

Quantum Inverse Scattering Method. Selected Topics

Four lectures given at Nankai Institute of Mathematics, Tianjin, China, 5–13 April 1991

E K Sklyanin

* † Research Institute for Theoretical Physics, University of Helsinki, Siltavuorenpenger 20C, 00170 Helsinki 17, Finland

Abstract. The lectures present an elementary introduction into the quantum integrable models aimed for mathematical physicists and mathematicians. The stress is made on the algebraic aspects of the theory and the problem of determining the spectrum of quantum integrals of motion. The XXX magnetic chain is used as the basic example. Two lectures are devoted to a detailed exposition of the Functional Bethe Ansatz — a new technique alternative to the Algebraic Bethe Ansatz — and its relation to the separation of variable method. A possibility to extend FBA to the $SL(3)$ is discussed.

Published in “Quantum Group and Quantum Integrable Systems” (Nankai Lectures in Mathematical Physics), ed. Mo-Lin Ge, Singapore: World Scientific, 1992, pp. 63–97.

*The work supported by the Academy of Finland

†Permanent address: St.Petersburg Branch of Steklov Mathematical Institute, Fontanka 27, St.Petersburg 191011, USSR

Lecture 1

It is great pleasure for me to visit China and to take part in this school. I would like to thank the Nankai Institute of Mathematics and Professor Ge Mo-Lin for hospitality.

My lectures will be devoted to the Quantum Inverse Scattering Method (QISM). The QISM is a direction in the theory of quantum integrable systems which starts its history from the summer 1978 when 3 groups: in Leningrad, USSR (Faddeev *et al.*), Fermilab, USA (Thacker, Creamer, Wilkinson) and in Freiburg, Germany (Honerkamp *et al.*), studying the quantum nonlinear Schrödinger equation, had found some striking and puzzling connections between the famous Bethe Ansatz and no less famous Classical Inverse Scattering Method (CISM).

In other words, QISM has arisen as a result of a synthesis of two traditions in the theory of integrable systems which, up to 1978, were developing quite independently. The first tradition originates from the famous paper by Hans Bethe (1931) from which the Bethe Ansatz took its name. It was developed in the works of L.Hulthen, E.Lieb, C.N.Yang, C.P.Yang, R.J.Baxter and many other scholars devoted mainly to exactly soluble models of lattice statistical mechanics and of quantum mechanics. The second, more young, tradition originates from the paper of Gardner, Green, Kruskal and Miura on the KdV equation which gave rise to CISM (Lax, Ablowitz, Kaup, Newell, Zakharov, Shabat, Faddeev,...). Still, the roots of this tradition can be traced even deeper into the mathematics of 19th century (Liouville, Jacobi, Kowalewsky,...).

One should mention also two more directions which contributed considerably to QISM. These are the Factorizable S -matrices theory due to A. B. Zamolodchikov and Al. B. Zamolodchikov and the various group-theoretical approaches (Adler, Kostant, Olshanetsky, Perelomov, Semenov–Tian–Shansky, Reyman *et al.*).

I am not going to present here the complete history of QISM. However, I would like to mention the main successes of QISM: exact quantization of the sine-Gordon equation (Faddeev, Takhtajan) and calculation of correlators for various quantum integrable models (V. E. Korepin, N. M. Bogoliubov, A. G. Izergin, F. A. Smirnov). Witnessing the contemporary Quantum Groups boom I cannot help reminding you that the whole QG business has arisen as a by-product of QISM, being now quite independent discipline. Let me add also one more remark concerning the development of QISM. There is a general trend in QISM which appears to become stronger last years: shift of stress from applications and related analytical questions to algebraic structures underlying the integrability.

In my lectures I am going to give an elementary introduction to QISM and to touch also some more special questions (Functional Bethe Ansatz). Since I would like to concentrate on the mathematical methods involved rather than applications of QISM, in the center of our attention there will be the only problem: calculation of the spectrum

of integrals of motion. Various approaches will be illustrated on the sole example: XXX magnetic chain.

To begin with, let me discuss briefly the concept itself of integrability. In the classical mechanics there is well-known definition of integrability due to Liouville (1855). According to Liouville, the classical Hamiltonian finite-dimensional system is called integrable if it possesses a set of independent integrals of motion commuting with respect to the Poisson bracket

$$\{I_j, I_k\} = 0.$$

The total number of the integrals of motion (including Hamiltonian) should be the half of the dimension of the phase space.

The Liouville's theorem provides also a way of constructing the action-angle variables for an integrable system in terms of the multi-variable curvilinear integrals. Being quite good for theoretical studies, this construction, unfortunately, does not help to perform an effective integration of concrete models. In practice, one needs usually to resort to more special techniques such as CISM or some algebraic methods.

The situation with the definition of the quantum integrability is even worse. The first thing which comes to mind is to mimic the classical Liouville's definition, namely, to require the existence of N (= number of degrees of freedom) commuting operators

$$[I_j, I_k] = 0.$$

Unfortunately, it seems to be very hard to develop this idea up to a satisfactory level of rigor. The main obstacle is the difficulty with correct definition of functional independence of integrals of motion in the quantum case. It is certain that it is necessary to restrict somehow the class of allowed functions but, as far as I know, at the moment there is no consistent theory of that kind. To sum up, I must confess that I don't know any good definition of the quantum complete integrability.

Let me take during these lectures the most pragmatist point of view: I shall call a quantum system "integrable" if it is possible to calculate exactly some quantities of physical interest, such as the common spectrum of commuting quantum integrals of motion or some correlators (in these lectures I'll concentrate on the first problem: the spectrum). The word "exactly" needs, of course, some comment.

An excursion into the history of classical mechanics and mathematical physics shows that the concept of "exact solubility" has been changed during the course of time. The main trend was the permanent extension of the class of functions used. First, elementary functions, then integrals of them, then solutions to certain second order differential equations, elliptic functions *etc.*. The most general concept of exact solubility elaborated in 19th century is, in my opinion, the concept of *separation of variables*, that is, reduction of a multidimensional problem to a series of one-dimensional ones. In my 3rd and 4th

lectures devoted to the “Functional Bethe Ansatz” I’ll try to show you that this old concept works quite effectively in the framework of QISM.

Now let us start the systematic introduction to QISM. The basic idea of QISM, as it has become clear the last years, is purely algebraic. Its roots can be traced up to the very dawn of quantum mechanics. I mean the early matrix formulation of quantum mechanics due to Heisenberg which led to a purely algebraic treatment of the quantum harmonic oscillator (Heisenberg, 1925) and the hydrogen atom (Pauli, 1926). The idea is to include the commutative algebra of the quantum integrals of motion $\{I_n\}$ into some bigger algebra \mathcal{A} . The space of the quantum states of the system in question is considered then as a representation (usually, irreducible) of that bigger algebra \mathcal{A} whose elements produce, roughly speaking, the transitions between the eigenstates of the quantum conserved quantities $\{I_n\}$. The common spectrum of $\{I_n\}$ can be found then by purely algebraic means.

In case of the harmonic oscillator there is only one integral of motion, the number of particles operator N , and it can be included into the Heisenberg Lie algebra \mathcal{A} generated by N and three extra generators: h (central element), a , a^\dagger

$$[h, a] = [h, a^\dagger] = [h, N] = 0$$

$$[a, a^\dagger] = h$$

$$[N, a] = -a \quad [N, a^\dagger] = a^\dagger$$

The procedure of finding the spectrum of N using the creation/annihilation operators a , a^\dagger is well known.

In case of the hydrogen atom (Coulomb problem) the components of the angular momentum vector and the so-called Laplace vector form an algebra whose factor over the relation *Hamiltonian=const* is the $so(4)$ Lie algebra. This remarkable fact allows to determine the spectrum of the Hamiltonian by purely algebraic methods.

In the above examples the bigger algebra \mathcal{A} was, up to minor reservations in the hydrogen atom case, a finite-dimensional Lie algebra. The fundamental peculiarity of QISM consists in using a new class of algebras to describe the dynamical symmetry of quantum integrable systems. These algebras are neither finite-dimensional nor Lie algebras.

The algebras used in QISM are described in terms of the generators $T_{\alpha\beta}(u)$, $\alpha, \beta \in \{1 \dots d\}$ which can be considered as the elements of the square $d \times d$ matrix $T(u)$ depending on the continuous parameter u frequently called the spectral parameter. The associative algebra \mathcal{T}_R is generated then by the quadratic relations

$$\begin{aligned}
& \sum_{\beta_1, \beta_2=1}^d R_{\alpha_1 \alpha_2, \beta_1 \beta_2}(u-v) T_{\beta_1 \gamma_1}(u) T_{\beta_2 \gamma_2}(v) \\
&= \sum_{\beta_1, \beta_2=1}^d T_{\alpha_2 \beta_2}(v) T_{\alpha_1 \beta_1}(u) R_{\beta_1 \beta_2, \gamma_1 \gamma_2}(u-v) \quad \forall u, v
\end{aligned}$$

where $R_{\alpha_1 \alpha_2, \beta_1 \beta_2}(u)$ is some given function (“structure constants tensor”). Using the matrix notation

$$\overset{1}{T} \equiv T \otimes \text{id} \quad \overset{2}{T} \equiv \text{id} \otimes T$$

the former relation can be written down in the compact form

$$R(u-v) \overset{1}{T}(u) \overset{2}{T}(v) = \overset{2}{T}(v) \overset{1}{T}(u) R(u-v) \quad (1.1)$$

The “structure constants” R are required to satisfy the consistency condition (the well known nowadays Yang–Baxter equation)

$$\begin{aligned}
& \sum_{\beta_1, \beta_2, \beta_3=1}^d R_{\alpha_1 \alpha_2, \beta_1 \beta_2}(u) R_{\beta_1 \alpha_3, \gamma_1 \beta_3}(u+v) R_{\beta_2 \beta_3, \gamma_2 \gamma_3}(v) \\
&= \sum_{\beta_1, \beta_2, \beta_3=1}^d R_{\alpha_2 \alpha_3, \beta_2 \beta_3}(v) R_{\alpha_1 \beta_3, \beta_1 \gamma_3}(u+v) R_{\beta_1 \beta_2, \gamma_1 \gamma_2}(u) \quad \forall u, v
\end{aligned}$$

or, briefly,

$$R_{12}(u) R_{13}(u+v) R_{23}(v) = R_{23}(v) R_{13}(u+v) R_{12}(u) \quad (1.2)$$

(the notation is obvious).

It is easy to see that the matrix trace $t(u)$ of $T(u)$

$$t(u) = \text{tr} T(u) \equiv \sum_{\alpha=1}^d T_{\alpha\alpha}(u)$$

forms a commutative family of operators

$$[t(u), t(v)] = 0 \quad \forall u, v$$

which can be thus considered as integrals of motion of some quantum integrable system. Generally speaking, there can be other independent integrals of motion but in the case $d = 2$, to which I’ll restrict my attention in these lectures, $t(u)$ turns out to be the maximal commutative subalgebra of \mathcal{T}_R .

So, given a solution $R(u)$ of the Yang–Baxter equation one can define the quadratic algebra \mathcal{T}_R . Given a representation of the algebra \mathcal{T}_R one obtains a quantum integrable system whose quantum space is the representation space of \mathcal{T}_R and the commutative integrals of motion are $t(u)$. The main problem of QISM is to find their common spectrum and, possibly, correlators of some physically interesting operators.

The main steps of QISM can be summarized as follows:

1. Take an R matrix.
2. Take a representation of \mathcal{T}_R .
3. Find spectrum of $t(u)$.
4. Find correlators.

The first step implies solving the Yang–Baxter equation (YBE). Many particular solutions has been found by trial-and-error method. As for for the general theory of YBE, the major contribution has been made by V. Drinfeld who gave an axiomatics of QISM based on the concept of *Hopf algebras*. In Drinfeld’s axiomatics the steps 1 and 2 of our scheme are intertwined inseparably. In the base of his theory lies the concept of the *quasitriangular Hopf algebra* whose representations produce particular R -matrices. Drinfeld has constructed also an important family of quasitriangular Hopf algebras called *Yangians* $\mathcal{Y}[\mathcal{G}]$ and parametrized by a simple Lie algebra \mathcal{G} .

