Indag. Mathem., N.S., 15 (1), 129–132 March 29, 2004

Short series over simple zeros of the Riemann zeta-function

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Communicated by Prof. R. Tijdeman at the meeting of September 29, 2003

ABSTRACT

Recently, Garaev showed that the series $\sum_\rho |\rho\zeta'(\rho)|^{-1}$ diverges, where the sum is taken over the simple zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function $\zeta(s)$. More precisely, he proved $\sum_{0<\gamma<T} |\rho\zeta'(\rho)|^{-1} \gg (\log T)^{1/2}$. Using a mean-value estimate due to Ramachandra and some result on the distribution of simple zeros $1/2 + i\gamma$ in short intervals on the critical line, we prove $\sum_{T<\gamma<T+H} |\zeta(1/2 + i\gamma)|^{-1} \gg H(\log T)^{-1/4}$ for $T^{0.552} \le H \le T$. This leads to a slight improvement of Garaev's result in replacing his lower bound by $(\log T)^{3/4}$.

1. INTRODUCTION AND STATEMENT OF RESULTS

The distribution of the nontrivial (non-real) zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function $\zeta(s)$ is a very interesting topic in number theory; for details we refer to the monography [7]. Let $N(T)$ count the number of nontrivial zeros with $0 < \gamma \le T$ (according multiplicities), then the Riemann-von Mangoldt formula states

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

The famous yet unproved Riemann hypothesis claims that all nontrivial zeros lie on the so-called critical line $Re\, s = 1/2$. It is conjectured that all or at least

Keywords: Riemann zeta-function, simple zeros.
AMS subject classification: 11M06.

almost all zeros of the zeta-function are simple. Recently, Garaev [2] showed that the series

$$\sum_{\substack{\rho \\ \zeta'(\rho)\neq 0}} |\rho\zeta'(\rho)|^{-1}$$

is divergent; this remarkable result was before only known subject to the truth of the Riemann hypothesis (see [7], page 374). Actually, Garaev proved a stronger result, namely that for every positive integer n

$$\sum_{\substack{10^n<\gamma<10^{n+1} \\ \zeta'(1/2+i\gamma)\neq 0}} |\gamma\zeta'(1/2+i\gamma)|^{-1} \gg n^{-1/2}.$$

This implies

$$(1) \qquad \sum_{\substack{0<\gamma<T \\ \zeta'(\rho)\neq 0}} |\rho\zeta'(\rho)|^{-1} \gg (\log T)^{1/2}.$$

Following Garaev's argument we study this series with respect to short intervals on the critical line. It will be shown that the use of a mean-value estimate due to Ramachandra yields

Theorem 1 *Let $T^{0.552} \leq H \leq T$. Then, for sufficiently large T,*

$$\sum_{\substack{T<\gamma<T+H \\ \zeta'(1/2+i\gamma)\neq 0}} |\zeta'(1/2+i\gamma)|^{-1} \gg H(\log T)^{-1/4}.$$

This leads to a slight improvement of Garaev's lower bound (1).

Corollary 2 *For sufficiently large T,*

$$\sum_{\substack{0<\gamma<T \\ \zeta'(1/2+i\gamma)\neq 0}} |\gamma\zeta'(1/2+i\gamma)|^{-1} \gg (\log T)^{3/4}.$$

Garaev's argument depends on the fact that a positive proportion of the zeros on the critical line are simple. Levinson [4] localized more than one third of the nontrivial zeros of the zeta-function on the critical line, and as Heath-Brown [3] observed, they are all simple. By optimizing the technique Levinson himself and others improved the proportion sligthly. Introducing Kloosterman sums Conrey [1] proved that more than two fifths of the zeros are simple and on the critical line. In order to prove Theorem 1 we have to consider short intervals. Let $N_1(T)$ denote the number of simple zeros $\rho = 1/2 + i\gamma$ of $\zeta(s)$ on the critical line with $0 < \gamma < T$. In [6] it was shown that

$$(2) \qquad N_1(T+H) - N_1(T) \gg N(T+H) - N(T) \gg H\log T$$

whenever $T^{0.552} \leq H \leq T$. This result relies on Levinson's method combined with a mean-square estimate for the zeta-function multiplied with a suitable mollifier valid for short intervals.

