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Minimal kernels of weakly complete spaces

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Abstract

Let X be a weakly complete space i.e. X a complex space endowed with a C^k-smooth, $k \geqslant 0$, plurisubharmonic exhaustion function. We give the notion of *minimal kernel* $\Sigma^1 = \Sigma^1(X)$ of X by the following property: $x \in \Sigma^1$ if no continuous plurisubharmonic exhaustion function is strictly plurisubharmonic near x. The study of the geometric properties of the minimal kernels is the aim of present paper. After stating that the minimal kernel Σ^1 of a weakly complete space can be defined by a single plurisubharmonic exhaustion function φ, called *minimal*, using the characterization in terms of Bremermann envelopes, we prove the following, crucial, result: if X is a weakly complete manifold and φ a minimal function for X, the nonempty level sets $\Sigma^1_c = \Sigma^1 \cap \{\varphi = c\}$ have the local maximum property. In the last section we discuss the special case of weakly complete surfaces. We prove that if $\dim_c X = 2$ and c is a regular value of a minimal function φ then the nonempty level sets $\Sigma^1_c = \Sigma^1 \cap \{\varphi = c\}$ are compact spaces foliated by holomorphic curves.

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1. Introduction

Let us define a *weakly complete space* as a complex space endowed with a C^k-smooth, $k \geqslant 0$, plurisubharmonic exhaustion function. Weakly complete subdomains of Stein spaces are Stein. Trivially, proper complex spaces over Stein spaces are weakly complete. More interesting examples are provided by the pseudoconvex subdomains of complex tori constructed by Grauert (cf. [A,N2]). With respect to their geometric-functional structure these domains have Levi flat boundary and all the level sets (except for the minimum ones) of the plurisubharmonic functions displaying the completeness are Levi flat, are foliated by everywhere dense complex hypersurfaces. In particular, they contain no compact complex subspace of positive dimension and have no nonconstant holomorphic function. We refer to them as *Grauert-type* spaces.

It is evident, proper complex spaces over Stein spaces and Grauert-type spaces constitute two opposite families of weakly complete spaces; thus, it is natural to ask whether these are the "only possible" such families. We are dealing with this general problem (definitely nontrivial even for dimension 2) in a forthcoming paper [ST2]. The results there are obtained by a very careful analysis of the geometry of the level sets of special plurisubharmonic exhaustion functions called *minimal*. The basic properties of these functions are the object of the present paper.

This paper is organized into four sections. After some generalities, in Section 3 we give the notion of *minimal kernel* $\Sigma^1 = \Sigma^1(X)$ of a weakly complete space X by the following property: $x \in \Sigma^1$ if no plurisubharmonic exhaustion function is strictly plurisubharmonic near x. A priori each regularity class of exhaustion functions could lead to a different minimal kernel. We do not think so, in fact we believe that for $k \leqslant +\infty$ they all determine the same minimal kernel (cf. Remark 3.1). Minimal functions then, are those plurisubharmonic exhaustion functions which fail to be strictly plurisubharmonic exactly at points of Σ^1. They always exist for each class of regularity (cf. Lemma 3.1).

The rest of the section is devoted to the study of the interplay between minimal functions and minimal kernel. We first prove that if $\varphi : X \to \mathbb{R}$ is any C^k-smooth, $k \leqslant +\infty$, minimal function, $\overline{B} \subset X$ any closed coordinate ball and $u : \overline{B} \to \mathbb{R}$ the Bremermann function on $\overline{B}$, with boundary condition $u = \varphi_{|bB}$ (cf. [B]), then

$$\Sigma^1 \cap B = Y = \{z \in B \colon u(z) = \varphi(z)\}$$

(cf. Corollary 3.5).

This allows us to prove the following, crucial, result (cf. Theorem 3.6)

(a) if X is a weakly complete manifold and φ a minimal function for X, the nonempty level sets $\Sigma^1_c = \Sigma^1 \cap \{\varphi = c\}$ have the local maximum property.

As a consequence, a weakly complete space with a discrete singular set is Stein if and only if it does not have compact local maximum sets (cf. Corollary 3.7).

Some additional results about differential–topological properties of level sets of plurisubharmonic exhaustion functions are proved at the end of the section.

Finally, in Section 4, we discuss the special case of weakly complete surfaces. In view of the main theorem of [Sh], on foliations by analytic discs of the polynomial hulls of a real 2-dimensional compact graph in $\mathbb{C}^2$, the structure of Σ^1 can be made further precise; which we do in Lemma 4.1, by proving that

(b) if $\dim_{\mathbb{C}} X = 2$ and c is a regular value of a minimal function φ, then the nonempty level sets $\Sigma^1_c = \Sigma^1 \cap \{\varphi = c\}$ are compact spaces foliated by Riemann surfaces.

2. General facts

1. Let X be a (countably at ∞) n-dimensional complex manifold. A smooth function ψ is called *q-plurisubharmonic* if its Levi form does not have more than q negative eigenvalues at any point.

Let Y be a locally closed set in X. We say that Y has *q-local maximum property* or that Y is a *q-local maximum set* if for every point $x \in Y$ there is a neighbourhood U of x with the following property: for every compact set $K \subset U$ and every function ψ which is q-plurisubharmonic in a neighbourhood of K,

$$\max_{Y \cap K} \psi = \max_{Y \cap bK} \psi,$$

where $\max_{Y \cap bK} \psi$ is meant to be $-\infty$ whenever $Y \cap bK = \emptyset$. Accordingly, a q-local maximum set has no isolated point.

If $q = 0$, we say that Y is a *local maximum set*.

The following is clear. Let $X \subset Z$, Z be a complex manifold. A subset Y of X is a q-local maximum set in X if and only if it is a q-local maximum set in Z.

Let us recall a useful characterization of local maximum sets (cf. [Sl3]).

A subset Y of $\mathbb{C}^n$ is a local maximum set if and only if do not exist $z^\circ \in Y$, $\varepsilon > 0$ and a plurisubharmonic function ψ on the ball $B = B(z^\circ, r) = \{|z - z^\circ| < r\}$ such that $\psi(z^\circ) = 0$ and $\psi(z) < -\varepsilon|z - z^\circ|^2$ for $z \in Y \cap B$.

Analytic subsets of X of pure dimension $d \geqslant 1$ are $(d-1)$-local maximum sets.

Proposition 2.1. *Let X be an n-dimensional complex manifold and $Y \subset X$ an analytic subset with* $\dim_{\mathbb{C}} Y_y \geqslant d \geqslant 1$ *for every $y \in Y$ (i.e. Y is a $(d-1)$-local maximum set). Let φ be a C^2 plurisubharmonic function on a neighbourhood of Y in X such that* $\varphi_{|Y} = \text{const}$. *Then the Levi form of φ can have at most $n - d$ positive eigenvalues at any point $y \in Y$.*

Proof. It is enough to verify the property at regular points of Y.

Let y be a regular point of Y and $z_1, \dots, z_n$ holomorphic coordinates on a neighbourhood U of y such that $U \cap Y = \{z_{k+1} = \dots = z_n = 0\}$, $k \geqslant d$. With respect to these coordinates, the hermitian matrix $\mathscr{H} = (\varphi_{\alpha\bar{\beta}}(y))$ of $\partial\bar{\partial}\varphi(y)$ has the following form:

$$\begin{pmatrix} \mathbf{0}_k & * \\ * & * \end{pmatrix},$$

where $\mathbf{0}_k$ is the $k \times k$ zero matrix. Then the associate hermitian form $\mathscr{H}$ is vanishing on a k-dimensional subspace $\mathbf{V}$ of $\mathbb{C}^n$. On the other hand, if $\mathscr{H}$ has $p = p(y)$ positive eigenvalues, $\mathscr{H}$ is positive on a p-dimensional subspace $\mathbf{W}$. It follows that $\mathbf{V} \cap \mathbf{W} = \{0\}$ and consequently $p + k \leqslant n$ i.e. $p(y) \leqslant n - k \leqslant n - d$. □

The above definitions can be given in the context of complex spaces. Let us just observe the following immediate consequence of the characterization of local maximum sets: a local maximum set in a Stein space X has no compact connected component.

More generally, let $\pi : \tilde{X} \to X$ be a proper morphism between irreducible, pure dimensional complex spaces. Let $\tilde{E} \subset \tilde{X}$ be the exceptional set of π i.e. $x \in \tilde{E}$ if and only if π is not a local isomorphism at x. Then if X is Stein every compact local maximum set of $\tilde{X}$ belongs to $\tilde{E}$.

2. Let X be a complex manifold. We say that a subset of XY has *q-maximum property*, or that Y is a *q-maximum set*, if it is a q-local maximum set and, for every function ψ which is q-plurisubharmonic in a neighbourhood of $\overline{Y}$,

$$\max_{\overline{Y}} \psi = \max_{\overline{Y} \setminus Y} \psi.$$

A closed subset is not a *q-maximum set*.