I don’t intend to go further into details of Drinfeld’s theory. For these lectures I have chosen another approach supposing that a solution to the YBE is given from the very beginning. This standpoint lies closer to the original form of QISM as it appears in the works of Leningrad group and is more convenient for applications and for discussing the main topic of these lectures — the spectrum of $t(u)$ (Step 3).

From this very moment I’ll consider only the case $d = 2$ that is $T(u)$ being a 2×2 matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Moreover, I’ll restrict my attention to the simplest solution $R(u)$ of the YBE, R -matrix of the XXX–magnet, corresponding to the Yangian $\mathcal{Y}[sl(2)]$. The solution is expressed in terms of the permutation operator \mathcal{P} in the tensor product $\mathbf{C}^2 \otimes \mathbf{C}^2$

$$\mathcal{P}x \otimes y = y \otimes x \quad \forall x, y \in \mathbf{C}^2$$

and reads as

$$R(u) = u + \eta \mathcal{P} = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix} \quad \begin{array}{l} a = u + \eta \\ b = u \\ c = \eta \end{array} \quad (1.3)$$

in the natural basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ in $\mathbf{C}^2 \otimes \mathbf{C}^2$.

Let us discuss now the second step of our scheme: finding representations of \mathcal{T}_R for given $R(u)$. The algebra \mathcal{T}_R possesses an important property which is called *comultiplication*. Let $T_1(u)$ and $T_2(u)$ be two representations of \mathcal{T}_R in the spaces V_1 and V_2

respectively. Then the matrix

$$T(u) = T_1(u)T_2(u) \quad T_{\alpha\gamma}(u) \equiv \sum_{\beta=1}^d T_{1,\alpha\beta}(u)T_{2,\beta\gamma}(u)$$

is also a representation of \mathcal{T}_R in the space $V_1 \otimes V_2$ called *tensor product* of representations $T_1(u)$ and $T_2(u)$. The possibility to multiply the representations of \mathcal{T}_R provides immediately an opportunity to construct infinitely many representations given a set of particular representations. Usually such basic representations are chosen to have some simple dependence (say, polynomial) on the spectral parameter u . Traditionally, in QISM such elementary representations are called *L-operators* and their product

$$T(u) = L_N(u) \dots L_2(u)L_1(u)$$

resp. *monodromy matrix*. Every new representation of this type gives rise to a new quantum integrable system. Putting aside the question of completeness of such representations, I would like to remark that this family of representations serves perfectly all the models important for applications.

Let us turn now to our basic example, the algebra \mathcal{T}_R corresponding to the XXX R-matrix described above. This algebra possesses the remarkable central element (Casimir operator) called *quantum determinant*

$$\begin{aligned} \Delta(u) \equiv \text{q-det}T(u) &= \text{tr}_{12} \frac{1-\mathcal{P}}{2} \hat{T}(u - \frac{\eta}{2}) \hat{T}(u + \frac{\eta}{2}) \\ &= \text{tr}_{12} \hat{T}(u + \frac{\eta}{2}) \hat{T}(u - \frac{\eta}{2}) \frac{1-\mathcal{P}}{2} \\ &= D(u - \frac{\eta}{2})A(u + \frac{\eta}{2}) - B(u - \frac{\eta}{2})C(u + \frac{\eta}{2}) \\ &= A(u - \frac{\eta}{2})D(u + \frac{\eta}{2}) - C(u - \frac{\eta}{2})B(u + \frac{\eta}{2}) \\ &= A(u + \frac{\eta}{2})D(u - \frac{\eta}{2}) - B(u + \frac{\eta}{2})C(u - \frac{\eta}{2}) \\ &= D(u + \frac{\eta}{2})A(u - \frac{\eta}{2}) - C(u + \frac{\eta}{2})B(u - \frac{\eta}{2}) \end{aligned} \quad (1.4)$$

Note that the quantum determinant respects the comultiplication

$$\text{q-det}T_1(u)T_2(u) = \text{q-det}T_1(u)\text{q-det}T_2(u)$$

The simplest, one-dimensional representation of \mathcal{T}_R is provided by a constant number matrix K satisfying the identity

$$[R(u), K \otimes K] = 0 \quad \forall u.$$

Actually, this condition is fulfilled for any matrix K due to the $SL(2)$ symmetry of the R-matrix. Note that the quantum determinant of K coincides with its ordinary determinant

$$\text{q-det}K = \det K$$

We shall see in the next lecture that such representations describe the boundary conditions for the integrable chains.

The next in complexity goes the L-operator $L(u)$ which is linear in the spectral parameter u . It is constructed in terms of three operators S_1, S_2, S_3 or $S_3, S_{\pm} \equiv S_1 \pm iS_2$ belonging to the irreducible finite-dimensional ($\dim = 2l + 1$) representation of the Lie algebra $sl(2)$

$$L(u) = u + \eta \sum_{\alpha=1}^3 S_{\alpha} \sigma_{\alpha} = \begin{pmatrix} u + \eta S_3 & \eta S_- \\ \eta S_+ & u - \eta S_3 \end{pmatrix} \quad S_{\pm} = S_1 \pm iS_2 \quad (1.5)$$

$$[S_{\alpha}, S_{\beta}] = i \sum_{\gamma=1}^3 \varepsilon_{\alpha\beta\gamma} S_{\gamma} \quad \begin{aligned} [S_3, S_{\pm}] &= \pm S_{\pm} \\ [S_+, S_-] &= 2S_3 \end{aligned}$$

$$S_1^2 + S_2^2 + S_3^2 = S_3^2 + \frac{1}{2}(S_+ S_- + S_- S_+) = l(l+1)$$

$$\text{q-det} L(u) = (u - l\eta - \frac{\eta}{2})(u + l\eta + \frac{\eta}{2})$$

Since the R-matrix $R(u-v)$ in the quadratic relation (1.1) determining the algebra \mathcal{T}_R depends only on the difference of the spectral parameters, it follows immediately that shift of the spectral parameter by a constant $L(u) \rightarrow L(u - \delta)$ is an automorphism of \mathcal{T}_R and provides thus a little bit richer family of representations.

Now, the comultiplication property allows immediately to construct plenty of representations of \mathcal{T}_R

$$T(u) = KL_N(u - \delta_N) \dots L_2(u - \delta_2) L_1(u - \delta_1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} (u) \quad (1.6)$$

The corresponding quantum determinant reads

$$\Delta(u) = \det K \prod_{n=1}^N \text{q-det} L_n(u - \delta_n) = \det K \prod_{n=1}^N (u - \delta_n - l_n \eta - \frac{\eta}{2})(u - \delta_n + l_n \eta + \frac{\eta}{2})$$

To sum up, we have constructed a family of finite-dimensional ($\dim = \prod_{n=1}^N (2l_n + 1)$) representations of \mathcal{T}_R

$$T(u|K \in \text{Mat}(2, 2), N \in \mathbf{Z}_+, \{l_n \in \mathbf{Z}/2\}_{n=1}^N, \{\delta_n \in \mathbf{C}\}_{n=1}^N)$$

parametrized by the matrix K , number N of L-operators, spins l_n and shifts δ_n . By the reasons which will be explained in the next lecture the corresponding quantum integrable system is called *inhomogeneous XXX spin chain*.

As a matter of fact, the representations constructed turn out to be irreducible for almost all values of δ_n . Moreover, the whole family contains all the irreducible finite-dimensional representations of the Yangian $\mathcal{Y}[sl(2)]$, see prof. M. Jimbo's lectures on the present school. Let me remark that Yangian case corresponds to $K = 1$ only.

In the next lecture we shall study these representations in more details.

Lecture 2

In the last lecture we have constructed a huge family (1.6) of representations of the algebra \mathcal{T}_R associated to the $sl(2)$ -invariant R-matrix (1.3). What I am going to do now is to show that some of the representations constructed, namely, the representations differing by the order of factors $L_n(u - \delta_n)$ in the product (1.6), are equivalent. Obviously, it is enough to prove the statement for the products $L_1(u)L_2(u)$ and $L_2(u)L_1(u)$. In fact, it follows from a general theorem of Drinfeld about the universal R-matrix claiming the equivalence $T_1(u)T_2(u) \simeq T_2(u)T_1(u)$ for any two representations $T_{1,2}(u)$ of Yangian, but it seems me instructive to show the equivalence manifestly on the simplest example.

However, let us consider first more simple representations K which are absent in Drinfeld's theory. It is obvious that the products K_1K_2 and K_2K_1 are not equivalent unless K 's commute. However, for the products of K and L situation is better.

Theorem 2.1 *If $\det K \neq 0$ then $KL(u) \simeq L(u)K$.*

Proof. It is necessary to find such an invertible operator \mathcal{K} in the representation space V of $L(u)$ that

$$KL(u) = \mathcal{K}^{-1}L(u)K\mathcal{K}$$

Let us use the $sl(2)$ -invariance of the L-operator (1.5):

$$[L(u), S^\alpha + \frac{1}{2}\sigma_\alpha] = 0. \tag{2.1}$$

Since the matrix K is invertible it can be represented as an exponent

$$K = k_0 \exp \sum_{\alpha=1}^3 k_\alpha \frac{1}{2} \sigma_\alpha.$$

Define now \mathcal{K} as

$$\mathcal{K} \equiv k_0 \exp \sum_{\alpha=1}^3 k_\alpha S^\alpha.$$

From (2.1) there follows immediately the identity

$$[L(u), K\mathcal{K}] = 0$$

proving the theorem.

It follows from the theorem that one can transform equivalently any product of L 's and invertible K 's to the form (1.6) K being the product of K 's in the same order.

Let us return now to the products of L 's.

Theorem 2.2 *Let two L-operators (1.5) represent the algebra \mathcal{T}_R in the spaces V_1 and V_2 and be characterised by the spins $l_{1,2}$ and shifts $\delta_{1,2}$ respectively. Then an invertible*

operator \mathcal{R} in the space $V_1 \otimes V_2$ intertwining the products $L_1(u - \delta_1)L_2(u - \delta_2)$ and $L_2(u - \delta_2)L_1(u - \delta_1)$

$$\mathcal{R}L_1(u - \delta_1)L_2(u - \delta_2) = L_2(u - \delta_2)L_1(u - \delta_1)\mathcal{R} \quad (2.2)$$

and providing therefore the equivalence of representations

$$L_1(u - \delta_1)L_2(u - \delta_2) \simeq L_2(u - \delta_2)L_1(u - \delta_1)$$

exists if and only if the difference $\delta_{21} \equiv \delta_2 - \delta_1$ does not take one of the following values

$$\delta_{21} \neq \pm(|l_2 - l_1| + 1)\eta, \pm(|l_2 - l_1| + 2)\eta, \dots, \pm(l_2 + l_1)\eta \quad (2.3)$$

Proof. Substituting the expression (1.5) for L into equation (2.2) for \mathcal{R} we obtain the equation

$$\mathcal{R}(u - \delta_1 + S_1^\alpha \sigma_\alpha)(u - \delta_2 + S_2^\beta \sigma_\beta) = (u - \delta_2 + S_2^\beta \sigma_\beta)(u - \delta_1 + S_1^\alpha \sigma_\alpha)\mathcal{R}$$

(here and further the summation over repeated indices is always supposed). Separating the terms containing u one notices that \mathcal{R} should be $SL(2)$ -invariant

$$[\mathcal{R}, S_1^\alpha + S_2^\alpha] = 0 \quad \forall \alpha$$

and depend in fact on the difference $\delta_{21} = \delta_2 - \delta_1$ only. The remaining terms result in the equations

$$\mathcal{R}[\delta_{21}(S_1^\alpha - S_2^\alpha) + 2i\varepsilon^{\alpha\beta\gamma}S_1^\beta S_2^\gamma] = [\delta_{21}(S_1^\alpha - S_2^\alpha) - 2i\varepsilon^{\alpha\beta\gamma}S_1^\beta S_2^\gamma]\mathcal{R} \quad \forall \alpha \quad (2.4)$$

The $SL(2)$ -invariance of \mathcal{R} allows to look for \mathcal{R} in the form

$$\mathcal{R} = \sum_{j=|l_1-l_2|}^{l_1+l_2} \rho_j(\delta_{21})P_j \quad (2.5)$$

where P_j are the projectors corresponding to the expansion of the tensor product of two finite-dimensional irreducible representations of $SL(2)$ labelled by spins $l_{1,2}$ into the sum of irreducible representations labelled by spin j .