The Cauchy-Schwarz inequality yields

$$N_1(T+H)-N_1(T)=\sum_{\substack{T<\gamma<T+H\\ \zeta'(1/2+i\gamma)\neq 0}}|\zeta'(1/2+i\gamma)|^{-1/2}|\zeta'(1/2+i\gamma)|^{1/2}$$

$$\leq\left(\sum_{\substack{T<\gamma<T+H\\ \zeta'(1/2+i\gamma)\neq 0}}|\zeta'(1/2+i\gamma)|^{-1}\right)^{1/2}\left(\sum_{\substack{T<\gamma<T+H\\ \zeta'(1/2+i\gamma)\neq 0}}|\zeta'(1/2+i\gamma)|\right)^{1/2}. \tag{3}$$

For the second sum on the right hand side of (3) we shall use Garaev's lemma:

Lemma 3 ([2]) *Suppose that $S(t)$ is a complex-valued twice continuously differentiable function on the closed interval $[t_0,t_k]$. Further, suppose that $t_0<t_1<\ldots<t_k$ and $S(t_j)=0$ for $0\leq j\leq k$. Then*

$$\sum_{1\leq j\leq k}|S'(t_j)|\ll\int_{t_1}^{t_k}|S''(t)|dt.$$

This is a simple consequence of the mean-value theorem in real analysis. We deduce

$$\sum_{\substack{T<\gamma<T+H\\ \zeta'(1/2+i\gamma)\neq 0}}|\zeta'(1/2+i\gamma)|\ll\int_{T-H}^{T+H}|\zeta''(1/2+it)|dt, \tag{4}$$

where we enlarged the region for the imaginary parts to assure the existence of a $t_0=\gamma$ with respect to (2). Ramachandra [5] proved that for $H\geq T^{1/2+\epsilon}$

$$\int_{T}^{T+H}|\zeta''(1/2+it)|dt\ll H(\log T)^{9/4}. \tag{5}$$

(Using the first moment instead of the second one saves a factor $(\log T)^{1/4}$; Garaev argued here with a mean-square estimate based on an approximate functional equation for the second derivative.) Substituting (5) in (4) leads to

$$\sum_{\substack{T<\gamma<T+H\\ \zeta'(1/2+i\gamma)\neq 0}}|\zeta'(1/2+i\gamma)|\ll H(\log T)^{9/4}.$$

This and the estimate (2) give via (3) the assertion of the Theorem.
Since we may divide the interval $[T,2T]$ into $\gg T/H$ many disjoint intervals of length H, Theorem 1 yields

$$\sum_{\substack{T<\gamma<2T\\ \zeta'(1/2+i\gamma)\neq 0}}|\gamma\zeta'(1/2+i\gamma)|^{-1}\gg(\log T)^{-1/4}. \tag{6}$$

Let c be a positive constant less than $(\log 2)^{-1}$. Using (7) with $2^{-n}T$ instead of T and summing up over all positive integers $n\leq c\log T$, we obtain

$$\sum_{\substack{T^{1-c\log 2}<\gamma<T \\ \zeta'(1/2+i\gamma)\neq 0}} |\gamma\zeta'(1/2+i\gamma)|^{-1} \geq \sum_{n\leq c\log T} \sum_{\substack{2^{-n}T<\gamma<2^{1-n}T \\ \zeta'(1/2+i\gamma)\neq 0}} |\gamma\zeta'(1/2+i\gamma)|^{-1}$$
$$\gg \sum_{n\leq c\log T} (\log(2^{-n}T))^{-1/4}.$$

For each term $n \leq c\log T$ we have $\log(2^{-n}T) \ll \log T$. This implies the assertion of the corollary.

Remark. After acceptance of the present paper, the authors were kindly informed that also Garaev found the improvement in Collary 2.

ACKNOWLEDGEMENTS

The authors would like to thank the referee for valuable comments and suggestions.

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