An Interesting Series Transformation

Andrew Snowden

Thursday, August 16th, 2001

Throughout this derivation, we assume that the zeta function has only simple zeros.

Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{\sqrt{n}} e^{-nx} \tag{1}$$

where $\mu(n)$ is the mobius function. Taking the Mellin Transform of this series gives

$$\int_0^{\infty} f(t) t^{s-1} \mathrm{d}t = \frac{\Gamma(s)}{\zeta(s+\frac{1}{2})}$$

and so we must have

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{\zeta(s+\frac{1}{2})} x^{-s} \mathrm{d}s.$$

Now, expanding $\left(\zeta(s+\frac{1}{2})\right)^{-1}$ into an Dirichlet series gives (1), so that is not very interesting. Instead, we will calculate the integral by examining the poles of the integrand and summing the residues. The poles of the integrand occur at the poles of the gamma function and the zeros of the zeta function (offset by a half). The zeta function has zeros at every negative even integer (so there is a pole at $s = -\frac{1}{2} - 2n$) and at the zeros in the critical strip (so at $s = \rho - \frac{1}{2}$). The gamma function has a pole at zero and at every negative integer. Therefore, we get

$$f(x) = \frac{1}{\zeta(\frac{1}{2})} + f_1(x) + f_2(x) + f_3(x) \tag{2}$$

where the constant term is due to the pole of the gamma function at 0,

$$f_1(x) = \sum_{\rho} \frac{\Gamma(\rho - \frac{1}{2})}{\zeta'(\rho)} x^{\frac{1}{2}-\rho}$$

is due to the zeros of the zeta function in the critical stip,

$$f_2(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!\zeta(\frac{1}{2} - n)} \tag{3}$$

is due to the poles of the gamma function at negative integers and

$$f_3(x) = \sum_{n=1}^{\infty} \frac{\Gamma(-\frac{1}{2} - 2n)}{\zeta'(-2n)} x^{2n+\frac{1}{2}} \tag{4}$$

is due to the zeros of the zeta function at negative even integers.

Let us start with $f_2(x)$. By using the functional equation for the zeta function and the duplication formula for the gamma function, it is easy to show that

$$\begin{aligned} (2n)!\zeta\big(1/2 - 2n\big) &= (-1)^n (8\pi)^{-2n} (4n)!\zeta\big(1/2 + 2n\big) \\ (2n+1)!\zeta\big(1/2 - 2n - 1\big) &= (-1)^{n+1} (8\pi)^{-2n-1} (2n+2)!\zeta\big(1/2 + 2n + 1\big). \end{aligned}$$

By taking (3), splitting the sum over the even terms and odd terms and using the above formula, we obtain

$$f_2(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (8\pi x)^{2n}}{(4n)!\zeta(\frac{1}{2} + 2n)} + \sum_{n=0}^{\infty} \frac{(-1)^n (8\pi x)^{2n+1}}{(4n+2)!\zeta(\frac{1}{2} + 2n + 1)}.$$

Now, expand the reciprocal of the zeta function into an Dirichlet series and reverse the order of summation:

$$f_2(x) = \sum_{j=1}^{\infty} \frac{\mu(j)}{\sqrt{j}} \left[\sum_{n=1}^{\infty} \frac{(-1)^n (\frac{8\pi x}{j})^{2n}}{(4n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{8\pi x}{j})^{2n+1}}{(4n+2)!} \right].$$

Now, by using the Laplace transform, or by other means, it can be shown that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(4n)!} &= \cosh y \cos y \\ \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(4n+2)!} &= \sinh y \sin y. \end{aligned}$$

where $y = x/\sqrt{2}$. By applying this to our most recent formula for $f_2(x)$, we obtain

$$f_2(x) = \sum_{j=1}^{\infty} \frac{\mu(j)}{\sqrt{j}} \left[\cosh a_j \cos a_j - 1 + \sinh a_j \sin a_j \right] \tag{5}$$

where $a_j = 2\sqrt{\pi x/j}$ (the -1 is due to the fact that the sum over the even series started at $n = 1$).

We now move on to $f_3(x)$. By using the functional equations for the zeta and gamma function, we initially arrive at

$$f_3(x) = -\frac{1}{2\sqrt{\pi x}} \sum_{n=1}^{\infty} \frac{(-1)^n (8\pi x)^{2n+1}}{(4n+1)!\zeta(2n+1)}.$$

Again, we will now expand the reciprocal of the zeta function into its Dirichlet series to obatin

$$f_3(x) = -\frac{1}{2\sqrt{\pi x}} \sum_{j=1}^{\infty} \mu(j) \sum_{n=1}^{\infty} \frac{(-1)^n (\frac{8\pi x}{j})^{2n+1}}{(4n+1)!}.$$

By differentiating a similar formula above, it is easy to see that

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{(4n+1)!} = y \cosh y \sin y + y \sinh \cos y$$

where again, $y = x/\sqrt{2}$. Using this in the previous series for $f_3(x)$ gives

$$f_3(x) = -\sum_{j=1}^{\infty} \frac{\mu(j)}{\sqrt{j}} \left[\cosh a_j \sin a_j + \sinh a_j \cos a_j \right]. \tag{6}$$

Now, by a stroke of good luck, $f_2(x)$ and $f_3(x)$ given by (5) and (6) add very nicely. By factoring the common sin and cos out and using $\cosh x - \sinh x = e^{-x}$, we obtain

$$f_2(x) + f_3(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{\sqrt{n}} \left[e^{-a_n} \cos a_n - e^{-a_n} \sin a_n - 1 \right]. \tag{7}$$

The final result is

$$\sum_{\rho} \frac{\Gamma(-\frac{1}{2} + \rho)}{\zeta'(\rho)} x^{\frac{1}{2} - \rho} = -\frac{1}{\zeta(\frac{1}{2})} + \sum_{n=1}^{\infty} \frac{\mu(n)}{\sqrt{n}} e^{-nx} + \sum_{n=1}^{\infty} \frac{\mu(n)}{\sqrt{n}} \left[e^{-a_n} \sin a_n + 1 - e^{-a_n} \cos a_n \right]. \tag{8}$$

where $a_n = 2\sqrt{\pi x/n}$, or replacing x by πx^2 gives

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{\sqrt{n}} \left[e^{-n\pi x^2} + e^{-2\pi x/\sqrt{n}} \big(\sin(2\pi x/\sqrt{n}) - \cos(2\pi x/\sqrt{n}) \big) + 1 \right] =$$

$$\frac{1}{\zeta(\frac{1}{2})} + \sum_{\rho} \frac{\pi^{\frac{1}{2}-\rho}\Gamma(-\frac{1}{2}+\rho)}{\zeta'(\rho)} x^{1-2\rho}. \qquad (9)$$

Assuming the RH so that $\sum \mu(n)/\sqrt{n}$ converges, the left hand of (8) goes to 0, so that

$$\lim_{x\to\infty} \sum_{\rho} \frac{\Gamma(-\frac{1}{2}+\rho)}{\zeta'(\rho)} x^{\frac{1}{2}-\rho} = 0.$$

For this to happen, an infinity of zeros would be needed on the line $\sigma = 1/2$. It seems that if the exponent of $1/2$ in our definition of f is replaced by α between $1/2$ and 1 and the assumption is made that there exists a zero with maximal real part, say σ_m then by the above reasoning, it seems plausible that a theorem stating that there would then in fact be an infinite number of zeros on the line $\sigma = \sigma_m$ would be possible to prove.