Using the last expansion (2.5) together with the equations (2.4) for \mathcal{R} one obtains, after some calculation, the recurrence relation for the eigenvalues $\rho_j(\delta_{21})$ of \mathcal{R}

$$\rho_{j+1}(\delta_{21}) = \frac{\delta_{21} + \eta(j+1)}{\delta_{21} - \eta(j+1)}\rho_j(\delta_{21}) \quad (2.6)$$

which determines \mathcal{R} up to an insignificant scalar factor. It remains to notice that the necessary and sufficient condition for ρ_j to be nonzero and therefore for \mathcal{R} to be

invertible is the condition (2.3) for $\delta_{1,2}$ and $l_{1,2}$. The same condition, in fact, ensures the irreducibility of the product $L_1(u - \delta_1)L_2(u - \delta_2)$ (for more details see Prof. Jimbo's lectures at the present school). In what follows the nondegeneracy condition (2.3) is always supposed to be fulfilled.

Let us turn now to the quantum integrals of motion $t(u) = \text{tr}T(u)$. The last theorem shows that the spectrum $t(u)$ does not depend on order of L 's in the product (1.6). However, the explicit expression of $t(u)$ in terms of the spin operators S_n is not of course invariant under arbitrary permutation of S_n .

For physical applications it is convenient to think of S_n as of spins of some atoms arranged as a one-dimensional closed chain (ring), so that it is natural to use the periodicity condition $n + N \equiv n$ in all the formulas. The integrable models corresponding to the representation (1.6) of the algebra \mathcal{T}_R would look more realistic if one could extract from $t(u)$ the *local integrals of motion* that is the quantities $H^{(k)}$ $k = 1, 2, 3, \dots$ expressible as the sums

$$H^{(k)} = \sum_{n=1}^N H_{n,n-1,\dots,n-k+1}^{(k)}$$

(the periodicity $N + 1 \equiv 1$ is supposed). The local densities $H_{n,n-1,\dots,n-k+1}^{(k)}$ should involve only k adjacent spins $S_n, S_{n-1}, \dots, S_{n-k+1}$

$$[H_{n,n-1,\dots,n-k+1}^{(k)}, S_m^\alpha] = 0 \quad m > n \text{ or } m < n + k - 1$$

An important case when such local integrals exist is that of the *homogeneous spin chain* corresponding to equal spins and shifts: $l_n \equiv l, \quad \delta_n \equiv \delta = 0$

$$T(u) = KL_N(u)L_{N-1}(u) \dots L_2(u)L_1(u)$$

The homogeneous chain has the important property of *translational invariance*: the similarity transformation $U(\cdot)U^{-1}$ defined by the relations

$$US_n^\alpha U^{-1} = S_{n+1}^\alpha \quad US_N U^{-1} = \mathcal{K}_1 S_1 \mathcal{K}_1^{-1} \quad (2.7)$$

leaves $t(u)$ invariant: $Ut(u)U^{-1} = t(u)$. The invariance of $t(u)$ follows directly from the cyclic invariance of the trace and from the theorem 2.1. The transformation $U(\cdot)U^{-1}$ generalizes the ordinary translation for the periodic chain ($K = 1$) to the twisted periodic boundary condition specified by the matrix K . Note that, generally speaking, $U^N \neq 1$ in contrast with the case $K = 1$.

The translational invariance of $t(u)$ suggests that the local integrals $H^{(k)}$ should also be translationally invariant:

$$UH_{n,n-1,\dots,n-k+1}^{(k)}U^{-1} = H_{n+1,n,\dots,n-k+2}^{(k)}$$

The simplest one-point density is

$$H_n^{(1)} = \ln \mathcal{K}_n$$

I leave the proof of the translational invariance $[U, H^{(1)}] = 0$ and commutativity $[H^{(1)}, t(u)] = 0$ for you as an exercise and pass to the problem of two-point Hamiltonian $H^{(2)}$.

The following criterion proposed by Sutherland (1970) will help us in our search for $H^{(2)}$.

Theorem 2.3 *If for some translationally invariant two-point density $H_{n,n-1}^{(2)}$ there exists a one-point translationally invariant density $Q_n(u)$ such that*

$$[H_{21}, L_2(u)L_1(u)] = L_2(u)Q_1(u) - Q_2(u)L_1(u) \quad (2.8)$$

then $H^{(2)}$ commutes with $t(u)$.

Proof. The following calculation shows how the terms cancel in the commutator $[H^{(2)}, t(u)]$

$$\begin{aligned} [\dots + H_{n+1,n}^{(2)} + H_{n,n-1}^{(2)} + \dots, \dots L_{n+1}(u)L_n(u)L_{n-1}(u) \dots] = \\ \dots + L_{n+1}(u)L_n(u)Q_{n-1}(u) - L_{n+1}(u)Q_n(u)L_{n-1}(u) + \dots \\ \dots + L_{n+1}(u)Q_n(u)L_{n-1}(u) - Q_{n+1}(u)L_n(u)L_{n-1}(u) + \dots \end{aligned}$$

To finish the argument it remains to use the cyclic invariance of trace and the translational invariance of $H^{(2)}$ to show the cancellation of the boundary terms $n = 1$ and $n = N$ which I leave to you as an exercise.

The density $H^{(2)}$ can be described in terms of the operator $\mathcal{R}(\delta_{21})$ constructed in the theorem 2.2. Revising the proof of the theorem one observes that the same $\mathcal{R}(\delta_{21})$ intertwines operators $L_1(u)$ and $L_2(v)$

$$\mathcal{R}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)\mathcal{R}(u-v) \quad (2.9)$$

Using the formula (2.6) it is easy to see that $\rho_{j+1}(0) = -\rho_j(0)$ and hence $\mathcal{R}(0)$ is proportional to the permutation operator \mathcal{P} in $V \otimes V$. Let us normalise $\mathcal{R}(u)$ multiplying it by appropriate scalar factor such that $\mathcal{R}(0) = \mathcal{P}_{12}$. Then the wanted expression for $H_{21}^{(2)}$ is given by $H_{21}^{(2)} = \dot{\mathcal{R}}(0)\mathcal{P}_{12}$ where $\dot{\mathcal{R}}$ stands for the derivative $d\mathcal{R}/du$.

To prove the statement let us differentiate (2.9) with respect to u at $u = v$. Multiplying the result

$$H_{21}\mathcal{P}_{12}L_1(u)L_2(u) + \mathcal{P}_{12}\dot{L}_1(u)L_2(u) = L_2(u)\dot{L}_1(u)\mathcal{P}_{12} + L_2(u)L_1(u)H_{21}\mathcal{P}_{12}$$

from the right hand side by \mathcal{P}_{12} and using the obvious identities $\mathcal{P}_{12}\mathcal{P}_{12} = 1$, $\mathcal{P}_{12}L_1\mathcal{P}_{12} = L_2$, $\mathcal{P}_{12}L_2\mathcal{P}_{12} = L_1$ we arrive at the Sutherland equation (2.8) for $Q_n(u) = \dot{L}_n(u)$.

Using the formula (2.6) for the eigenvalues of \mathcal{R} one can derive the eigenvalue expansion for $H_{21}^{(2)}$

$$H_{21}^{(2)} = \text{const} + \text{coeff} \times \sum_{j=|l_1-l_2|}^{l_1+l_2} P_j \sum_{k=1}^j \frac{1}{k}$$

One can show that the higher local integrals $H^{(3)}$ etc. also exist for the homogeneous XXX spin chain. The most popular method to produce them belongs to Baxter (1972) and Lüscher (1976) and gives an expression for $H^{(k)}$ in terms of higher derivatives of $\mathcal{R}(u)$ at $u = 0$. I'll not go into details of the calculation which are well described in the literature and simply reproduce the result.

Let us construct first the quantities $t^{(k)}(u)$ which are easier to describe by words then to write a compact formula. Let $\tilde{t}(u) = t(u + \frac{\eta}{2})$. To construct $t^{(k)}(u)$ take the product $\tilde{t}(u)\tilde{t}(u + \eta) \dots \tilde{t}(u + (k-1)\eta)$. Then subtract all possible terms obtained from it after replacing two adjacent factors $t(u + k\eta)t(u + (k+1)\eta)$ by $\tilde{\Delta}(u + (k + \frac{1}{2})\eta)$ where $\tilde{\Delta}(u) = \Delta(u + \frac{\eta}{2})$ is obtained from the quantum determinant (1.4) by the same shift as $\tilde{t}(u)$ from $t(u)$. Then add all terms obtained from the original product after replacing in the same way two pairs of \tilde{t} 's by $\tilde{\Delta}$'s and continue changing sign at each step until the t 's will be exhausted. The result is $t^{(k)}(u + (k-1)\frac{\eta}{2})$. For example

$$\begin{aligned} t^{(1)}(u) &= \tilde{t}(u) \\ t^{(2)}(u + \frac{\eta}{2}) &= \tilde{t}(u)\tilde{t}(u + \eta) - \tilde{\Delta}(u + \frac{\eta}{2}) \\ t^{(3)}(u + \eta) &= \tilde{t}(u)\tilde{t}(u + \eta)\tilde{t}(u + 2\eta) - \tilde{\Delta}(u + \frac{\eta}{2})\tilde{t}(u + 2\eta) - \tilde{t}(u)\tilde{\Delta}(u + \frac{3\eta}{2}) \\ t^{(4)}(u) &= \tilde{t}(u)\tilde{t}(u + \eta)\tilde{t}(u + 2\eta)\tilde{t}(u + 3\eta) - \tilde{\Delta}(u + \frac{\eta}{2})\tilde{t}(u + 2\eta)\tilde{t}(u + 3\eta) \\ &\quad - \tilde{t}(u)\tilde{\Delta}(u + \frac{3\eta}{2})\tilde{t}(u + 3\eta) - \tilde{t}(u)\tilde{t}(u + \eta)\tilde{\Delta}(u + \frac{5\eta}{2}) \\ &\quad + \tilde{\Delta}(u + \frac{\eta}{2})\tilde{\Delta}(u + \frac{5\eta}{2}) \end{aligned}$$

and so on.

It turns out that if l is the spin of the homogeneous XXX spin chain then $t^{(2l)}(u)$ taken at $u = 0$ is proportional to the generalised translation operator U defined previously (2.7). Finally, the local integrals of motion $H^{(k)}$ are obtained as the coefficients of the power series

$$\tau(u) \equiv \ln U^{-1}t^{(2l)}(u) = \sum_{k=1}^{\infty} \frac{u^k}{k!} H^{(k+1)} \quad (2.10)$$

In conclusion, I want to show you a simple method for generating higher integrals which is not so widely known. The method is based on the concept of *master symmetry* which can be defined as an operator \mathcal{B} producing one integral of motion from another

$$[\mathcal{B}, H^{(k)}] = H^{(k+1)} \quad (2.11)$$

Tetelman (1982) has shown that for the integrable spin chains the master symmetry \mathcal{B} can be taken as the following sum (“boost” operator)

$$\mathcal{B} = \sum_n n H_{n,n-1}^{(2)}$$

Strictly speaking, the above formula makes sense only for the infinite chain since the linearly growing coefficient n contradicts obviously the periodicity of the spin chain. Nevertheless, in the periodic case one can use it quite formally in commutators which “differentiate” the linear dependence on n and yield periodic expressions.