If $q = 0$ we say that Y is a *maximum set*. In general q-local maximum property does not imply q-maximum property. As an example take $Z = (\overline{D} \times \mathbb{C}) \cup (\{0\} \times D)$, where D is the unit disc in $\mathbb{C}$. Z is a local maximum set in $\mathbb{C} \times \mathbb{C}$ but is not a maximum set, because for the function $\psi = |z_1|$ $((z_1, z_2) \in \mathbb{C}^2)$ one has $\max_{\overline{Z}} \psi = 1$ and $\psi = 0$ on $\overline{Z} \setminus Z$.

We observe the following two properties:

- with the above notations, if ψ varies in the space of continuous plurisubharmonic functions in a neighbourhood of $\overline{Y}$, then the subset

  $$Y_{\bigcirc} = \bigcap_{\psi} \left\{ x \in X \colon \psi(x) = \max_{\overline{Y}} \psi \right\}$$

  has local maximum property, provided it is nonempty;
- if $Y_{\bigcirc}$ is *minimal* i.e. has no proper local maximum subset then it is a maximum set.

3. Minimal kernels

1. Let $\mathscr{F}^k(X)$, $k \geq 2$, be the family of all C^k plurisubharmonic exhaustion functions on X. We say that X is *weakly complete* (with respect to $\mathscr{F}^k(X)$) if $\mathscr{F}^k(X)$ is nonempty for some $k \geq 2$. For every $\varphi \in \mathscr{F}^k(X)$ denote Z^j_φ the set of those points $x \in X$ such that the Levi form of φ at x has at least j zero eigenvalues. We define

$$\Sigma^j = \Sigma^j(X) = \bigcap_\varphi Z^j_\varphi,$$

where φ varies in $\mathscr{F}^k(X)$. The subset Σ^j, $1 \leq j \leq n$, is said to be the *j-minimal kernel* of X (with respect to the chosen family $\mathscr{F}^k(X)$). The set Σ^1 is also called the *minimal kernel* of X.

Observe that, by Proposition 2.1, the set Σ^j contains every compact complex analytic subset of X of pure dimension $d \geq j$, $j = 1, \ldots, n$. Furthermore, every compact local maximum subset Y of X intersects Σ^1 (along $Y_\bigcirc$), by the properties referred to in Section 2. In contrast, we will see below (cf. Theorem 3.6) that Σ^1 is the union of a family of pairwise disjoint, compact, local maximum sets.

Lemma 3.1. *There is a C^k plurisubharmonic function $\varphi : X \to \mathbb{R}$ such that*

$$\Sigma^j = Z^j_\varphi$$

for every $j = 1, \ldots, n$. In particular, X is Stein if and only if $\Sigma^1 = \phi$.

Proof. For the purpose of proof fix any C^∞ hermitian metric on X. Nothing more is assumed about the metric.

Denote by $\mathrm{SI}(\varphi)$ (*singularity indicator of* φ) a subset of the tangent bundle $T(X)$ defined as follows:

$$\mathrm{SI}(\varphi) = \bigcup_{x \in X} \{\xi \in T_x(X) : i\partial\overline{\partial}\varphi(x)(\xi, \xi) = 0, ||\xi||_x = 1\},$$

where $||\xi||_x$ is the norm of ξ with respect to the fixed hermitian metric.

Denote $\mathrm{SI}(X) = \bigcap_\varphi \mathrm{SI}(\varphi)$. Clearly for every $x \in X$, $T_x(X) \cap \mathrm{SI}(X)$ is the unit sphere of a complex finite-dimensional subspace of $T_x(X)$ and $x \in Z^j_\varphi$ if and only if the (complex) dimension of this subspace is at least j.

Let $F_1 \subset F_2 \subset \cdots \subset F_l \ldots$, $F_l \subset \overset{\circ}{F}_{l+1}$, be a sequence of compact sets such that $\bigcup_{l \geq 0} F_l = X$ (where $\overset{\circ}{F}_{l+1}$ is the interior of F_{l+1}).

Assertion. Every $\mathrm{SI}(\varphi)$ is closed and so $\mathrm{SI}(X)$ is a closed subset of $T(X)$.

For every point $\xi \in T_x(X) \setminus \mathrm{SI}(X)$ there is a C^k exhaustion function $\varphi^{x,\xi} : X \to \mathbb{R}$ such that $i\partial\overline{\partial}\varphi(x)(\xi, \xi) > 0$. Consequently, there is a neighbourhood $V = V_{x,\xi}$ of ξ in $T(X)$ such that $i\partial\overline{\partial}\varphi(y)(\eta, \eta) > 0$ for $\eta \in \overline{V}$, $\eta \in T_y(X)$.

Clearly, $\overline{V} \cap \mathrm{SI}(X) = \emptyset$. The sets $V_{x,\xi}$ form an open covering of $T(X)\backslash\mathrm{SI}(X)$. Choose a countable subcovering by $\{V_{x_l,\xi_l}\}_{l\geqslant 1}$ which we denote $\{V_l\}_{l\geqslant 1}$ and denote the corresponding exhaustion functions $\varphi_l : X \to \mathbb{R}$. Then $i\partial\overline{\partial}\varphi_l(y)(\eta,\eta)>0$ for $\eta \in \overline{V_l}$.

Choose a positive ε_l, small enough so that

$$\varepsilon_l ||\varphi_{|F_l}||_{C^k(F_l)} < \frac{1}{2^l}, \quad l = 1, 2, \dots$$

if $k \leqslant +\infty$, or

$$\varepsilon_l ||\varphi_{|F_l}||_{C^l(F_l)} < \frac{1}{2^l}, \quad l = 1, 2, \dots$$

if $k = +\infty$ (here, of course, the hermitian norm is used.)

Clearly, the series

$$\sum_{l=1}^{+\infty} \varepsilon_l \varphi_l$$

converges in the C^k (respectively, C^∞) topology to some C^k (respectively, C^∞) function ψ, which is plurisubharmonic. Furthermore, for every fixed l, and for $\eta \in V_l$, $\eta \in T_y(X)$

$$i\partial\overline{\partial}\psi(y)(\eta,\eta) \geqslant \varepsilon_l (i\partial\overline{\partial}\varphi_l)(y)(\eta,\eta) > 0$$

on $\overline{V_l}$, since $\{V_l\}_{l\geqslant 1}$ covers $T(X)\backslash\mathrm{SI}(X)$, and so

$$i\partial\overline{\partial}\varphi(y)(\eta,\eta) > 0$$

on the whole $T(X)\backslash\mathrm{SI}(X)$, i.e.

$$\mathrm{SI}(\psi) = \mathrm{SI}(X).$$

It follows that $Z^j_\psi = \Sigma^j$ for $j = 1, \dots, n$. $\square$

Remark 3.1. A priori each of these regularity classes could lead to a different minimal kernel. We do not think so, in fact we believe that for $2 \leqslant k \leqslant +\infty$ they all determine the same minimal kernels Σ^j, $j = 1, \dots, n$. It should also be observed that, even if the above proof can be adapted in the real analytic case, it is not clear at all whether or not a weakly complete manifold admits a real analytic plurisubharmonic exhaustion function.

A C^k-smooth, $k \geqslant 2$, plurisubharmonic exhaustion function φ on X is said to be *minimal* if $Z^1_\varphi = \Sigma^1$.

2. Both notions of minimal kernel and minimal function make sense, more generally, in the context of *weakly complete spaces* i.e. complex spaces endowed with continuous and plurisubharmonic exhaustion functions. We recall that a continuous real-valued function φ on a complex space X is said to be plurisubharmonic if for every holomorphic map $h: D \to X$, where D is the unit disc in $\mathbb{C}$, $\varphi \circ h$ is subharmonic on D. φ is said to be strictly plurisubharmonic if, for any $x \in X$ and smooth function v on X, the function $\varphi + cv$ is plurisubharmonic around x for $c \in \mathbb{R}$ sufficiently small. Equivalently (cf. [FN, Theorem 5.3.1]) if for any local embedding of X in some $\mathbb{C}^N$, φ is locally the restriction of a local strictly plurisubharmonic in $\mathbb{C}^N$. Lemma 3.1 then extends with the same proof.

