+1} 2^{-n} \zeta(n) - P_n\right) \frac{s^n}{n}$$

Thus, if $A_1 = \log(2\pi) - 1$, $A_n = (-1)^{n+1} 2^{-n} \zeta(n) - P_n$ and $B_n = [A_1, -A_2, \ldots, (-1)^{n+1} A_n]$ we have

$$2(s-1)\zeta(s) = 1 + \sum_{n=1}^{\infty} B_n \frac{x^n}{n!}.$$

Furthermore, by multiplying both sides by $(1-s)^{-1}$, expanding this into a power series and then equating coefficients, we find that

$$g_n = -\frac{n!}{2}\left(1 + \sum_{k=1}^{n} \frac{B_k}{k!}\right). \tag{3}$$

Thus, we have an expression for g_n in terms of $P_2, \ldots, P_n$. If we have the n expressions for $g_1, \ldots, g_n$, then the 2nd equation will give P_2 in terms of g_1 and g_2. Inserting this into the 3rd equation and solving for P_3 gives P_3 in terms of g_1, g_2 and g_3. Continuing this process, P_n can clearly be expressed in terms of $g_1, \ldots, g_n$.

Now, we have

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s)$$

and so

$$s(s-1)\zeta(1-s) = \frac{e^{cs}\cos(\pi s/2)\Gamma(s+1)}{(2\pi)^s\Gamma(s/2+1)} \prod_\rho \left(1-\frac{s}{\rho}\right)e^{s/\rho}.$$

By taking logarithms and expanding the right hand side into a power series, we have

$$\log\left[s(s-1)\zeta(1-s)\right] = -(1+\gamma)s + \sum_{n=2}^{\infty}\left(P_n - (1-2^{-n})\zeta(n)\right)\frac{s^n}{n}.$$

Thus, letting $A'_1 = -(1+\gamma)$, $A'_n = P_n - (1-2^{-n})\zeta(n)$ and $B'_n = [A'_1, -A'_2, \ldots, (-1)^{n+1}A'_n]$ gives

$$s(s-1)\zeta(1-s) = 1 + \sum_{n=1}^{\infty} B'_n \frac{s^n}{n!}.$$

Again, on multipltying by $(1-s)^{-1}$ and equating coefficients, we find that

$$\gamma_n = -n!\left(1 + \sum_{k=1}^{n+1}\frac{B'_k}{k!}\right).$$

Clearly, this gives γ_n in terms of $P_2, \ldots, P_n$. Likewise, P_n can be solved for in terms of $\gamma_1, \ldots, \gamma_n$.

From the formula connecting g_n with $P_1, \ldots, P_n$ and γ_n with $P_1, \ldots, P_n$, clearly another formula can be derived connecting g_n with $\gamma_1, \ldots, \gamma_n$ or γ_n with $g_1, \ldots, g_n$. These formula can also be obtained more directly from the functional equation for $\zeta(s)$ as well.

Here are some examples of the results just obtained:

$$\gamma_1 = g_2 + \frac{\log(2\pi)^2}{2} - \frac{\gamma^2}{2} + \frac{\pi^2}{24} \tag{4}$$

$$\gamma_1 = -\frac{1}{2}\left(\gamma^2 + 1 + P_2 - \frac{\pi^2}{8}\right) \tag{5}$$

$$g_2 = -\frac{1}{2}\left(\log(2\pi)^2 + 1 + \frac{\pi^2}{24} - P_2\right) \tag{6}$$

$$P_2 = \frac{\pi^2}{8} - 1 - \gamma^2 - 2\gamma_1 \tag{7}$$

and

$$\gamma_2 = -\frac{1}{3} + \frac{\gamma}{2} + \frac{\gamma^3}{6} - \frac{\pi^2\gamma}{16} + \frac{7}{36}\zeta(3) + \frac{\gamma}{2}P_2 - \frac{1}{3}P_3 \tag{8}$$

$$P_3 = -1 - \frac{\gamma^3}{2} + \frac{7}{12}\zeta(3) - 3\gamma\gamma_1 - 3\gamma_2 \tag{9}$$

$$\begin{aligned} g_3 = & -1 - \frac{\pi^2}{8} - \frac{\zeta(3)}{8} - \frac{3}{2}\log(2\pi)\left(1 - \frac{\pi^2}{24}\right) \\ & + \frac{1}{2}(6 - 3\log(2\pi))P_2 + P_3 \end{aligned} \tag{10}$$