The possibility to use \mathcal{B} as a master symmetry is based on the obvious identity

$$[\mathcal{B}, U^n] = n U^n H^{(2)} \quad (2.12)$$

and the identity

$$[\mathcal{B}, t(u)] = \dot{t}(u) \quad (2.13)$$

which is proven in the same way as the theorem 2.3 with the only difference that due to the factor n in the definition of \mathcal{B} the terms do not cancel completely but add up to $\dot{t}(u)$

$$\begin{aligned} [\dots + (n+1)H_{n+1,n}^{(2)} + nH_{n,n-1}^{(2)} + \dots, \dots L_{n+1}(u)L_n(u)L_{n-1}(u)\dots] = \\ \dots + nL_{n+1}(u)L_n(u)\dot{L}_{n-1}(u) - nL_{n+1}(u)\dot{L}_n(u)L_{n-1}(u) + \dots \\ \dots + (n+1)L_{n+1}(u)\dot{L}_n(u)L_{n-1}(u) - (n+1)\dot{L}_{n+1}(u)L_n(u)L_{n-1}(u) + \dots \end{aligned}$$

Using the identities (2.12) and (2.13) one obtains the commutation relation

$$[\mathcal{B}, \tau(u)] = \dot{\tau}(u) - H^{(2)}$$

between \mathcal{B} and the generating function (2.10) of $H^{(k)}$. Expanding the last equality in powers of u one arrives at the wanted relation (2.11).

Lecture 3

Now we are in a position to attack the main problem of QISM — the spectral analysis of the commuting family $t(u)$, see Step 3 of the general scheme given in the Lecture 1. Traditionally, various methods of solving the problem have the common name: Bethe Ansatz (BA), with the corresponding adjective: coordinate BA, algebraic BA, analytic BA, functional BA, nested BA *etc.*. In the last two lectures I’ll touch briefly the algebraic BA (ABA) described at length in Prof. L. D. Faddeev’s Nankai lectures (1987) and concentrate on the new variant of BA — the functional BA.

The ABA works for the highest vector representations $T(u)$ of the Yangian $\mathcal{Y}[sl(2)]$ that is the representations possessing the heighest (vacuum) vector $|0\rangle$ which is annihilated by the off-diagonal element $C(u)$ of the matrix $T(u)$ and which is an eigenvector of its diagonal elements $A(u)$ and $D(u)$

$$C(u) |0\rangle = 0 \quad \begin{aligned} A(u) |0\rangle &= \Delta_-(u) |0\rangle \\ D(u) |0\rangle &= \Delta_+(u) |0\rangle \end{aligned} \quad \forall u$$

It is easy to show that the eigenvalues $\Delta_{\pm}(u \mp \frac{\eta}{2})$ factorize the quantum determinant $\Delta(u)$ of $T(u)$

$$\Delta_+(u - \frac{\eta}{2}) \Delta_-(u + \frac{\eta}{2}) = \Delta(u)$$

Let us look now for the eigenvectors of $t(u)$ in the form

$$|v_1, v_2, \dots, v_M\rangle = \prod_{m=1}^M B(v_m) |0\rangle$$

Then it is possible to show that the eigenvalue problem

$$t(u) |v_1, v_2, \dots, v_M\rangle = \tau(u) |v_1, v_2, \dots, v_M\rangle$$

is equivalent to the set of Bethe equations

$$\frac{\Delta_+(v_m)}{\Delta_-(v_m)} = \prod_{\substack{n=1 \\ n \neq m}}^M \frac{v_m - v_n - \eta}{v_m - v_n + \eta} \quad m = 1, \dots, M \quad (3.1)$$

the corresponding eigenvalue $\tau(u)$ of $t(u)$ being

$$\tau(u) = \Delta_-(u) \prod_{m=1}^M \frac{u - v_m - \eta}{u - v_m} + \Delta_+(u) \prod_{m=1}^M \frac{u - v_m + \eta}{u - v_m} \quad (3.2)$$

Note that one can obtain the Bethe equations (3.1) from the formula (3.2) for $\tau(u)$ by taking residue at $u = v_m$ and using the smoothness of the polynomial $\tau(u)$. The last equation (3.2) is in turn equivalent to the linear finite-difference spectral problem (Baxter's equation)

$$\tau(u)Q(u) = \Delta_+(u)Q(u + \eta) + \Delta_-(u)Q(u - \eta) \quad (3.3)$$

for the polynomial $Q(u)$ determined by its zeroes $\{v_m\}_{m=1}^M$

$$Q(u) \equiv \prod_{m=1}^M (u - v_m)$$

We shall return to the Baxter's equation when speaking about the Functional Bethe Ansatz.

Let us apply the ABA construction to our favorite example — the inhomogeneous XXX magnetic chain. It is a simple exercise to show that the vacuum vector exists if and only if the matrix K determining the boundary condition is triangular ($K_{21} = 0$). For the sake of simplicity we shall assume K to be diagonal

$$K = \begin{pmatrix} \xi_- & 0 \\ 0 & \xi_+ \end{pmatrix}$$

In that case the vacuum $|0\rangle$ is the common highest vector of the local representations S_n of $su(2)$

$$S_n^- |0\rangle = 0 \quad S_n^3 |0\rangle = -l_n |0\rangle \quad \forall n \in \{1, \dots, N\}$$

The corresponding eigenvalues of $A(u)$ and $D(u)$ are polynomials of degree N

$$\Delta_{\pm}(u) = \xi_{\pm} \prod_{n=1}^N (u - \delta_n \mp l_n \eta)$$

The ABA proved to be quite powerful method which helped to solve such models as sine-Gordon, XXX and XXZ magnets and others. However, being restricted to the highest vector representations, it fails for representations which do not satisfy this condition. In particular, such interesting integrable models as sinh-Gordon, Toda chain, quantum tops fall beyond the reach of ABA. One should mention also the so-called *completeness problem* that is the question if the Bethe eigenstates $|v_1, v_2, \dots, v_M\rangle$ are complete. There are simple examples when this is not the fact and some special investigation is needed in order to gain the missing eigenstates.

In the rest of this lecture and in the last one I'll describe an alternative method, the Functional Bethe Ansatz (FBA), which is free of the restrictions inherent in ABA.

The Functional Bethe Ansatz was born from a fortunate marriage of two ideas. The first one is the central idea of QISM that is using quadratic R matrix algebras (Yangians *etc.*) to describe the dynamic symmetry of integrable systems. The second one is the very old idea of separation of variables. So, before I'll proceed to FBA itself, let me remind you some elementary facts concerning the separation of variables.

Briefly, the separation of variables is a reduction of a multidimensional spectral problem to a system of one-dimensional (multiparameter) spectral problems. The simplest example of separation of variables is provided by the Cartesian coordinates on the plane $\mathbf{R}^2 \ni (x_1, x_2)$.

Let us introduce two commuting differential operators of second order

$$D_1 = \frac{\partial^2}{\partial x_1^2} \quad D_2 = \frac{\partial^2}{\partial x_2^2}$$

$$[D_1, D_2] = 0$$

which can be thought of as the integrals of motion of some integrable quantum system. It follows from the commutativity that the operators D_i possess common eigenfunctions

$$\begin{cases} D_1\Phi = \lambda_1\Phi \\ D_2\Phi = \lambda_2\Phi \end{cases}$$

It is well known that the eigenfunctions factorize

$$\Phi(x_1, x_2) = X_1(x_1)X_2(x_2)$$

the original two-dimensional spectral problem being thus reduced to two independent one-dimensional spectral problems

$$\begin{cases} X_1'' = \lambda_1 X_1 \\ X_2'' = \lambda_2 X_2 \end{cases}$$

Less trivial example is presented by the parabolic coordinates (y_1, y_2)

$$\begin{cases} x_1 = \frac{y_1^2 - y_2^2}{2} \\ x_2 = y_1 y_2 \end{cases}$$

The corresponding commuting second-order differential operators are

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} &= \frac{1}{y_1^2 + y_2^2} \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \\ D &= -2x_1 \frac{\partial^2}{\partial x_2^2} + 2x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_1} &= \frac{1}{y_1^2 + y_2^2} \left(y_2^2 \frac{\partial^2}{\partial y_1^2} - y_1^2 \frac{\partial^2}{\partial y_2^2} \right) \end{aligned}$$

As previously, their common eigenfunctions

$$\Delta\Phi = \lambda\Phi \quad D\Phi = \mu\Phi$$

factorize

$$\Phi(y_1, y_2) = Y_1(y_1)Y_2(y_2)$$

The corresponding separated equations are

$$\begin{cases} Y_1'' - (\lambda y^2 + \mu)Y_1 = 0 \\ Y_2'' - (\lambda y^2 - \mu)Y_2 = 0 \end{cases}$$

Note that, in contrast with the previous example, the spectral parameters λ and μ cannot be decoupled, and both equations should be solved together. This situation of the so called *Multiparameter spectral problem* is typical for the separation of variables in the general case. Note that the Baxter's equation (3.3) arisen in ABA is another example of (finite-difference) multiparameter spectral problem.

In the above examples the separation of variables was produced by a pure change of coordinates. In the most general case, however, the separation should be obtained via

generic canonical transformation involving not only coordinates but also momenta. The simplest example of this kind is the Goryachev-Chaplygin top.

The quantum GC top is described in terms of the operators J_α, x_α $\alpha = 1, 2, 3$ belonging to the $e(3)$ Lie algebra

$$\begin{aligned} [J_\alpha, J_\beta] &= -i\varepsilon_{\alpha\beta\gamma}J_\gamma \\ [J_\alpha, x_\beta] &= -i\varepsilon_{\alpha\beta\gamma}x_\gamma \\ [x_\alpha, x_\beta] &= 0 \end{aligned}$$

The values of the Casimir operators are assumed to be fixed

$$\sum_{\alpha=1}^3 x_\alpha^2 = 1 \quad \sum_{\alpha=1}^3 x_\alpha J_\alpha = 0$$

Note that the second constraint is crucial for integrability.

The Hamiltonian of the model is

$$H = \frac{1}{2}(J_1^2 + J_2^2 + 4J_3^2) - bx_1 = \frac{1}{2}(J^2 + 3J_3^2) - bx_1 \quad (3.4)$$

where b is a parameter (magnitude of the external field).

The second integral of motion commuting with H is

$$G = 2J_3(J^2 - J_3^2 + \frac{1}{4}) + b(x_3J_1 + J_1x_3) \quad (3.5)$$

(note the quantum correction $\frac{1}{4}$).

The separated variables (u_1, u_2) in the classical case were found by Chaplygin himself

$$\begin{aligned} u_1 &= J_3 + \sqrt{J^2} \\ u_2 &= J_3 - \sqrt{J^2} \end{aligned}$$

Their quantum generalization has been found few years ago by Komarov. They are defined as the “operator roots” of the quadratic polynomial with commuting coefficients

$$u^2 - 2J_3u - (J^2 - J_3^2 + \frac{1}{4})$$

(note the quantum correction $\frac{1}{4}$). The common spectrum of (u_1, u_2) turns out to form an equidistant square lattice

$$\text{spec}(u_1, u_2) = \{1 \mp \frac{1}{2} + 2n_1, -1 \pm \frac{1}{2} - 2n_2\}$$

The common eigenfunctions of H and G factorize in the u -representation

$$\Phi(u_1, u_2) = \varphi(u_1)\varphi(u_2)$$

The corresponding one-dimensional separated equation is the same for u_1 and u_2

$$\tau(u)\varphi(u) = d(u+1)\varphi(u+2) + d(u-1)\varphi(u-2)$$

where

$$\tau(u) = u^3 - 2\left(H + \frac{1}{8}\right)u - G$$

$$d(u) = b\sqrt{u^2 - \frac{1}{4}}$$

The example of GC top shows how nontrivial can be the task of finding the separation of variables. There is no general recipe guaranteeing success for any given integrable model. Fortunately, for the integrable systems subject to QISM such a recipe, to which I have given the name “Functional Bethe ansatz (FBA)”, exists, at least in the case of the Yangian $\mathcal{Y}[sl(2)]$. The success of FBA lies in using the algebraic machinery of QISM which provides a powerful tool for finding the separated variables.

So, I am starting the detailed exposition of FBA. Let me remind you that throughout these lectures we consider the algebra \mathcal{T}_R associated to the R matrix of the XXX model. Suppose that the 2×2 matrix $T(u)$ is a polynomial in the spectral parameter u

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = \sum_{n=0}^N u^n T_n = \sum_{n=0}^N u^n \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$$

and defines a representation of \mathcal{T}_R in a finite-dimensional space W .