It is worth observing that no point of the minimal kernel of a weakly complete space X can be isolated. Indeed assume, for a contradiction, that $x \in \Sigma^1$ is isolated and let U be an open neighbourhood of x such that $U \cap \Sigma^1 = \{x\}$ and U is isomorphic to an analytic subset of a ball in some $\mathbb{C}^N$. Let V, W, $V \Subset W \Subset U$ be open neighbourhoods of x. Choose v continuous and strictly plurisubharmonic on U and $\varrho \in C^0(U)$ with $\varrho \geqslant 0$, $\operatorname{supp} \varrho \subset U$, $\varrho \equiv 1$ on a neighbourhood of $\overline{W}$. Let $\psi_\varepsilon = \psi + \varepsilon \varrho v$, $\varepsilon > 0$, with ψ minimal. Then $\psi_\varepsilon = \psi$ on $U \backslash V$ and is strictly plurisubharmonic on a neighbourhood of $\overline{W}$. Moreover, since ψ is strictly plurisubharmonic on a neighbourhood of $\overline{V} \backslash W$, ψ_ε is also strictly plurisubharmonic, for ε sufficiently small. It follows that, taking ψ_ε in U and ψ on $X \backslash \overline{V}$, we obtain a plurisubharmonic exhaustion function on X which is strictly plurisubharmonic near x. This is a contradiction.

3. Let Ω be a bounded, strictly pseudoconvex domain in $\mathbb{C}^n$ and g a continuous function $b\Omega \to \mathbb{R}$. Let us denote $\mathscr{P}(\Omega; g)$ the family of all functions $u: \overline{\Omega} \to \mathbb{R}$ which are plurisubharmonic in Ω and such that $u_{|b\Omega} \leqslant g$. Then the upper envelope u of $\mathscr{P}(\Omega; g)$ exists, is plurisubharmonic in Ω and $u = g$ on $b\Omega$ (cf. [B]). We call it the *Bremermann function* (in $\overline{\Omega}$) with boundary condition $u = g$, i.e. the continuous solution of the Bremermann–Dirichlet problem in $\overline{\Omega}$ with boundary condition $u = g$.

Lemma 3.2. *Let U be a domain in $\mathbb{C}^n$ and $\varphi: U \to \mathbb{R}$ be a continuous plurisubharmonic function. Let $B \Subset U$ be an open ball and $u: \overline{B} \to \mathbb{R}$ be the continuous solution of the Bremermann–Dirichlet problem in $\overline{B}$, with boundary condition $u = \varphi$. Denote*

$$Y = \{z \in B:\ u(z) = \varphi(z)\}.$$

Then

(a) *Y is a local maximum set provided it is nonempty.*
(b) *$Y \subset Z^1_\varphi$.*

Proof. (a) Suppose $Y \neq \emptyset$, then $Y = \{z \in B:\ v(z) = 0\}$, where $v(z) = -u(z) + \varphi(z)$. By [Sl1], $-u$ is $(n-1)$-plurisubharmonic [HM] and since φ is 0-plurisubharmonic, the sum $v = -u + \varphi$ is $(n-1)$-plurisubharmonic.

Thus, v is a (continuous) $(n-1)$-plurisubharmonic function such that $\max v = 0$. Again in view of [Sl1], the maximum level set $Y = \{z \in B\colon v(z) = 0\}$ has local maximum property.

(b) Suppose now that $z^\circ \in Y \backslash Z^1_\varphi$ i.e. φ is strictly plurisubharmonic in some open ball $B' = B(z^\circ, r)$, $B' \Subset B$. Choose $\varrho \in C^\infty(\overline{B})$ with $\varrho \geqslant 0$ on $\overline{B}$, $\operatorname{supp} \varrho \subset B'$ and such that $\varrho(z^\circ) > 0$. For ε positive and sufficiently small $\varphi_1 = \varphi + \varepsilon\varrho$ is plurisubharmonic in B, $\varphi_1 = \varphi$ on bB and $\varphi_1(z^\circ) > u(z^\circ)$: this is in contradiction with the definition of u. $\square$

Lemma 3.3 (C^∞ regularization of $\max(s,t)$). *For every $\varepsilon > 0$, there is a convex C^∞ smooth function $\varrho^\varepsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with the following properties*:

(a) *ϱ^ε is convex*;
(b) *if $|s-t| \geqslant \varepsilon$, $\varrho^\varepsilon(s,t) = \max(s,t)$*;
(c) $\max(s,t) \leqslant \varrho^\varepsilon(s,t) \leqslant \max(s,t) + \varepsilon$.

Proof. Let $\chi \in C^\infty(\mathbb{R})$, $\chi \geqslant 0$ with $\operatorname{supp} \varrho \subset (-\varepsilon, \varepsilon)$ be such that

$$\int_{-\infty}^{+\infty} \chi(\tau)\, d\tau = 1, \qquad \int_{-\infty}^{+\infty} \tau\chi(\tau)\, d\tau = 0$$

and ϱ^ε be defined by

$$\begin{aligned}\varrho^\varepsilon &= \int_{-\infty}^{+\infty} \sup(s, t+\tau)\chi(\tau)\, d\tau \\ &= s\int_{-\infty}^{s-t} \chi(\tau)\, d\tau + \int_{s-t}^{+\infty} \chi(\tau)(t+\tau)\, d\tau.\end{aligned}$$

Properties (b) and (c) are immediate. Convexity comes from the explicit computation of the Hessian form

$$\operatorname{Hess}(\varrho^\varepsilon)(s,t)(\xi,\eta) = \chi(s-t)|\xi - \eta|^2. \qquad \square$$

Remark 3.2. If φ, ψ are two C^k-smooth, $k \leqslant +\infty$, plurisubharmonic functions with the same domain, then $z \mapsto \varrho^\varepsilon(\varphi(z), \psi(z))$ is plurisubharmonic and C^k-smooth; furthermore, $\varrho^\varepsilon(\varphi, \psi)$ converges uniformly to $\max\,(\varphi, \psi)$.

Lemma 3.4. *Let X be a complex manifold and $\varphi : X \to \mathbb{R}$ a C^k-smooth, $k \leqslant +\infty$, plurisubharmonic function. Let $U \subset X$ be a coordinate domain, $\zeta \in X$ and $B = B(\zeta, r)$, $B \Subset U$, an open ball. Let $u : \overline{B} \to \mathbb{R}$ be the Bremermann function such that $u_{|bB} = \varphi_{|bB}$. Denote*

$$Y = \{z \in B\colon u(z) = \varphi(z)\}.$$

Then for every compact subset K of the open set $B\setminus Y$, there is a C^k-smooth plurisubharmonic function ψ on X such that

(i) $\psi = \varphi$ *on* $X\setminus\overline{B}$;

(ii) ψ is C^∞*-smooth and strictly plurisubharmonic in* $\overset{\circ}{K}$

Proof. Fix a positive $\varepsilon > 0$ such that $u(z) - 5\varepsilon > \varphi(z)$ for $z \in K$.

Choose a C^∞-smooth and strictly plurisubharmonic function χ defined in some open ball $B' = B(\zeta, R)$ containing $\overline{B}$ and such that

$$u(z) < \chi(z) < u(z) + \varepsilon$$

for $z \in \overline{B}$. Let $\varrho^\varepsilon(s,t)$ be the function of the previous lemma. Set

$$\psi^\varepsilon = \varrho^\varepsilon(\chi - 3\varepsilon, \varphi).$$

Then $\psi^\varepsilon : B' \to \mathbb{R}$ is a C^k-smooth plurisubharmonic function. Observe that

$$\chi(z) - 3\varepsilon > u(z) - 3\varepsilon \quad \text{and} \quad u(z) - 5\varepsilon > \varphi(z)$$

on K. Thus,

$$|(\chi(z) - 3\varepsilon) - \varphi(z)| \geqslant 2\varepsilon;$$

hence,

$$\varrho^\varepsilon(\chi(z) - 3\varepsilon, \varphi(z)) = \max(\chi(z) - 3\varepsilon, \varphi(z)) = \chi(z) - 3\varepsilon,$$

i.e. $\psi^\varepsilon = \chi$ in K.

On the other hand,

$$\chi(\zeta) - 3\varepsilon < u(\zeta) - 2\varepsilon < \varphi(\zeta) - \varepsilon$$

for $\zeta \in bB$. Thus,

$$\varphi - (\chi - 3\varepsilon) > \varepsilon$$

and $\varrho^\varepsilon(\chi - 3\varepsilon, \varphi) = \varphi$ in a neighbourhood of bB. Therefore, in some spherical shell $\{r - \delta < |z - \zeta| < r + \delta\}$, we have $\psi^\varepsilon = \varphi$.

Thus, if we set

$$\psi(z) = \begin{cases} \psi^\varepsilon(z) & \text{if } |z - \zeta| < r + \delta, \\ \varphi(z) & \text{if } z \in X\setminus\overline{B}, \end{cases}$$

we obtain a well-defined plurisubharmonic function with all the required properties. $\square$

Remark 3.3. If φ from the last lemma is a smooth plurisubharmonic exhaustion function, then ψ is also a smooth plurisubharmonic exhaustion function; therefore, $Z^1_\psi \cap K = \phi$.