The recipe which constitutes the heart of FBA can be expressed in few words. Take the polynomial $B(u)$ having commuting operator coefficients (the commutativity

$$[B(u), B(v)] = 0 \quad \forall u, v$$

is a simple consequence of the basic quadratic relation (1.1) for the T matrix). Then the operator zeroes x_n of $B(u)$ provide the wanted separated variables. The rest of my lectures will be devoted to the deciphering and adjustment of this obscure remark.

The first problem which arises immediately is to give the precise meaning to the roots of the operator polynomial $B(u)$. In order to simplify the task and to avoid troubles with degenerate cases I’ll impose few conditions on the representation $T(u)$. The first condition is always fulfilled for irreducible representations of \mathcal{T}_R .

Condition 1 *The senior coefficient T_N is a number matrix. The quantum determinant $\Delta(u)$ of $T(u)$ is a number-valued function.*

The second one is a nondegeneracy condition. It ensures that $B(u)$ and $\Delta(u)$ are polynomials of the maximal degree.

Condition 2

$$B_N \neq 0 \quad \det T_N \neq 0$$

Though it is not yet clear what are the operator roots of $B(u)$ their symmetric polynomials \hat{b}_n are easy to define

$$\hat{b}_n \equiv (-1)^n B_{N-n}/B_N \quad n = 1, 2, \dots, N$$

Using \hat{b}_n one can express $B(u)$ as

$$B(u) = B_N(u^N - \hat{b}_1 u^{N-1} + \hat{b}_2 u^{N-2} - \dots)$$

It follows from the commutativity of $B(u)$ that the operators $\{\hat{b}_n\}_{n=1}^N$ also commute. Let \mathbf{B} be their common spectrum.

$$\mathbf{B} = \text{spec}\{\hat{b}_n\}_{n=1}^N$$

The following condition of semisimplicity is the most restrictive one. Fortunately, it holds for many interesting models.

Condition 3 *The operators $\{\hat{b}_n\}_{n=1}^N$ have a complete set of common eigenfunctions and to every point $\mathbf{b} = (b_1, \dots, b_N) \in \mathbf{B} \subset \mathbf{C}^N$ there corresponds only one eigenfunction.*

It follows from the last condition that the representation space W is isomorphic to the space $\text{Fun}\mathbf{B}$ of functions on the finite set $\mathbf{B} \subset \mathbf{C}^N$. Of course, the isomorphism is not unique. It is defined up to multiplication by some nonzero function on \mathbf{B} . So, let us suppose that some isomorphism is fixed and the operators \hat{b}_n are realized as the operators of multiplication by the corresponding coordinates b_n in \mathbf{C}^N

$$\mathbf{b} = (b_1, \dots, b_N) \in \mathbf{C}^N \quad f \in \text{Fun}\mathbf{B}$$

$$[\hat{b}_n f](\mathbf{b}) = b_n f(\mathbf{b})$$

In what follows in such cases I will not make difference between the operators like \hat{b}_n and the corresponding functions like b_n and always will omit hats over operators (see later operators x_n).

Since b_n are the symmetric polynomials of the roots of $B(u)$ which are yet to be defined we are led to consider the mapping

$$\Theta : \mathbf{C}^N \rightarrow \mathbf{C}^N : \mathbf{x} \rightarrow \mathbf{b}$$

given by the formula

$$b_n(\mathbf{x}) = s_n(\mathbf{x})$$

where $s_n(\mathbf{x})$ is the elementary symmetric polynomial of degree $n \in \{1, \dots, N\}$ of N variables $\{x_n\}_{n=1}^N$.

$$\begin{aligned} s_1(\mathbf{x}) &= x_1 + x_2 + \dots + x_N \\ s_2(\mathbf{x}) &= x_1 x_2 + \dots \\ &\dots \\ s_N(\mathbf{x}) &= x_1 x_2 \dots x_N \end{aligned}$$

Note that the pre-image $\Theta^{-1}(\mathbf{B})$ is symmetric under permutations of the coordinates $\{x_n\}_{n=1}^N$.

And finally, the very last condition.

Condition 4 *Pre-image $\mathbf{X} = \Theta^{-1}(\mathbf{B})$ contains no multiple points that is each $\mathbf{b} \in \mathbf{B}$ has exactly $N!$ pre-images.*

In principle this condition can be omitted but in that case one is urged to use rather sophisticated language of jets in order to work with the multiple points.

It follows from the last condition that the mapping Θ induces isomorphism between $W = \text{Fun}\mathbf{B}$ and the space $\text{Sym Fun}\mathbf{X}$. Now it is easy to define the operator roots x_n of $B(u)$ as multiplication operators in the *extended representation space*

$$\widetilde{W} = \text{Fun}\mathbf{X} \supset W = \text{Sym Fun}\mathbf{X}$$

It is obvious that $\text{spec}\{x_n\}_{n=1}^N = \mathbf{X}$ and that

$$B(u) = B_N \prod_{n=1}^N (u - x_n) \quad (3.6)$$

For the last equality to be correct its right hand side should be restricted to the space W since $B(u)$ is originally is defined only on W . However, the same equality can be considered as definition of the natural extension of $B(u)$ from W to \widetilde{W} .

Working with \widetilde{W} one should always keep in mind that this is a “nonphysical space” and all the final results should use only the original space $W = \text{Fun}\mathbf{B} = \text{Sym Fun}\mathbf{X}$.

The operators x_n being defined the next problem is to calculate the expression for the commuting integrals of motion $t(u) = \text{tr}T(u) = A(u) + D(u)$ in the x -representation and to observe the resulting separation of variables. This task will be solved in several steps. First, let us introduce the “momenta” conjugated to the “coordinates” x_n . Consider the operators X_n^\pm

$$X_n^- = \sum_{p=1}^N x_n^p A_p \equiv [A(u)]_{u=x_n} \quad (3.7)$$

$$X_n^+ = \sum_{p=1}^N x_n^p D_p \equiv [D(u)]_{u=x_n} \quad (3.8)$$

which are obtained from the polynomials $A(u)$ and $D(u)$ as their values at the “points” $u = x_n$. The substitution of operator values for u will be defined correctly if one prescribes some rule for operator ordering. Here the ordering of x ’s to the left is chosen. I shall call it “substitution from the left”. Note that the operators X_n^\pm act from W to \widetilde{W} .

It is convenient to introduce the shift operators $E_n^\pm = e^{\pm\eta\partial/\partial x_n}$

$$E_n^\pm : \mathbf{C}^N \rightarrow \mathbf{C}^N : (x_1, \dots, x_n, \dots, x_N) \rightarrow (x_1, \dots, x_n \pm \eta, \dots, x_N)$$

It turns out that the operators X_n^\pm have nice commutation relations with x_n .

Theorem 3.1

$$X_m^\pm x_n = (x_n \pm \eta \delta_{mn}) X_m^\pm \quad \forall m, n \quad (3.9)$$

Proof. Consider the following commutation relation between $A(u)$ and $B(u)$ which can be extracted from the basic commutation relation (1.1)

$$(u - v)A(v)B(u) + \eta B(v)A(u) = (u - v + \eta)B(u)A(v)$$

Now substitute x_n for v from the left. The second term in the left hand side will vanish since $B(x_n) = 0$.

$$(u - x_n)X_n^- B(u) = (u - x_n + \eta)B(u)X_n^-$$

After substituting the expansion (3.6) for $B(u)$, cancelling the factors B_N and $(u - x_n)$, and expanding both sides in powers of u we obtain the relation

$$X_n^- s(\mathbf{x}) = s(E_n^- \mathbf{x}) X_n^- \quad (3.10)$$

first for any elementary symmetric and hence for any symmetric polynomial $s(\mathbf{x})$. Note that, since \mathbf{X} is a finite set, $s(\mathbf{x})$ can be in fact any symmetric function on \mathbf{X} .

Quite similarly, the commutation relation between X_m^+ and x_n is obtained from the identity

$$(u - v)D(u)B(v) + \eta D(u)B(v) = (u - v + \eta)B(v)D(u)$$

The next natural step would be establishing the commutation relations between X^\pm 's. However, it cannot be done immediately because of the fact that the operators X_n^\pm act by definition from W to \widetilde{W} . So, we need first to extend them from W to \widetilde{W} as we did it with $B(u)$. To this end, consider the constant function ω on \mathbf{X}

$$\omega(\mathbf{x}) = 1 \quad \forall \mathbf{x} \in \mathbf{X}$$

Obviously, ω is symmetric under permutations and thus belongs to $\text{Sym Fun } \mathbf{X} = W$. Consider now the functions Δ_n^\pm on \mathbf{X}

$$\Delta_n^\pm(\mathbf{x}) = [X_n^\pm \omega](\mathbf{x}) \quad \forall \mathbf{x} \in \mathbf{X}$$

defined as the images of ω for the operators X_n^\pm .

Let us show now that due to the commutation relations (3.9) between X_m^\pm and x_n the functions Δ_n^\pm on \mathbf{X} determine uniquely the action of X_n^\pm on any vector s of W . Note that any vector $s \in W$ which is identified due to the isomorphism $W = \text{Fun } \mathbf{X}$ with some symmetric function $s(x_1, \dots, x_N)$ can be created from the cyclic vector ω by

the operator $\hat{s} = s(\hat{x}_1, \dots, \hat{x}_N)$. Then, applying X_n^\pm to s and using the identity (3.10) for the function $s(\mathbf{x})$ we obtain

$$[X_n^\pm s](\mathbf{x}) = [X_n^\pm \hat{s}\omega](\mathbf{x}) = s(E_n^\pm \mathbf{x})[X_n^\pm \omega](\mathbf{x}) = s(E_n^\pm \mathbf{x})\Delta_n^\pm(\mathbf{x})$$

Now we can consider the last equality as the definition of action of X_n^\pm on *arbitrary*, not necessarily symmetric, function $s \in \text{Fun}\mathbf{X} = \widetilde{W}$. Obviously the commutation relations (3.9) with x 's will still be valid for X^\pm 's thus extended on \widetilde{W} . Consequently, the relation (3.10) is now valid for any (not necessarily symmetric) function $s(\mathbf{x})$. In other words, the extended operators X_n^\pm can be expressed in terms of multiplication and shift operators

$$X_n^\pm = \Delta_n^\pm E_n^\pm \quad (3.11)$$

Now we can calculate the commutation relations for extended X^\pm 's.

Theorem 3.2

$$\begin{aligned} [X_m^\pm, X_n^\pm] &= 0 \quad \forall m, n \\ [X_m^+, X_n^-] &= 0 \quad \forall m, n \quad m \neq n \\ X_n^\pm X_n^\mp &= \Delta(x_n \pm \frac{\eta}{2}) \quad \forall n \end{aligned} \quad (3.12)$$

Proof. Let us prove first the commutativity of X_n^- 's. The assertion being obvious for $m = n$, it is enough to consider the case $m = 1, n = 2$. Take the product $A(u)A(v)$ and substitute in it $u = x_1$ and $v = x_2$

$$[A(u)A(v)]_{\substack{u=x_1 \\ v=x_2}} = \sum_{n_1 n_2} x_1^{n_1} x_2^{n_2} A_{n_1} A_{n_2} = \dots$$

Then, using the commutativity of x 's and the definition of X_1^- we obtain

$$\dots = \sum_{n_1 n_2} x_2^{n_2} x_1^{n_1} A_{n_1} A_{n_2} = \sum_n x_2^n X_1^- A_n = \dots$$

Finally, using commutativity of x_2 and X_1^- and the definition of X_2^- we arrive at $X_1^- X_2^-$

$$\dots = \sum_n X_1^- x_2^n A_n = X_1^- X_2^-$$

In the same way, starting from $A(v)A(u)$ one obtains $X_2^- X_1^-$. Since $A(u)$ and $A(v)$ commute due to (1.1) so do X_1^- and X_2^- , the first assertion of the theorem being thus proven. Quite analogously, the commutativity of X^+ 's follows from the commutativity of $D(u)$. Similarly, commutativity of X_m^+ and X_n^- for $m \neq n$ is derived from the identity

$$(u - v)D(u)A(v) + \eta B(u)C(v) = (u - v)A(v)D(u) + \eta B(v)C(u)$$

Consider now one of equalities (1.4) for the quantum determinant

$$\Delta(u - \frac{\eta}{2}) = A(u)D(u - \eta) - B(u)C(u - \eta)$$

Substituting $u = x_n$ from the left leaves only the first term in the right hand side. Proceeding in the same way as above we obtain

$$\begin{aligned} \Delta(x_n - \frac{\eta}{2}) &= [A(u)D(u - \eta)]_{u=x_n} = \sum_{p,q} x_n^p (x_n - \eta)^q A_p D_q = \sum_{p,q} (x_n - \eta)^q x_n^p A_p D_q \\ &= \sum_q (x_n - \eta)^q X_n^- D_q = \dots \end{aligned}$$

Now it remains to use the commutation relation (3.9) between X_n^- and x_n to arrive at the wanted result

$$\dots = \sum_q X_n^- x_n^q D_q = X_n^- X_n^+$$

The analogous equality for $X_n^+ X_n^-$ is obtained in the same way starting from the identity

$$\Delta(u + \frac{\eta}{2}) = D(u)A(u + \eta) - B(u)C(u + \eta)$$

I propose as a home exercise to look once more the proof of the theorem through and to trace where the extended definition of X^\pm 's was used and where the original one (3.7), (3.8).

Let us write down the whole set of commutation relations between x 's and X^\pm 's

$$\begin{aligned} [x_m, x_n] &= 0 & \forall m, n \\ X_m^\pm x_n &= (x_n \pm \eta \delta_{mn}) X_m^\pm & \forall m, n \\ [X_m^\pm, X_n^\pm] &= 0 & \forall m, n \\ [X_m^+, X_n^-] &= 0 & \forall m, n \quad m \neq n \\ X_n^\pm X_n^\mp &= \Delta(x_n \pm \frac{\eta}{2}) & \forall n \end{aligned}$$

It is quite natural to inquire about the representation theory for the associative algebra \mathcal{X}_Δ defined by the generators $\{x_n, X_n^\pm\}_{n=1}^N$ and the above relations, and labelled by the polynomial $\Delta(u)$ of degree $2N$.

In the case of finite-dimensional representations such that the spectrum \mathbf{X} of the operators $\{x_n\}_{n=1}^N$ is simple and has no multiple points, as throughout my lecture due to the Conditions 1–4, the problem of constructing a representation of \mathcal{X}_Δ is equivalent to that of finding the functions $\{\Delta_n^\pm\}_{n=1}^N$ on \mathbf{X} satisfying certain relations which follow from the relations (3.12) between X^\pm 's

$$\begin{aligned} \Delta_m^\pm(\mathbf{x}) \Delta_n^\pm(E_m^\pm \mathbf{x}) &= \Delta_n^\pm(\mathbf{x}) \Delta_m^\pm(E_n^\pm \mathbf{x}) & \forall m, n \quad \forall \mathbf{x} \in \mathbf{X} \\ \Delta_m^+(\mathbf{x}) \Delta_n^-(E_m^+ \mathbf{x}) &= \Delta_n^-(\mathbf{x}) \Delta_m^+(E_n^- \mathbf{x}) & \forall m, n \quad m \neq n \quad \forall \mathbf{x} \in \mathbf{X} \\ \Delta_n^\pm(\mathbf{x}) \Delta_n^\mp(E_n^\pm \mathbf{x}) &= \Delta(x_n \pm \frac{\eta}{2}) & \forall n \quad \forall \mathbf{x} \in \mathbf{X} \end{aligned} \quad (3.13)$$

Strictly speaking, the above relations are not defined when the shift E_n^\pm moves the point \mathbf{x} out of the set \mathbf{X} . It means that the functions $\{\Delta_n^\pm\}_{n=1}^N$ must vanish on the boundary \mathbf{X}_n^\pm of the set \mathbf{X} with respect to the shift E_n^\pm

$$\Delta_n^\pm(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \mathbf{X}_n^\pm \quad (3.14)$$

$$\mathbf{X}_n^\pm = \{\mathbf{x} \in \mathbf{X} \mid E_n^\pm \mathbf{x} \in (\mathbf{C} \setminus \mathbf{X})\}$$

As a consequence, the function $\Delta(u)$ must vanish on the set

$$\bigcup_{n=1}^N \left((\mathbf{X}_n^+ + \frac{\eta}{2}) \cup (\mathbf{X}_n^- - \frac{\eta}{2}) \right)$$

Irreducibility and finite-dimensionality must put severe restrictions on the set \mathbf{X} and the functions $\Delta(u)$, Δ_n^\pm . It seems highly probable that a kind of Stone-von Neumann theorem should exist for the algebra \mathcal{X}_Δ that is there should exist essentially unique irreducible representation for every allowed $\Delta(u)$.

Let us consider a simple example of irreducible representation for \mathcal{X}_Δ . Let numbers $\{\lambda_n^\pm\}_{n=1}^N$ be such that $(\lambda_n^+ - \lambda_n^-)/\eta = 2l_n$ is a positive integer $\forall n$. Let the sets Λ_n be the equidistant (step η) strings connecting λ_n^\pm

$$\Lambda_n = \{\lambda_n^-, \lambda_n^- + \eta, \dots, \lambda_n^+ - \eta, \lambda_n^+\} \quad |\Lambda_n| = 2l_n + 1 \quad (3.15)$$

Let us suppose, in addition, that the sets Λ_n do not intersect

$$m \neq n \quad \Rightarrow \quad \Lambda_m \cap \Lambda_n = \emptyset \quad (3.16)$$

Define now \mathbf{X} as the parallelepiped

$$\mathbf{X} = \Lambda_1 \times \Lambda_2 \times \dots \times \Lambda_N \quad |\mathbf{X}| = \prod_{n=1}^N (2l_n + 1) \quad (3.17)$$

and the functions Δ_n^\pm , $\Delta(u)$ as

$$\begin{aligned} \Delta_n^\pm(\mathbf{x}) &= \Delta_\pm(x_n) \\ \Delta_\pm(u) &= \xi_\pm \prod_{n=1}^N (u - \lambda_n^\pm) \quad \xi_\pm \neq 0 \\ \Delta(u) &= \Delta_-(u + \frac{\eta}{2}) \Delta_+(u - \frac{\eta}{2}) \end{aligned} \quad (3.18)$$

where ξ_\pm are arbitrary nonzero numbers.

Theorem 3.3 *The functions Δ_n^\pm define an irreducible representation of the algebra \mathcal{X}_Δ in the space $\text{Fun}\mathbf{X}$.*

Proof. One can verify easily that the functions Δ_n^\pm satisfy the conditions (3.13), (3.14) and hence define a representation of \mathcal{X}_Δ in $\text{Fun}\mathbf{X}$. Its irreducibility follows from the standard argument. Suppose $V \subset \text{Fun}\mathbf{X}$ is an invariant subspace. Being invariant, in particular, under the commutative subalgebra generated by $\{x_n\}_{n=1}^N$ the space V can consist only of the functions vanishing on some subset of \mathbf{X} . However, such a subspace cannot be invariant under the operators X_n^\pm because of the condition (3.16) which ensures that the functions $\Delta_n^\pm(\mathbf{x})$ have no zeroes on \mathbf{X} other than in \mathbf{X}_n^\pm .

Theorem 3.4 *Let functions $\tilde{\Delta}_n^\pm$ define a finite-dimensional representation of \mathcal{X}_Δ with the same $\Delta(u)$ and \mathbf{X} as for the above described standard irreducible representation. Then the two representations are equivalent.*

Proof. We are going to show that the equivalence is provided by the multiplication operator $\rho : \text{Fun}\mathbf{X} \rightarrow \text{Fun}\mathbf{X} : f(\mathbf{x}) \rightarrow \rho(\mathbf{x})f(\mathbf{x})$ where $\rho(\mathbf{x})$ is a function having no zeroes on \mathbf{X} . The equivalence of operators X_n^\pm leads to the equation

$$\rho(E_n^\pm \mathbf{x}) \Delta_n^\pm(\mathbf{x}) = \rho(\mathbf{x}) \tilde{\Delta}_n^\pm(\mathbf{x})$$

which can be considered as a set of recurrence relations for $\rho(\mathbf{x})$. Note that the recurrence relations are compatible because of the conditions (3.13) for Δ_n^\pm and $\tilde{\Delta}_n^\pm$. The function $\rho(\mathbf{x})$ is thus defined uniquely on \mathbf{X} up to an insignificant coefficient. It remains to show that $\rho(\mathbf{x}) \neq 0$ on \mathbf{X} which follows from the fact that the functions $\tilde{\Delta}_n^\pm(\mathbf{x})$ have no zeroes on \mathbf{X} other than on \mathbf{X}_n^\pm . The last assertion follows in turn from the third of the equalities (3.13), definition (3.18) of $\Delta(u)$ and the condition (3.16).

It is natural to ask whether the above constructed sample representations exhaust all the irreducible finite-dimensional representations of \mathcal{X}_Δ . I don't know the answer and hope that somebody of the audience will be interested in this problem. Fortunately, the results already obtained are enough to help solution of our main problem — spectral analysis of $\tau(u)$.

Lecture 4

Let us apply now the results of the previous lecture concerning the representation theory for the algebras \mathcal{T}_R and \mathcal{X}_Δ to our permanent example — XXX magnet.

Theorem 4.1 *Let the representation $T(u)$ of \mathcal{T}_R be given by the product (1.6)*

$$T(u) = KL_N(u) \dots L_2(u)L_1(u)$$

and the following nondegeneracy conditions be fulfilled. The first one concerns the matrix K

$$\det K \neq 0 \quad K_{12} \neq 0$$

and ensures the Condition 3.1 of Lecture 3. The second condition coincides with the nonintersection condition (3.16) for the sets $\{\Lambda_n\}_{n=1}^N$ defined by the formula (3.15) with

$$\lambda_n^\pm = \delta_n \pm l_n$$

Then all the conditions 3.1–4 of Lecture 3 are satisfied. The spectrum of operators $\{x_n\}_{n=1}^N$ denoted now as $\widetilde{\mathbf{X}}$ is the union of parallelepipeds

$$\widetilde{\mathbf{X}} = \bigcup_{\sigma \in S_N} \sigma \mathbf{X}$$

taken over all permutations σ of coordinates $\{x_n\}_{n=1}^N$. The set \mathbf{X} is defined by the formula (3.17). Note that the sets $\sigma \mathbf{X}$ do not intersect due to condition (3.16). The corresponding representation of the algebra \mathcal{X}_Δ is the direct sum of $N!$ standard irreducible representations with $\text{spec}\{x_n\}_{n=1}^N = \sigma \mathbf{X}$ described in the end of previous lecture, the parameters λ_n being defined above and ξ_\pm being arbitrary nonzero numbers satisfying

$$\xi_+ \xi_- = \det K$$

The proof is performed by induction in N . Let $N = P + Q$ and

$$T(u) = T_P(u)T_Q(u)$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_P & B_P \\ C_P & D_P \end{pmatrix} \begin{pmatrix} A_Q & B_Q \\ C_Q & D_Q \end{pmatrix}$$

Let the generators of \mathcal{X}_Δ corresponding to T, T_P, T_Q be, respectively, zZ^\pm, xX^\pm, yY^\pm and the corresponding functions $\Delta_\pm(u)$ be $\Delta_\pm(u), \Delta_\pm^{(P)}, \Delta_\pm^{(Q)}$.

Consider the eigenvalue problem

$$B(u)\Phi = \beta(u)\Phi$$

for the matrix element $B(u)$ of $T(u)$

$$B(u) = A_P(u)B_Q(u) + B_P(u)D_Q(u)$$

Note that since the spaces $\text{SymFun}\widetilde{\mathbf{X}} \otimes \text{SymFun}\widetilde{\mathbf{Y}}$ and $\text{Fun}\mathbf{X} \otimes \text{Fun}\mathbf{Y}$ are isomorphic one can think of Φ as belonging to $\text{Fun}\mathbf{X} \otimes \text{Fun}\mathbf{Y} \equiv \text{Fun}(\mathbf{X} \times \mathbf{Y})$.