Corollary 3.5. *Let X be a weakly complete manifold and $\varphi : X \to \mathbb{R}$ any C^k-smooth, $k \leqslant +\infty$, minimal function i.e. $Z^1_\varphi = \Sigma^1$. Let $\overline{B} \subset X$ be any closed coordinate ball and $u : \overline{B} \to \mathbb{R}$ be the Bremermann function on $\overline{B}$, with boundary condition $\varphi_{|bB}$. Then it holds that*

$$\Sigma^1 \cap B = Y = \{z \in B:\ u(z) = \varphi(z)\}.$$

Proof. Applying the last remark to a sequence of compacts $K_\nu \subset B \backslash Y$ such that $\bigcup_{\nu \geqslant 0} \overset{\circ}{K}_\nu = B \backslash Y$, we obtain $(B \backslash Y) \cap \Sigma^1 = \emptyset$ i.e. $\Sigma^1 \cap B \subset Y$. On the other hand $Y \subset Z^1_\varphi = \Sigma^1$ by Lemma 3.2(b). Thus, $Z^1_\varphi \cap B = Y$. $\square$

Remark 3.4. If X is a weakly complete space the conclusion of Corollary 3.5 holds true for $\Sigma^1 \cap (X \backslash \mathrm{Sing}\, X)$.

4. The following theorem states a crucial property of the structure of the minimal singular set of a weakly complete manifold:

Theorem 3.6. *Let $\varphi : X \to \mathbb{R}$ be a continuous minimal function. Then, for every value c such that $\Sigma^1_c = \Sigma^1 \cap \{\varphi = c\}$ is nonempty, the set Σ^1_c is compact and has local maximum property.*

Proof. Let φ as in the theorem and denote $m = \inf_{z \in X} \varphi(z)$. Then m is finite.

Fix c as above and consider a smooth increasing and strictly convex function $\chi : [m-1, +\infty) \to \mathbb{R}$ with the additional property that: $\chi(c) = c$, $\chi(t) > t$ for all $t \neq c$, $t \geqslant m-1$ (we can simply take $\chi(t) = t + \delta(t-c)^2$, with $\delta > 0$ small enough so that $1 + 2\delta(m-1-c) > 0$, in order to guarantee $\chi'(t) > 0$).

Let $\psi(z) = (\chi \circ \varphi)(z)$. Then ψ is again a plurisubharmonic exhaustion function. Since $\chi'(t) > 0$, $Z^1_\psi \subset Z^1_\varphi$, and since Z^1_φ was minimal, $Z^1_\psi = Z^1_\varphi = \Sigma^1$. On the other hand, $\psi - \varphi = 0$ on $\{\varphi = c\}$, $\psi - \varphi \geqslant 0$ on X and $\psi - \varphi > 0$ on $X \backslash \{\varphi = c\}$.

Take now an arbitrary point $z^\circ \in \Sigma^1 \cap \{\psi = c\} = \Sigma^1 \cap \{\varphi = c\}$, and its coordinate closed ball $\overline{B}$ centred at z°. Let $u : \overline{B} \to \mathbb{R}$ be the Bremermann function u on $\overline{B}$, with boundary condition $\psi_{|bB}$. Then $\varphi \leqslant \phi \leqslant u$ in B.

Denote

$$Y = \{z \in B:\ u(z) = \psi(z)\}.$$

By Corollary 3.5, we have $Y = \Sigma^1 \cap B$, and so

$$Y \cap \{\psi = c\} = \Sigma^1_c \cap Y.$$

Consider now an open ball $B' \Subset B$ centred at z° and recall that

$$u(z^\circ) = \min_{\mu \in \mathscr{J}_{z^\circ}} \int_{bB'} u(\zeta)\, d\mu(\zeta),$$

where $\mathscr{J}_{z^\circ}$ denotes the set of all Jensen measures for z°, supported by bB' (cf. [G]). Choose one, say μ, that realizes the minimum, i.e.

$$u(z^\circ) = \int_{bB'} u(\zeta) d\mu(\zeta).$$

Since $u(z^\circ) = \varphi(z^\circ)$, $\varphi, u \leqslant \psi$ we have

$$\begin{aligned} 0 = u(z^\circ) - \varphi(z^\circ) &\geqslant \int_{bB'} u(\zeta)\, d\mu(\zeta) - \int_{bB'} \varphi(\zeta)\, d\mu(\zeta) \\ &= \int_{bB'} (u(\zeta) - \varphi(\zeta))\, d\mu(\zeta) \end{aligned}$$

because

$$\varphi(z^\circ) \leqslant \int_{bB'} \varphi(\zeta)\, d\mu(\zeta),$$

μ being a Jensen measure. Since $u - \varphi$ is a nonnegative continuous function on bB', positive on $bB' \setminus Y$, with zero integral, we must have $\operatorname{supp} \mu \subset Y \cap bB' = \Sigma^1_c \cap bB'$.

We now apply the same argument to functions ψ and φ

$$\begin{aligned} 0 = \psi(z^\circ) - \varphi(z^\circ) = u(z^\circ) - \varphi(z^\circ) = \int_{bB'} u(\zeta)\, d\mu(\zeta) - \varphi(z^\circ) \\ \geqslant \int_{bB'} (\psi(\zeta) - \varphi(\zeta))\, d\mu(\zeta) \geqslant 0. \end{aligned}$$

Again, $\psi - \varphi$ is a continuous nonnegative function, with integral zero; thus,

$$\operatorname{supp} \mu \subset bB' \cap \{\psi - \varphi = 0\} \subset \{\varphi = c\} \cap bB'.$$

It follows that

$$\operatorname{supp} \mu \subset \Sigma^1 \cap \{\varphi = c\} \cap bB' = \Sigma^1_c \cap bB'.$$

Thus, we have obtained:

- for every $z^\circ \subset \Sigma^1_c$ and for every small coordinate ball B centred at z° there is a Jensen measure μ for z° supported by $\Sigma^1_c \cap bB$.

It is an immediate observation that this condition implies that the set Σ^1_c is a local maximum set i.e. has local maximum property. $\square$

As a consequence we find that Σ^1_c has no isolated point. Also a C^k-smooth, $k \leqslant +\infty$, minimal function φ is strictly plurisubharmonic near an isolated point of a level set.

Remark 3.5. It does not seem easy to show that the sets Σ^1_c for a minimal real analytic function $\alpha : X \to \mathbb{R}$ have local maximum property; in particular the analogue of regularization procedure is not readily available in the real analytic category.

Under the hypothesis of the above theorem, if X is a complex space the local maximum property holds true for the subsets $\Sigma^1_c \backslash \operatorname{Sing} X$ (provided nonempty).

Corollary 3.7. *A weakly complete space X with discrete singular set is Stein if and only if it has no compact local maximum set.*

Proof. Let φ be a minimal function on X i.e. $Z^1_\varphi = \Sigma^1$ and assume $\Sigma^1 \neq \emptyset$. We know that $\Sigma \not\subset \operatorname{Sing} X$. By hypothesis X has no compact local maximum subset so we can choose c in such a way that $\emptyset \neq \Sigma^1_c \subset X \backslash \operatorname{Sing} X$. Then, in view of Theorem 3.6, Σ^1_c is a local maximum set and consequently $\Sigma^1_c = \emptyset$. This is a contradiction. Thus $\Sigma^1 = \emptyset$. In view of [N1] X is Stein. □

5. We end this section on qualitative properties of level sets of plurisubharmonic functions with two results. The first one is a weak form of Hopf lemma.

Lemma 3.8. *Let $\Omega \subset \mathbb{C}^n$ be open, $\varphi \in C^1(\Omega)$ a plurisubharmonic function. Assume that the set $S = \{\varphi = 0\}$ is a $C^{1+\delta}$-smooth hypersurface, $\delta > 0$ and $\Omega_- = \{\varphi < 0\}$ is nonempty with $(\overline{\Omega}_- \backslash \Omega_-) \cap \Omega = S$. Then $\partial\varphi(z) \neq 0$ for every $z \in S$.*

Proof. Fix $\zeta \in S$ and let L be the complex line passing through ζ and containing the normal vector to S at ζ. Without loss of generality we can make a complex affine change of coordinates so that $\zeta = (0, \dots, 0) \in \mathbb{C}^n$, $L = \mathbb{C} \times (0, \dots, 0)$. The normal vector at ζ is $(1, 0, \dots, 0)$. Consider a small ball B_ε of radius ε, centred at 0, such that $\overline{B}_\varepsilon \subset \Omega$ and denote

$$D_\varepsilon = \{z = x + iy \in \mathbb{C} \colon (z, 0, \dots, 0) \in B_\varepsilon \cap W_-\},$$

and

$$\gamma = \{z \in \mathbb{C} \colon (z, 0, \dots, 0) \in S \cap B_\varepsilon\},$$

with $0 \in \gamma$. Then $\overline{D}_\varepsilon \backslash D_\varepsilon = \gamma \cup \tau$, where $\tau \subset \{|z| = \varepsilon\}$. We claim that, for sufficiently small ε, D_ε is a topological disc bounded by an open arc γ, $C^{1+\delta}$-smooth and a closed real analytic arc τ. The function $\varphi^*(z) = \varphi(z, 0, \dots, 0)$ is subharmonic inside D_ε and continuous on $\overline{D}_\varepsilon$. Let $u : \overline{D}_\varepsilon \to \mathbb{R}$ be a harmonic function in D_ε with continuous boundary value equal to $\varphi^*_{|bD_\varepsilon}$. Since $\varphi < 0$ in W_- it follows that $\varphi^* < 0$ on τ, except for the endpoints, while $\varphi^*_{|\bar{\gamma}} = 0$. It follows that $u<0$ on D_ε, with $u_{|\gamma} = 0$. By the standard Hopf Lemma, the normal derivative of u at $0\in\gamma$ is positive, i.e. $\partial u/\partial v(0)>0$, and so $\partial\varphi^*/\partial x(0)>0$, i.e. $\partial\varphi/\partial x(0)\neq 0$. That is $\partial\varphi(\zeta)\neq 0$, as required. □

The second one consists of

Theorem 3.9. *Let X be a weakly complete manifold of complex dimension $n\geqslant 2$ and $\varphi : X \to \mathbb{R}$ be a C^2-smooth plurisubharmonic exhaustion function. Let $r>\min\varphi$ and let Y be a connected component of the level set $\{\varphi = r\}$. Assume Y is relatively open in $\{\varphi = r\}$ and that Y does not contain local minimum points of φ. Assume further that Y is a local maximum set. Then there is an $s<r$ such that the topological boundary of the connected component of the set $K := \{s\leqslant\varphi\leqslant r\}$ containing Y is contained in $\{\varphi = s\}\cup Y$. Thus $K := \{s\leqslant\varphi\leqslant r\}$, $bK = Z\cup Y$, where $Z\subset\{z\in K\colon \varphi(z) = s\}$. Any such s and K satisfy the following properties*:

(a) *K is a connected compact set with nonempty interior.*
(b) *The forms*

$$(dd^c\varphi)^{n-1}\wedge d\varphi\wedge d^c\varphi, (dd^c\varphi)^{n-1}\wedge d\varphi, (dd^c\varphi)^{n-1}\wedge d^c\varphi$$

vanish identically on K and $(dd^c\varphi)^n$ vanishes on $K\backslash Y$.
(c) *Every level set $\{z\in K\colon \varphi(z) = t\}$, with $s\leqslant t\leqslant r$, has the local maximum property.*

Remark 3.6. The assumption that Y is relatively open in $\{\varphi = r\}$ (i.e. that $\{\varphi = r\}\backslash Y$ is compact) is automatically satisfied when r is a noncritical value of φ; the statement is, however, true, also when r is a critical value.

The proof of the theorem is based on the following lemmas.

Lemma 3.10. *Let φ be a C^2-smooth plurisubharmonic function defined in the neighbourhood of a fixed point $p\in X$, where X is a complex manifold,* $\dim_{\mathbb{C}} X = n$, *$n\geqslant 2$. Assume that $(dd^c\varphi)^{n-1}\wedge d\varphi\wedge d^c\varphi = 0$ at the point p. Then*

(i) $[(dd^c\varphi)^{n-1}\wedge d\varphi](p) = 0$, $[(dd^c\varphi)^{n-1}\wedge d^c\varphi](p) = 0$,
(ii) $(dd^c\varphi)^n(p) = 0$, *provided $d\varphi(p) = 0$.*

Proof. Assume that $d\varphi(p)\neq 0$. (The statement is trivial otherwise).
Since

$$(dd^c\varphi)^{n-1}\wedge d\varphi\wedge d^c\varphi = (2i)^n(\partial\bar{\partial}\varphi)^{n-1}\wedge\partial\varphi\wedge\bar{\partial}\varphi$$

we have $[(\partial\bar{\partial}\varphi)^{n-1}\wedge\partial\varphi\wedge\bar{\partial}\varphi](p) = 0$. As it is well known, for a plurisubharmonic function φ with $\partial\varphi(p)\neq 0$, the last condition means that the complex Hessian of φ at p, say $\mathscr{H}(\cdot,\cdot)$, is not positive definite on $T_p(Y)$, the complex tangent space (of dimension $n-1$) to the level set $Y = \{\varphi = \varphi(p)\}$, at p. That is, there is a nonzero vector $\xi \in T_p(Y)$ such that $\mathscr{H}(\xi, \xi) = 0$, and so $\mathscr{H}(\lambda\xi, \mu\xi) = 0$, $\lambda, \mu \in \mathbb{C}$. Since $\mathscr{H} \geqslant 0$, by Schwarz inequality, $\mathscr{H}(\xi, x) = 0$ for all $x \in T_p(Y)$. Thus, $\mathscr{H}$ induces a hermitian form $\tilde{\mathscr{H}}$ on $T_p(X)/\{\mathbb{C}\,\xi\}$. Since $\dim_{\mathbb{C}} T_p(X)/\{\mathbb{C}\,\xi\} = n-1$, there are $\mathbb{C}$-linear forms

$$\mu_1, \mu_2, \ldots, \mu_{n-1} : T_p(X)/\{\mathbb{C}\,\xi\} \to \mathbb{C}$$

such that

$$\tilde{\mathscr{H}}(x, y) = \sum_{j=1}^{n-1} \mu_j(x)\overline{\mu}_j(y).$$

Let $\lambda_j := \mu_j \circ \pi$, $j = 1, 2, \ldots, n-1$, where

$$\pi : T_p(X) \to T_p(X)/\{\mathbb{C}\xi\}$$

is the standard projection. Then

$$\mathscr{H}(x, y) = \sum_{j=1}^{n-1} \lambda_j(x)\overline{\lambda}_j(y)$$

and

$$dd^c\varphi(p) = \text{const} \sum_{j=1}^{n-1} \lambda_j \wedge \overline{\lambda}_j.$$

Consequently,

$$dd^c\varphi^{n-1}(p) = \text{const}\, \lambda_1 \wedge \lambda_2 \wedge \cdots \lambda_{n-1} \wedge \overline{\lambda}_1 \wedge \overline{\lambda}_2 \wedge \cdots \overline{\lambda}_{n-1}. \tag{$*$}$$

This implies immediately that $(dd^c\varphi)^n(p) = 0$.

Let $\chi := \partial\varphi(p) : T_p(X) \to \mathbb{C}$. Since χ is a $\mathbb{C}$-linear form vanishing on the subspace $\mathbb{C}\xi$ ($\xi \in T_p(Y) = \operatorname{Ker}\chi$), it has to be a linear combination of $\lambda_1, \ldots, \lambda_{n-1}$, $\chi = \sum_{j=1}^{n-1} c_j\lambda_j$. Since

$$d\varphi(p) = \partial\varphi(p) + \overline{\partial}(p) = \sum_{j=1}^{n-1} c_j\lambda_j + \sum_{j=1}^{n-1} \overline{c}_j\overline{\lambda}_j$$

and

$$d^c\varphi(p) = i[\overline{\partial}\varphi(p) - \partial\varphi(p)] = i\sum_{j=1}^{n-1} \overline{c}_j\overline{\lambda}_j - i\sum_{j=1}^{n-1} c_j\lambda_j$$

we obtain, by ($*$), that $[(dd^c\varphi)^{n-1} \wedge d\varphi](p) = [(dd^c\varphi)^{n-1} \wedge (d^c\varphi)](p) = 0$. $\square$

Lemma 3.11. *Let X be a real N-dimensional manifold and $K \subset X$ be a compact set. Let ω be an $(N-1)$-form, with C^1-smooth coefficients, defined in a neighbourhood of K. Assume that ω vanishes on the topological boundary bK of K in X. Then*

$$\int_K d\omega = 0.$$

Proof. *Step* 1. The statement is true for $K \subset \mathbb{R}^N$.