Substitutions $u = x_p$ ($p \in \{1, \dots, P\}$) and $u = y_q$ ($q \in \{1, \dots, Q\}$) result, respectively, in

$$\beta(x_p)\Phi(\mathbf{x}, \mathbf{y}) = b_Q(x_p - y_1) \dots (x_p - y_Q)\Delta_-^{(P)}(x_p)\Phi(E_p^- \mathbf{x}, \mathbf{y})$$

and

$$\beta(y_q)\Phi(\mathbf{x}, \mathbf{y}) = b_P(y_q - x_1) \dots (y_q - x_P)\Delta_+^{(Q)}(y_q)\Phi(\mathbf{x}, E_q^+ \mathbf{y})$$

where

$$\Delta_-^{(P)}(x) = \xi_-^{(P)} \prod_{p=1}^P (x - \lambda_p^-)$$

$$\Delta_+^{(Q)}(y) = \xi_+^{(Q)} \prod_{q=1}^Q (y - \lambda_q^+)$$

Let us extract out of Φ the factor ρ

$$\Phi(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{x}, \mathbf{y}) \Psi(\mathbf{x}, \mathbf{y})$$

satisfying the equations

$$b_Q(x_p - y_1) \dots (x_p - y_Q) \rho(E_p^- \mathbf{x}, \mathbf{y}) = \rho(\mathbf{x}, \mathbf{y})$$

$$b_P(y_q - x_1) \dots (y_q - x_P) \rho(\mathbf{x}, E_q^+ \mathbf{y}) = \rho(\mathbf{x}, \mathbf{y})$$

It is easy to verify that the equations are compatible and have unique solution up to a constant coefficient.

The original spectral problem is written in terms of Ψ as follows

$$\beta(x_p) \Psi(\mathbf{x}, \mathbf{y}) = \Delta_-^{(P)}(x_p) \Psi(E_p^- \mathbf{x}, \mathbf{y})$$

$$\beta(y_q) \Psi(\mathbf{x}, \mathbf{y}) = \Delta_+^{(Q)}(y_q) \Psi(\mathbf{x}, E_q^+ \mathbf{y})$$

and apparently allows the separation of variables

$$\Psi(\mathbf{x}, \mathbf{y}) = \left(\prod_{p=1}^P \theta_p(x_p) \right) \left(\prod_{q=1}^Q \chi_q(y_q) \right)$$

$$\beta(x) \theta_p(x) = \Delta_-^{(P)} \theta_p(x - \eta) \quad x \in \Lambda_p^{(P)}$$

$$\beta(y) \chi_q(y) = \Delta_+^{(Q)} \chi_q(y + \eta) \quad y \in \Lambda_q^{(Q)}$$

It remains to determine the spectrum of $\beta(u)$. Consider the equation

$$\beta(x) \theta(x) = \Delta_-^{(P)}(x) \theta(x - \eta)$$

There is the alternative: either $\theta(\lambda^-) \neq 0$ and hence $\beta(\lambda^-) = 0$, or $\theta(\lambda^-) = 0$. If $\theta(\lambda^-) = 0$ then there is the new alternative: either $\theta(\lambda^- + \eta) \neq 0$ and then $\beta(\lambda^- + \eta) = 0$, or $\theta(\lambda^- + \eta) = 0$, and so on. This argument shows that there exists $\lambda \in \Lambda$ such that $\beta(\lambda) = 0$. Consequently,

$$\beta(u) = \left(\prod_{p=1}^P (u - \lambda_p) \right) \left(\prod_{q=1}^Q (u - \lambda_q) \right) \quad \lambda_p \in \Lambda_p^{(P)} \quad \lambda_q \in \Lambda_q^{(Q)}$$

or

$$\text{spec}\{z_n\}_{n=1}^N \equiv \tilde{\mathbf{Z}} = \bigcup_{\sigma \in S_N} \sigma \mathbf{Z} \quad \mathbf{Z} = \mathbf{X} \times \mathbf{Y}$$

The spectrum of z_n being the same as expected, it remains to refer to the Theorem 3.4 claiming the uniqueness of the representations of the algebra \mathcal{X}_Δ corresponding to the sets $\sigma \mathbf{Z}$.

Let us return now to our main problem — the spectral analysis of $t(u) = A(u) + D(u)$. Consider the spectral problem

$$t(u)\varphi = \tau(u)\varphi$$

then substitute from the left $u = x_n$ and use the definitions (3.7), (3.7) of X_n^\pm together with the expression (3.11) for X_n^\pm . The resulting set of equations

$$\tau(x_n)\varphi(\mathbf{x}) = \Delta_n^+(\mathbf{x})\varphi(E_n^-\mathbf{x}) + \Delta_n^-(\mathbf{x})\varphi(E_n^+\mathbf{x}) \quad n \in \{1, \dots, N\}$$

allows, obviously, separation of variables

$$\varphi(x_1, \dots, x_N) = \prod_{n=1}^N Q_n(x_n)$$

which leads to the set of N one-dimensional finite-difference multiparameter spectral problems

$$\tau(x_n)Q_n(x_n) = \Delta_+(x_n)Q_n(x_n - \eta) + \Delta_-(x_n)Q_n(x_n + \eta) \quad x_n \in \Lambda_n \quad n \in \{1, \dots, N\} \quad (4.1)$$

which are identical in the form with Baxters's equation (3.3) arising in ABA. However, interpretation of the same equation is different. In the ABA case $Q(u)$ was polynomial and now Q 's are functions on discrete sets Λ_n . In the ABA the natural way to solve the equation is to use the Bethe equations (3.1) for the zeroes of $Q(u)$. Though Bethe equations as a rule cannot be solved exactly they are simplified substantially in the infinite-volume limit $N \rightarrow \infty$ which is the most interesting case for applications. In contrast, the FBA result suggests solving the system of recurrence relations for $Q(u)$ numerically which can be of interest for small N . So, for the XXX magnet ABA and FBA approaches are complementary. The FBA interpretation of the spectral problem (4.1) has however some advantage: since we have established the one-to-one correspondence between the eigenfunctions φ of the original multidimensional spectral problem and those of the related one-dimensional ones there is no *completeness problem* in FBA which arises in ABA due to the fact that, generally speaking, there could exist nonzero polynomial solutions to (4.1) having no counterpart among eigenvectors of $t(u)$ and vice versa there could be eigenvectors of $t(u)$ which cannot be expressed as $B(v_1) \dots B(v_M) |0\rangle$. To be just, it is necessary to remark that the incompleteness of

ABA is a matter of degeneration and can always be removed by a small variation of parameters.

Let us discuss now the possibility to apply FBA to the models other than XXX magnetic chains. A possible way to generalize the above results is to consider infinite-dimensional representations of the algebra \mathcal{T}_R . If the spectrum \mathbf{X} of the operators x_n is discrete then the generalization presents no difficulty. Consider for example the Goryachev-Chaplygin top discussed already in the previous lecture. It turns out that in order to construct the matrix $T(u)$ representing the algebra \mathcal{T}_R it is necessary to introduce additional dynamical canonical variables p and q

$$[p, q] = 1$$

commuting with the generators x_α and J_α of $e(3)$. Then the elements of $T(u)$ are defined as follows

$$\begin{aligned} A(u) &= b(x_+u - \frac{1}{2}\{x_3, J_+\}) \\ B(u) &= e^{-2iq}(u^2 - 2J_3u - (J_3^2 + \frac{1}{4})) \\ C(u) &= be^{2iq}[(x_+u - \frac{1}{2}\{x_3, J_+\})(u + p + 2J_3) - bx_3^2] \\ D(u) &= (u + p + 2J_3)(u^2 - 2J_3u - (J^2 - J_3^2 + \frac{1}{4})) + bx_-u - \frac{1}{2}\{x_3, J_-\} \end{aligned}$$

where $\{, \}$ is anticommutator, $x_\pm = x_1 \pm ix_2$, $J_\pm = J_1 \pm iJ_2$. The corresponding R-matrix is $R(u) = u - 2\mathcal{P}$. It follows from the above formulas that

$$t(u) = A(u) + D(u) = u^3 + pu^2 - 2(H_p + \frac{1}{8})u - G_p$$

where

$$H_p = H + pJ_3 \quad G_p = G + p(J^2 - J_3^2 + \frac{1}{4})$$

H and G being the integrals of motion of the quantum GC top (3.4), (3.5). Since p is one of integrals of motion it can be considered as a scalar parameter in H_p and G_p . In particular, for $p = 0$ the GC top is recovered. The general case $p \neq 0$ has also nice physical interpretation: it corresponds to the so-called gyrostat. The zeroes of u_1, u_2 of $B(u)$ and the corresponding separation of variables are the same as discussed already in the previous lecture.

Much more difficult is the case when the zeroes x_n of $B(u)$ have continuous spectrum. Such an example is provided by the periodic Toda chain. In that case the spectrum of x_n is real and continuous but the shift η in the Baxter's equations is purely imaginary. The rigorous mathematical justification of FBA presents in this situation serious analytical difficulties which are not yet overcome. However, quite formal application of FBA, "on the physical level of rigour", leads to reasonable results and agrees with the results for the infinite volume limite obtained by independent methods.

Another challenging problem is the generalization of FBA to Yangians of simple Lie algebras other than $sl(2)$ that is those of rank > 1 . I believe that solution of this

problem will clarify the algebraic roots of FBA scheme which at present is nothing that a misterious prescription: “take zeroes of $B(u)$ and you obtain what you want”. Let me conclude my lectures with a brief discussion of the problem for the $sl(3)$ case.

However, let us return first to the $sl(2)$ case and look once more on the Baxter’s equation (3.3) dividing it by $Q(x)$

$$\tau(u) = \Delta_+(u) \frac{Q(u-\eta)}{Q(u)} + \Delta_-(u) \frac{Q(u+\eta)}{Q(u)}$$

The formula can be rewritten as

$$\tau(u) = \Lambda_+(u) + \Lambda_-(u) \quad \Lambda_{\pm}(u) \equiv \Delta_{\mp}(u) \frac{Q(u \pm \eta)}{Q(u)}$$

It is easy to see that

$$\Delta(u) = \Lambda_-(u + \frac{\eta}{2}) \Lambda_+(u - \frac{\eta}{2})$$

The fact that the quantities $\Lambda_{\pm}(u)$ are expressed in terms of the spectral invariants of the matrix $T(u)$, its trace $t(u)$ and quantum determinant $\Delta(u)$, suggests the interpretation of $\Lambda_{\pm}(u)$ as the eigenvalues of some operators which could be called “the quantum eigenvalues of the 2×2 matrix $T(u)$ ”. One can write down also the quantum analogs of the secular equation for $\Lambda_{\pm}(u)$

$$\Lambda_+(u) \Lambda_+(u + \eta) - \tau(u + \eta) \Lambda_+(u) + \Delta(u + \frac{\eta}{2}) = 0$$

$$\Lambda_-(u - \eta) \Lambda_-(u) - \tau(u - \eta) \Lambda_-(u) + \Delta(u - \frac{\eta}{2}) = 0$$

which turn into the familiar quadratic equation

$$\Lambda^2 - \tau \Lambda + \Delta = 0$$

in the classical limit $\eta \rightarrow 0$. Note that, in contrast with the classical case, Λ_+ and Λ_- satisfy two different equations.

The operators \mathbf{X}_n^{\pm} also can be considered as “quantum eigenvalues” of $T(x_n)$. Note that after substitution $u = x_n$ the matrix $T(u)$ becomes triangular due to $B(x_n) = 0$. In the classical case it means that the diagonal elements $X_n^- = A(x_n)$ and $X_n^+ = D(x_n)$ of $T(u)$ coincide with its eigenvalues. In the quantum case one can also write down the quantum analogs of the secular equations for X_n^{\pm} . They are obtained by excluding X_n^+ or X_n^- from the equalities

$$t(x_n) = X_n^+ + X_n^-$$

$$\Delta(x_n \pm \frac{\eta}{2}) = X_n^{\pm} X_n^{\mp}$$

and read

$$[X_n^+]^2 - X_n^+ t(x_n) + \Delta(x_n + \frac{\eta}{2}) = 0$$

$$[X_n^-]^2 - X_n^- t(x_n) + \Delta(x_n - \frac{\eta}{2}) = 0$$

The above observations need, of course, more profound study aimed at understanding the algebraic meaning of the “quantum eigenvalues”. However, they have already enough heuristic power to make some conclusions concerning FBA for $sl(3)$ case.