In this case, $\omega = \sum_{j=1}^N \omega_j$, where

$$\omega_j = f_j \, dx_1 \wedge \cdots dx_{j-1} \wedge dx_{j+1} \wedge \cdots dx_N.$$

By hypothesis, $\omega_j(p) = 0$, $j = 1, \ldots, N$, for $p \in bK$. Thus, it is enough to prove the statement for every elementary form ω_j, $j = 1, \ldots, N$. We do it, without loss of generality, for the form

$$\omega_1(x_1, \ldots, x_N) = f(x_1, \ldots, x_N) \, dx_2 \wedge \cdots \wedge dx_N.$$

Let $\operatorname{supp} g$ denote the closed support of a generic function g. In our situation, we can assume (by replacing f by ϱf, where ϱ is a C^∞-cut-off function such that $\varrho_{|K} = 1$, $\operatorname{supp} \varrho$ compact and contained in the interior of $\operatorname{supp} f$), that $f \in C^1(\mathbb{R}^N)$ and $\operatorname{supp} f \subset [-A, A]^N$, where $[-A, A]^N := [-A, A]^{\times N}$. Then, by the Fubini theorem

$$\begin{aligned} \int_K d\omega &= \int_K \frac{\partial f}{\partial x_1}(x_1, \ldots, x_N) \, dx_1 \wedge \cdots \wedge dx_N \\ &= \int_{[-A,A]^{N-1}} \left(\int_{I_{x_2,\ldots,x_N}} \frac{\partial f}{\partial x_1}(x_1, \ldots, x_N) \, dx_1 \right) dx_2 \wedge \cdots \wedge dx_N, \end{aligned}$$

where

$$I_{x_2,\ldots,x_N} = \{t \in \mathbb{R}\colon \ (t, x_2, \ldots, x_{N_1})\},$$

is a compact subset of $\mathbb{R}$, for $(x_2, \ldots, x_N) \in [-A, A]^{N-1}$ (might be occasionally empty).

It is clearly enough to show that

$$\int_{I_{x_2,\ldots,x_N}} \frac{\partial f}{\partial x_1}(x_1, \ldots, x_N) \, dx_1 = 0,$$

for all $(x_2, \ldots, x_N) \in [-A, A]^{N-1}$.

Fix $(x_2, \ldots, x_N)$ such that $I := I_{x_2,\ldots,x_N}$ is nonempty and let

$$g(t) = \frac{\partial f}{\partial x_1}(t, x_2, \ldots, x_N).$$

Thus g is C^1-function of t. Denote

$$a = \min\{t\colon\ t \in I\}, \quad b = \max\{t\colon\ t \in I\}.$$

Then $I = [a,b] \setminus \bigcup_{j=1}^{M} (\alpha_j, \beta_j)$ where $0 \leqslant M \leqslant +\infty$. Observe that the set

$$\{a, b, \alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots\} \times (x_2, \dots, x_N)$$

is contained in the boundary of K; for example

$$(\alpha_j, x_2, \dots, x_N) \in \overline{(\alpha_j, \beta_j) \times (x_2, \dots, x_N)} \subset \overline{\mathbb{R}^N \setminus K}.$$

Thus, $g(t) = 0$ for $t = a, b, \alpha_j, \beta_j$, $j = 1, \dots, M$. Now

$$\begin{aligned}
\int_I \frac{\partial f}{\partial x_1}(t, x_2, \dots, x_N)\, dt &= \int_I \frac{d}{dt} g(t)\, dt \\
&= \int_a^b \frac{d}{dt} g(t)\, dt - \sum_{j=1}^{M} \int_{\alpha_j}^{\beta_j} \frac{d}{dt} g(t)\, dt \\
&= g(b) - g(a) - \sum_{j=1}^{M} (g(\beta_j) - g(\alpha_j)) = 0.
\end{aligned}$$

This completes the proof of Step 1.

Step 2. We now reduce the general case to the Euclidean case by localization.

There is a C^∞ partition of unity $\varrho_1, \dots, \varrho_s$ and finite coverings $\{V_j\}_{1 \leqslant j \leqslant s}$, $\{U_j\}_{1 \leqslant j \leqslant s}$ of K such that $\operatorname{supp} \varrho_j \subset \overset{\circ}{V}_j$, $V_j \subset U_j$, $j = 1, \dots, s$, V_j is a compact set, U_j is a coordinate neighbourhood in X (diffeomorphic to a subset of $\mathbb{R}^N$) and $\varrho_j \geqslant 0$, $\sum_{j=1}^{s} \varrho_j(x) = 1$ for $x \in W$, $W \subset \bigcup_{j=1}^{s} V_j$, where W is a neighbourhood of K, and $\bigcup_{j=1}^{s} V_j \subset \operatorname{supp} \omega$.

Now let $\omega_j = \varrho_j \omega$. Then $\omega_{|W} = (\sum_{j=1}^{s} \omega_j)_{|W}$, and $d\omega = \sum_{j=1}^{s} d\omega_j$ on W. Thus,

$$\int_K d\omega = \sum_{j=1}^{s} \int_K d\omega_j.$$

It suffices to show that

$$\int_K d\omega_j = 0.$$

Observe $d\omega_j = 0$ on $X \setminus \operatorname{supp} \varrho_j$ and so $d\omega_j = 0$ on $K \setminus K \cap V_j$. Consequently,

$$\int_K d\omega_j = \int_{K \cap V_j} d\omega_j = 0.$$

Now $K \cap V_j$ is a compact set and

$$b(K \cap V_j) \subset [(bK) \cap V_j] \cup (K \cap bV_j)$$
$$\subset [(bK) \cap V_j] \cup [K \backslash \operatorname{supp} \varrho_j].$$

Thus, $\omega_j = \varrho_j \omega$ vanishes on $b(K \cap V_j)$. Since $K \cap V_j \subset U_j$, which is diffeomorphic to an open subset of $\mathbb{R}^N$,

$$\int_K d\omega_j = 0$$

by Step 1. □

Lemma 3.12. *Let X be a complex manifold of (complex) dimension n and $\varphi : X \to \mathbb{R}$ be a C^3-smooth plurisubharmonic function. Let $K \subset X$ be a compact set such that $bK \subset \{\varphi = s\} \cup \{\varphi = r\}$, where $s < r$, $r, s \in \mathbb{R}$. Assume that the form $(dd^c\varphi)^{n-1} \wedge d^c\varphi$ vanishes on the set $Y = \{z \in K\colon\ \varphi(z) = r\}$. Then*

(i) $(dd^c\varphi)^n$ *on* $\mathring{K}$;

(ii) $(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi = 0$ *on* $\mathring{K}$.

Proof. Let $Z = \{\varphi = s\}$ and $\omega = (\varphi - s)(dd^c\varphi)^{n-1} \wedge d^c\varphi$. By the assumption, $\omega(p) = 0$ for $p \in Y$. On the other hand, for $p \in Z$, $\varphi(p) - s = 0$ and so $\omega(p) = 0$ as well. Thus, $\omega_{|K} = 0$. By Lemma 3.11

$$\int_K d\omega = 0.$$

Now

$$d\omega = (dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi + (\varphi - s)(dd^c\varphi)^n$$

and so

$$0 = \int_K (dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi + \int_K (\varphi - s)(dd^c\varphi)^n. \tag{$*$}$$

Since $\varphi - s \geqslant 0$ on K and $dd^c\varphi \geqslant 0$, both integrands in $(*)$ are nonnegative forms with continuous coefficients. By $(*)$, each of them has zero integral and so they have to vanish on $\mathring{K}$, i.e.

$$(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi = (\varphi - s)(dd^c\varphi)^n = 0$$

on $\mathring{K}$. As $\varphi - s > 0$ on $\mathring{K}$, $(dd^c\varphi)^n = 0$. □

Lemma 3.13. *Let $\varphi : X \to \mathbb{R}$ be a C^2-smooth plurisubharmonic function, where X is a complex manifold of dimension $n \geqslant 2$. Let $W \subset X$ be a relatively compact open subset. Assume that*

$$(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi = 0$$

on W, and that $d\varphi(p) \neq 0$ for every $p \in \overline{W}$. Then

(i) *for every $c \in \mathbb{R}$ and for a sufficiently large even integer $k = k(c)$ the function $\lambda = (c - \varphi)^k$ is $(n-2)$-plurisubharmonic in W;*
(ii) *for every $t \in \mathbb{R}$, the level set $Y = \{x \in W\colon\ \varphi(x) = t\}$ is a local maximum set, provided it is nonempty.*

Proof. (i) The complex Hessian of λ at $p \in \overline{W}$ is

$$\mathscr{H}(\lambda)(p) = k(\varphi - c)^{k-2}\{(k-1)\partial\varphi(p) \otimes \overline{\partial}\varphi(p) + (\varphi - c)\mathscr{H}(\varphi)(p)\},$$

where $\mathscr{H}(\varphi)(p)$ is the complex Hessian of λ at p.

Fixing any Riemannian metric, choose the unique vector $v_p \in T_p(X)$ which is orthogonal to $T_p(Y) \subset T_p(X)$ and satisfies $\partial\varphi(p)(v_p) = 1$, for every $p \in \overline{W}$. It is clear that $p \mapsto v_p$ is a continuous section on $\overline{W}$; by compactness of $\overline{W}$ there is an even integer k large enough so that

$$\begin{aligned}\mathscr{H}(\lambda)(p)(v_p, v_p) = & k[\varphi(p) - c]^{k-2}\{(k-1)|\partial\varphi(p)(v_p)|^2 \\ & + [\varphi(p) - c]\mathscr{H}(\varphi)(p)(v_p, v_p)\}\end{aligned} \qquad (*)$$

is nonnegative.

Now fix an arbitrary point $p \in W$. Since $(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi = 0$ at p, there is a nonzero vector $w_p \in T(X)$ tangent to $\{\varphi = \varphi(p)\}$ at p such that $\mathscr{H}(\varphi)(p)(w_p, w_p) = 0$ and $\partial\varphi(p)(w_p) = 0$. Since $\mathscr{H}(\varphi)(p) \geqslant 0$, by Schwarz Lemma $\mathscr{H}(\varphi)(p)(w_p, v_p) = 0$. In addition, w_p and v_p are linearly independent (being orthogonal). Since

$$\mathscr{H}(\varphi)(p)(\mu w_p + v v_p, \mu w_p + v v_p) = |v|^2 \mathscr{H}(\varphi)(p)(v_p, v_p)$$

and $\partial\varphi(p)(\mu w_p + v v_p) = v$, we obtain, by formula $(*)$, that

$$\mathscr{H}(\lambda)(p)(\mu w_p + v v_p, \mu w_p + v v_p) = |v|^2 \mathscr{H}(\lambda)(p)(v_p, v_p) \geqslant 0.$$

Thus, $\mathscr{H}(\lambda)(p)$ is positive semi-definite on the 2-dimensional subspace spanned by v_p, w_p for every $p \in W$. This means that λ is $(n-2)$-plurisubharmonic.

(ii) Fix $t \in \mathbb{R}$ such that $\{\varphi = t\} \cap W = \phi$ and let

$$W_+ = \{z \in W\colon\ \varphi > t\}, \quad W_- = \{z \in W\colon\ \varphi(z) < t\}.$$

Since φ is plurisubharmonic, W_- is pseudoconvex in W and so $(n-2)$-pseudoconvex in W as well (cf. [Sl2, Definition 4.1]). Choose now c real such that $c-\varphi>0$ in $\overline{W}$. Then $W_+ = \{z \in W\colon \lambda(z) < (t-c)^k\}$. By part (i) λ is $(n-2)$-plurisubharmonic, its sublevel set W_+ is $(n-2)$-pseudoconvex in W, by [Sl2, Theorem 4.3]. Since W_- and W_+ are disjoint, their union $W_- \cup W_+$ is also $(n-2)$-pseudoconvex (in W). Now, by a duality result (cf. [Sl2, Corollary 4.10]), the complement of a $(n-2)$-pseudoconvex set has the local maximum property of order $n-(n-2)-2=0$. Thus, the set $Y = W \backslash (W_- \cup W_+)$ has the usual local maximum property. □

Proof of Theorem 3.9. Choose disjoint, open and relatively compact sets U, V such that $Y \subset U$, $\{\varphi = r\} \backslash Y \subset V$.

There is an $s<r$ such that $\{z \in X\colon s \leqslant \varphi(z) \leqslant r\} \subset U \cup V$. Let $K_0 = \{s \leqslant \varphi(z) \leqslant r\} \cap U$. Clearly, K_0 is compact, $bK_0 \subset \{\varphi = s\} \cup Y$, and $bK_0 = \{\varphi = r\} \cap U = Y$. (The latter identity holds because φ is not constant on any neighbourhood.) Since none of the points of Y is a local minimum point of φ, the open set $\{s<\varphi<r\} \cap U$ is nonempty and contained in K_0, thus $\mathring{K}_0 \neq \emptyset$.

Finally, K_0 is a connected set. Suppose not, then there is a decomposition $K_0 = K_1 \cup L$, where K_1, L are compact and disjoint, $K_1 \supset Y$, and so $bL \subset \{\varphi = s\}$. If L is not contained in $\{\varphi = s\}$, then $\max_L \varphi = t_0 > s$. Let $z_0 \in L$ be any point where $\varphi(z_0) = t_0$. As $z_0 \in L \backslash bL$ i.e. is an interior point of L, there is a connected open set W such that $z_0 \in W \subset L$. Thus, φ has maximum at z_0 relative to W and so (being plurisubharmonic) must be constant on W, contrary to the assumptions. Consequently, $L \subset \{\varphi = s\}$. However, as $L \subset K_0 = U \cap \{s \leqslant \varphi \leqslant r\}$, we find that φ attains a constant maximum value $\varphi(s)$ on L relative to some connected open neighbourhood of L, and so (by plurisubharmonicity) φ is constant on this neighbourhood, contrary to the assumptions.

Thus, $L = \emptyset$, and K_0 is connected. We let $K := K_0$. It is clear that K has the properties required in (a).

(b) Consider an arbitrary point $p \in Y$. If $d\varphi(p) = 0$, then the equality $[(dd^c\varphi)^{n-1} \wedge d^c\varphi](p) = 0$ holds trivially.

Now assume $d\varphi(p) \neq 0$, and suppose $[(dd^c\varphi)^{n-1} \wedge d^c\varphi](p) \neq 0$. By Lemma 3.10 this means that $[(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi](p) \neq 0$, i.e. $[(\partial\overline{\partial}^c\varphi)^{n-1} \wedge \partial\varphi \wedge \overline{\partial}\varphi](p) > 0$. By the well-known facts this (together with the plurisubharmonicity of φ) implies that the hypersurface $\{\varphi = r\} \subset Y \ni p$ is strictly pseudoconvex in a neighbourhood of p, and that there is a local peak function for p at a neighbourhood of p. Then Y is not a local maximum set, contrary to the assumptions. This means that $[(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi](p) = 0$. By Lemma 3.10, $[(dd^c\varphi)^{n-1} \wedge d^c\varphi](p) = 0$. Consequently, the last identity holds at every point $p \in Y$, and by Lemma 3.12 $(dd^c\varphi)^n = 0$ on $\mathring{K}$ and $[(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi](p) = 0$ on $\mathring{K}$. Applying again Lemma 3.10 we find that $(dd^c\varphi)^{n-1} \wedge d^c\varphi$ and $(dd^c\varphi)^{n-1} \wedge d^c\varphi$ vanish on $\mathring{K} \cup Y$. Since it follows from the assertion below that $Z \subset \overline{\mathring{K}}$; therefore, all the four forms considered above vanish on Z as well (by continuity). (It is unclear at the moment whether $(dd^c\varphi)^n$ vanish at the singular points of Y.)

(c) We need the following assertion: if $t_n \searrow t_0$, $t_n, t_0 \in [s, r]$, and $Y_n = \{\varphi = t_n\} \cap K$, $Y_0 = \{\varphi = t_0\} \cap K$, then Y_n converges to Y_0 in the Hausdorff topology on compact sets.

Let $t_0 \in [s, r]$. By (b), $(dd^c\varphi)^{n-1} \wedge d^c\varphi = 0$ on $Y_0 = \{\varphi = t_0\} \cap K$. If t_0 is a noncritical value of φ, then by Lemma 3.13, Y_0 must have the local maximum property. Suppose now, that t_0 is a critical value. By Sard Theorem (cf. [N3]), the set of critical values is dense in $[s, r]$. We choose a decreasing sequence $\{t_n\}$ converging to t_0, $t_n \searrow t_0$. Then the Y_n's have the local maximum property and, by the assertion, $\lim_{n \to +\infty} Y_n = Y_0$. By [Sl2, Proposition 4.10 (iii)], the limit set Y_0 must have the local maximum property as well. □

4. Weakly complete surfaces

1. From now on we will assume that X is a *weakly complete surface* i.e. a connected weakly complete 2-dimensional complex manifold. By *holomorphic curve*, we intend a pure 1-dimensional complex space. A *complex curve* is a regular holomorphic curve.

For a weakly complete surface X the structure of the minimal kernel Σ^1 can be made further precise.