Consider the algebra \mathcal{T}_R generated by the 3×3 matrix

$$T(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{32}(u) & T_{33}(u) \end{pmatrix}$$

and the quadratic relations (1.1) with the R-matrix

$$R(u) = u + \eta \mathcal{P}$$

in $\mathbf{C}^3 \otimes \mathbf{C}^3$. The algebra is equivalent to the Yangian $\mathcal{Y}[sl(3)]$.

It turns out that the trace $t_1(u)$ of $T(u)$

$$t_1(u) = \text{tr}T(u)$$

does not provide the complete set of integrals of motion and should be accompanied with the operators

$$t_2(u) = \text{tr}_{12} P_{12}^- \overset{1}{T}(u) \overset{2}{T}(u + \eta)$$

where $P_{12}^- = (1 - \mathcal{P})/2$ is the antisymmetrizer in $\mathbf{C}^3 \otimes \mathbf{C}^3$, see (1.4).

The Casimir operator of \mathcal{T}_R (quantum determinant of $T(u)$) is given by the formula

$$\Delta(u) = \text{q-det}T(u) = \text{tr}_{123} P_{123}^- \overset{1}{T}(u) \overset{2}{T}(u + \eta) \overset{3}{T}(u + 2\eta)$$

where P_{123}^- is the antisymmetrizer in $\mathbf{C}^3 \otimes \mathbf{C}^3 \otimes \mathbf{C}^3$.

The commuting quantities $t_1(u)$, $t_2(u)$ and $\Delta(u)$ constitute three spectral invariants of the matrix $T(u)$.

The ABA for $sl(N)$ case was developed in the papers by Yang, Sutherland, Kulish and Reshetikhin. It is the results of the last two authors which will be especially useful for our purposes. As Kulish and Reshetikhin (1982) have shown, the eigenvalues $\tau_{1,2}(u)$ of $t_{1,2}(u)$ together with the quantum determinant $\Delta(u)$ can be written down in the form

$$\begin{aligned} \tau_1(u) &= \Lambda_1(u) + \Lambda_2(u) + \Lambda_3(u) \\ \tau_2(u) &= \Lambda_1(u)\Lambda_2(u + \eta) + \Lambda_1(u)\Lambda_3(u + \eta) + \Lambda_2(u)\Lambda_3(u + \eta) \\ \Delta(u) &= \Lambda_1(u)\Lambda_2(u + \eta)\Lambda_3(u + 2\eta) = \Delta_1(u)\Delta_2(u + \eta)\Delta_3(u + 2\eta) \end{aligned} \quad (4.2)$$

where the number polynomials $\Delta_{1,2,3}(u)$ like $\Delta_{\pm}(u)$ for $sl(2)$ case are expressed in terms of representation parameters.

The three “quantum eigenvalues” $\Lambda_{1,2,3}(u)$ of $T(u)$ can be expressed in terms of two polynomials $Q_{1,2}(u)$

$$\Lambda_1(u) = \Delta_1(u) \frac{Q_1(u+\eta)}{Q_1(u)} \quad \Lambda_2(u) = \Delta_2(u) \frac{Q_1(u-\eta)}{Q_1(u)} \frac{Q_2(u+\eta)}{Q_2(u)}$$

$$\Lambda_3(u) = \Delta_3(u) \frac{Q_2(u-\eta)}{Q_2(u)}$$

Kulish and Reshetikhin have derived also from (4.2) the finite-difference equations (analog of Baxter’s equation) for $Q_{1,2}(u)$ which with the use of the shift/multiplication operators

$$\Xi_1 = \Delta_1(x) e^{\eta\partial/\partial x}$$

$$\Xi_2 = \Delta_1(x-\eta) \Delta_2(x) e^{\eta\partial/\partial x}$$

can be put into the form

$$\begin{aligned} [\Xi_1^3 - \tau_1(x+2\eta)\Xi_1^2 + \tau_2(x+\eta)\Xi_1 - \Delta(x)]Q_1(x) &= 0 \\ [\Xi_2^3 - \tau_2(x+\eta)\Xi_2^2 + \tau_1(x+\eta)\Delta(x)\Xi_2 - \Delta(x-\eta)\Delta(x)]Q_2(x) &= 0 \end{aligned} \quad (4.3)$$

The last equations resemble very much the FBA separated equations and suggest, by analogy with the $sl(2)$ case, the following conjecture concerning the possible form of FBA in the $sl(3)$ case.

Conjecture. *There exist two sets of operators $x_1^{(n_1)}, X_1^{(n_1)}$ $n_1 \in \{1, \dots, N\}$ and $x_2^{(n_2)}, X_2^{(n_2)}$ $n_2 \in \{1, \dots, 2N\}$ such that $X_{1,2}$ satisfy the “secular equations”*

$$X_1^3 - X_1^2 t_1(x_1) + X_1 t_2(x_1) - \Delta(x_1) = 0 \quad \forall n_1$$

$$X_2^3 - X_2^2 t_2(x_2 - \eta) + X_2 \Delta(x_2 - \eta) t_1(x_2) - \Delta(x_2 - \eta) \Delta(x_2) = 0 \quad \forall n_2$$

which lead to the separated equations (4.3) for the eigenvalue problem for the operators $t_{1,2}(u)$ and the eigenfunction

$$\varphi(\mathbf{x}_1, \mathbf{x}_2) = \prod_{n_1=1}^N Q_1(x_1^{(n_1)}) \prod_{n_2=1}^{2N} Q_2(x_2^{(n_2)})$$

The problem which remains unsolved is how to construct such operators that is to find a generalisation of the recipe $B(u) = 0$ valid for the $sl(2)$ case. In my opinion, solution of this problem will contribute to better understanding of the algebraic roots of Bethe Ansatz.

Bibliographical Notes

Lecture 1

The main ideas of the algebraic approach to the quantum integrability are present already in the papers of fathers of quantum mechanics

W. Heisenberg (1925) *Z. Phys.* **33**, 879.

W. Pauli (1926) *Z. Phys.* **36**, 336–363.

An extensive survey of the Classical Inverse Scattering Method with special stress on its Hamiltonian aspects which are important for quantization can be found in the book

L. D. Faddeev, L. A. Takhtajan (1987) *Hamiltonian Methods in the Theory of Solitons*.
Berlin: Springer.

Reviews of the Quantum Inverse Scattering Method

L. A. Takhtajan, L. D. Faddeev (1979) *Russian Math. Surveys* **34**:5, 11–68.

P. P. Kulish, E. K. Sklyanin (1982a) *Lecture Notes in Physics* **151**, pp.61–119, Berlin:
Springer.

L. D. Faddeev (1984) *Les Houches, Session XXXIX, 1982, Recent Adv. in Field Theory
and Stat. Mech.*, eds. J.-B. Zuber, R. Stora, pp.563–608, Elsevier.

L. D. Faddeev (1987) in *Nankai Lectures on Mathematical Physics, Integrable Systems*,
ed. by X.-C. Song, pp.23–70, Singapore: World Scientific, 1990.

Elementary facts concerning the Yang-Baxter equation are contained in the surveys

P. P. Kulish, E. K. Sklyanin (1982b) *J. Sov. Math.* **19**, 1596–1620.

M. Jimbo (1989) *Int. J. Mod. Phys.* **4**, 3759–3777.

For more sophisticated treatment of the Yang-Baxter equation based on the Quantum
Group theory see

V. G. Drinfeld (1985) *Sov. Math. Dokl.* **32**, 254–258.

V. G. Drinfeld (1988) *Sov. Math. Dokl.* **36**, 212–216.

V. G. Drinfeld (1987) *Quantum Groups*, in *Proceedings of the International Congress
of Mathematicians*, Berkeley, 1986, American Mathematical Society, 798–820.

L. A. Takhtajan (1989) in *Nankai Lectures on Mathematical Physics, Introduction to
Quantum Group and Integrable Massive Models of Quantum Field Theory*, eds.
M.-L. Ge, B.-H. Zhao, pp.69–197, Singapore: World Scientific, 1990.

Lecture 2

The proof of the theorem 2.2 follows the paper

N. Yu. Reshetikhin, P. P. Kulish, E. K. Sklyanin (1981) *Lett. Math. Phys.* **5**, 393–403.

The theorem 2.3 is borrowed from

B. Sutherland (1970) *J. Math. Phys.* **11**, 3183–3186.

The construction of the local integrals of motion using the generating function $\tau(u)$, see formula (2.10), was proposed in

R. J. Baxter (1972) *Ann. Phys.* **70**, 323–337.

M. Lüscher (1976) *Nucl. Phys.* B**117**, 475–492.

The definition of $t^{(k)}(u)$ is taken from (Kulish, Sklyanin, 1982a), see above. For the master symmetries for quantum integrable chains see

M. G. Tetelman (1982) *Sov. Phys. JETP* **55**(2), 306–310.

E. Barouch, B. Fuchssteiner (1985) *Stud. Appl. Math.* **73**, 221–237.

H. Araki (1990) *Commun. Math. Phys.* **132**, 155–176.

The boost operator \mathcal{B} was shown recently to have close relation to Baxter’s corner transfer matrices

H. Itoyama, H. B. Thacker (1987) *Phys. Rev. Lett.* **58**, 1395–1398.

Lecture 3

The Baxter’s equation (3.3) was introduced in

R. J. Baxter (1971) *Stud. in Appl. Math.* **50**, 51–69.

An alternative derivation of Baxter’s equation using “Analytical Bethe Ansatz” is given in

N. Yu. Reshetikhin (1983) *Lett. Math. Phys.* **7**, 205–213.

For the Algebraic Bethe Ansatz see reviews on QISM cited above. Examples of incompleteness of ABA are presented in

L. V. Avdeev, A. A. Vladimirov (1986) *Theor. & Math. Phys.* **69**, 1071–1079.

T. Koma, H. Ezawa (1987) *Progr. Theor. Phys.* **78**, 1009–1021.

Separation of variables in the quantum Goryachev-Chaplygin top (gyrostat) was obtained in

I. V. Komarov (1982) *Theor. & Math. Phys.* **50**, 265–270.

I. V. Komarov, V. V. Zalipaev (1984) *J. Phys. A: Math. Gen.* **17**, 31–49.

The Functional Bethe Ansatz was proposed in papers

E. K. Sklyanin (1985a) *Lecture Notes in Physics* **226**, pp.196–233, Berlin: Springer.

E. K. Sklyanin (1985b) *J. Sov. Math.* **31**, 3417–3431.

influenced deeply by the articles

H. Flashka, D. W. McLaughlin (1976) *Progr. Theor. Phys.* **55**, 438–456.

M. Gutzwiller (1981) *Ann. Phys.* **133** 304–331.

The exposition of FBA in the present lectures is an improved and extended variant of the paper

E. K. Sklyanin (1990) “Functional Bethe Ansatz”, in *Integrable and Superintegrable Systems*, ed. by B. A. Kupershmidt, pp.8–33, Singapore: World Scientific.

The proof of Theorem 3.1 is given in full length. The idea of extended space \widetilde{W} as well as the Theorems 3.3 and 3.4 are new.

Lecture 4

The proof of the Theorem 4.1 sketched in (Sklyanin, 1990) is given here in full length. For the application of FBA to Goryachev-Chaplygin top and Toda chain see, respectively Sklyanin (1985b) and (1985a). The analysis of SL(3) case is based on the paper

P. P. Kulish, N. Yu. Reshetikhin (1982) *Zapiski Nauch. Semin. Leningr. Otd. Matem. Inst. Steklova* **120**, 92–121.