Lemma 4.1. *Let $\varphi : X \to \mathbb{R}$ be a C^2 minimal function. Assume that c is a regular value of φ. Then $\Sigma^1_c = \Sigma^1 \cap \{\varphi = c\}$ is a compact foliated space (foliated by complex curves) provided it is nonempty. Furthermore, for every point $p_0 \in \Sigma^1_c$, there is a special holomorphic coordinate chart (U, z, w), where $p \in U$, $z, w : U \to \mathbb{C}$, $z(p_0) = w(p_0) = 0$, and numbers $\varepsilon, \delta > 0$ such that:*

(i) $\overline{D}_\varepsilon \times \overline{D}_\delta := \{p \in U : |z(p)| \leqslant \varepsilon, |w(p)| \leqslant \delta\} \Subset U$;

(ii) *there is a family of holomorphic functions $f_t : D_\varepsilon \to D_\delta$, $t \in T$, such that, in the coordinates (z, w)*

$$\Sigma^1_c \cap (D_\varepsilon \times D_\delta) = \bigcup_{t \in T} \{(z, f_t(z)) : z \in D_\varepsilon\};$$

(iii) *there is a C^2-smooth function $v : D_\varepsilon \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that, in the coordinates (z, w)*

$$\{\varphi = c\} \cap (D_\varepsilon \times D_\delta) = \{(z, t + iv(z, t)) : |z| < \varepsilon, |w| < \delta\}.$$

Proof. The problem being local, we can assume, without loss of generality, that X is a neighbourhood in $\mathbb{C}^2$. We can make a complex-affine change of coordinates so that in the new coordinates (z, w), $z = x + iy$, $w = u + iv$, the point p becomes $(0, 0)$ and the (real) tangent hyperplane to $S = \{\varphi = c\}$ is identical with the hyperplane $\operatorname{Im} w = 0$, i.e. $T_p(S) = \mathbb{C} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{C}$.

Let us write $\mathbb{C}^2 = \mathbb{C} \times \mathbb{R} \times i\mathbb{R}$ and choose a ball B in $\mathbb{C} \times \mathbb{R}$,

$$B = B_r = \{(z, u) \in \mathbb{C} \times \mathbb{R}\colon\ |z|^2 + u^2 < r^2\},$$

$r > 0$, small enough so that the connected component of

$$(\overline{B} \times i\mathbb{R}) \cap \{\varphi = c\}$$

containing 0 is the graph of some C^2 function $v : \overline{B} \to \mathbb{R}$, i.e. equal to

$$\{(z, w) \in \overline{B} \times i\mathbb{R}\colon\ \operatorname{Im} w = v(z, u)\}.$$

Now let $u : \overline{B} \to \mathbb{R}$ be the unique continuous solution of the Dirichlet problem

$$\begin{cases} L_0 u = 0 & \text{in } B, \\ u = v & \text{on } bB \end{cases}$$

for the Levi operator, like in [ST1, Theorem 13]. Denote by Y the graph of u and by Y_0 the graph of $u_{|bB}$; let

$$Z = \Sigma_c^1 \cap (B \times i\mathbb{R}),$$

$$Z_0 = \Sigma_c^1 \cap (bB \times i\mathbb{R}).$$

Since Z has local maximum property and $\overline{Z} \backslash Z \subset Z_0 \subset Y_0$, and since $\widehat{Y_0} = Y$ ($\widehat{Y_0}$ denotes the polynomial hull), therefore $Z \subset Y \backslash Y_0$.

By [Sh], the hypersurface $Y \backslash Y_0$ is foliated by complex leaves so, in order to show that Σ_c^1 is foliated by complex leaves, it suffices to show the following.

Assertion. A local maximum set Z, relatively closed in a foliated hypersurface $Y \backslash Y_0$, is the union of a family of complex leaves of Y.

Indeed, suppose that there is a complex leaf l of Y which intersects Z but is not contained in Z. Let (z_0, w_0) be a point in l which belongs to the relative boundary in l of the set $Z \cap l$.

By the properties of Shcherbina's foliation of $Y \backslash Y_0$ (cf. [Sh, Main Theorem (ii)]), there is a disc $D_\varepsilon = \{z \in \mathbb{C}\colon\ |z - z_0| < \varepsilon\}$ and a holomorphic function $f_0 : D_\varepsilon \to \mathbb{C}$, such that $f_0(z_0) = w_0$ and $\Delta_0 := \{(z, f_0(z))\colon\ z \in D_\varepsilon\} \subset l$. Observe that Δ_0 is an analytic disc such that $(z_0, w_0) \in \Delta_0 \cap Z$, $\Delta_0 \subset Z$.

Consider now a family of analytic discs Δ_s, $0<s<k$, $k>0$, defined by "vertical" translation,

$$\Delta_s = \Delta_0 + is = \{(z, f_0(z) + is):\ z \in \Delta_\varepsilon\}.$$

Clearly,

$$(\overline{\Delta}_s, \overline{\Delta}_s \setminus \Delta_s) \to (\overline{\Delta}_0, \overline{\Delta}_0 \setminus \Delta_0)$$

in the Hausdorff metric, as $s \to 0^+$, and $\overline{\Delta}_s \cap Y = \emptyset$, for $s>0$.

Observe that the set

$$W = B \times i\mathbb{R} \setminus Z$$

is pseudoconvex, and clearly $\overline{\Delta}_s \subset V$, $s>0$, and $\Delta_0 \not\subset V$. By the strong continuity principle (strong disc theorem of Bremermann, [V, p. 151]) Δ_0 must be contained in bV, i.e. $\Delta_0 \subset Z$. This proves the assertion. $\square$

Remark 4.1. Vladimirov formulates the strong disc theorem for flat discs only. However, by applying the biholomorphical transformation $(z, w) \to (z, w - f_0(z))$ in $D_\varepsilon \times \mathbb{C}$, we can change our discs Δ_s, $s \geqslant 0$, into flat discs, and so reduce our case to his.

2. The method of proof of the above lemma can be employed to state the following.

Proposition 4.2. *Let M be a C^0 hypersurface in $\mathbb{C}^2$ and $Z \subset M$ a holomorphic curve (not necessarily locally closed). Assume that M is locally a graph of some continuous function. Then Z does not have any singular point.*

Proof. Suppose $p \in Z$ is a singular point of Z. Let the coordinates (z, w) and $B = B_r \subset \mathbb{C} \times \mathbb{R}$ be chosen in such a way that $p = (0, 0)$ and locally at p M is the graph of some continuous function $v : \overline{B} \to \mathbb{R}$.

Since $Z \cap (\overline{B} \times i\mathbb{R}) \subset \mathrm{graph}(v)$, the intersection $Z \cap \{0\} \times \mathbb{C}$ is a countable set (at worse). Let D_δ be the disc $\{z \in \mathbb{C}:\ |z| < \delta\}$. It is possible to choose $\delta > 0$ such that the connected component F of the intersection $(\overline{D}_\delta \times \mathbb{C}) \cap Z$ containing 0 is compact and $F \cap (D_\delta \times \mathbb{C})$ is a relatively closed holomorphic curve of $D_\delta \times \mathbb{C}$ and $F \cap \{0\} \times \mathbb{C} = \{(0, 0)\}$. We now apply the main result of [Sl3], to show that F must be an analytic disc.

Fix $\zeta \in bD_\delta$. The set $F_\zeta = F \cap (\{\zeta\} \times \mathbb{C})$ is finite and contained in the real curve $\gamma_\zeta = \{(\zeta, t + iv(\zeta, t)):\ t \in \mathbb{R}\}$. Let X_ζ be the smallest closed subarc of γ_ζ containing F_ζ. Let $X = \bigcup_{\zeta \in bD_\delta} \{\zeta\} \times X_\zeta$. Then X is a compact subset of $bD_\delta \times \mathbb{C}$ containing $F_0 = (bD_\delta \times \mathbb{C}) \cap F$, where $F \setminus F_0$ is a holomorphic curve. Hence, the polynomially convex hull $Y = \hat{X}$ of X contains F, and so is nontrivial (i.e. $Y \neq X$). By the quoted result [Sl3], $Y \setminus X$ is foliated by graphs of bounded holomorphic functions $D_\delta \to \mathbb{C}$. Observe now that by shrinking δ further, if necessary, we can assume without loss of generality that $Y \subset B \times i\mathbb{R}$, and so $Y \subset \text{graph}(v)$, i.e. Y is nowhere dense in $\mathbb{C}^2$ (in fact each $Y_\zeta = Y \cap (\{\zeta\} \times \mathbb{C}))$, $\zeta \in D_\zeta$ is contained in a curve $\{(\zeta, t + iv(\zeta, t)) : t \in \mathbb{R}\}$. Again by the quoted theorem of [Sl3], this implies that $Y \backslash X$ has unique foliation by analytic discs, graphs of holomorphic functions. Applying this theorem over smaller discs we find that the regular part of $F \backslash F_0$ is contained in the union of at most countably many disjoint analytic discs of the unique foliation of $Y \backslash X$. But since F is the closure of the regular part of $F \backslash F_0$, and since F is connected, we conclude that it is contained in exactly one analytic disc of the foliation of $Y \backslash X$, which is the graph of a bounded holomorphic function. Hence, F is regular. In particular, p is a regular point of Z. $\square$

Remark 4.2. The proof when M is C^1 is quite elementary using the Weierstrass preparation theorem, and it is valid for any dimension, i.e. a complex analytic subspace Z of $\mathbb{C}^n$ of pure dimension $n-1$, contained in a real C^1 hypersurface M, must be regular.

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