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On the spectral problem in the linear stability study of flows on a sphere

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Abstract

A normal mode instability study of a steady nondivergent flow on a rotating sphere is considered. A real-order derivative and family of the Hilbert spaces of smooth functions on the unit sphere are introduced, and some embedding theorems are given. It is shown that in a viscous fluid on a sphere, the operator linearized about a steady flow has a compact resolvent, that is, a discrete spectrum with the only possible accumulation point at infinity, and hence, the dimension of the unstable manifold of a steady flow is finite. Peculiarities of the operator spectrum in the case of an ideal flow on a rotating sphere are also considered. Finally, as examples, we consider the normal mode stability of polynomial (zonal) basic flows and discuss the role of the linear drag, turbulent diffusion and sphere rotation in the normal mode stability study. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

The barotropic instability of a nondivergent flow on a sphere caused by the existence of a sufficiently large horizontal shear of the flow and its role in the atmosphere dynamics have been studied for half a century [1–6]. Lorenz [2] noted that this type of instability accounted for the existence of a limit in the atmospheric predictability. On the other hand, Simmons et al. [4] showed that barotropic instability can be responsible for a low frequency variability of the

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large-scale atmospheric dynamics. The two most commonly used types of the barotropic instability are the linear (exponential or algebraic) instability of the flow subjected to infinitesimal perturbations [7] and the nonlinear instability of the flow to small but finite perturbations [8,9].

Since the famous work by Rayleigh [10], many papers have been devoted to the linear instability of the zonal flows [1,11–15]. Nevertheless, there are a lot of important questions that remain to be answered, especially on the stability of non-zonal flows. The application of numerical methods permits further insight into this problem, but the basic challenge here is the accuracy of the numerical stability results [16]. Therefore, theoretical results are very important for testing a numerical stability study algorithm [17,18]. In the case of an ideal fluid on a sphere, such testing can be carried out with the new instability conditions developed for the Legendre-polynomial flows, Rossby–Haurwitz waves, Wu–Verkley waves and modons in [19–22].

In this work, we continue the analytical study of the normal mode instability of a steady nondivergent flow on a rotating sphere. We examine the spectrum of the vorticity equation operator linearized about the flow. To this end, we introduce a real-order derivative Λ^r of a function on the unit sphere S by means of a multiplicative operator, and a family H^r of Hilbert spaces of smooth functions on S. Some embedding theorems for H^r considered are used to prove that in the case of a viscous flow, the linearized operator has a compact resolvent, that is, a discrete spectrum with the only possible accumulation point at infinity. Peculiarities of the operator spectrum in the case of an ideal flow on a rotating sphere are also considered. Finally, as examples, we consider the normal mode stability of polynomial (zonal) basic flows and discuss the role of the linear drag, turbulent diffusion and sphere rotation in the normal mode stability study.

2. Normal mode stability

We now briefly describe the method of normal modes used for the study of the linear instability of a two-dimensional steady flow in an incompressible fluid on a rotating unit sphere S (for more details about the normal mode method on a sphere, see [4,6,12,15,16,23,24]).

The motion of such a fluid is described by the nonlinear non-dimensional vorticity equation [6,25]

$$\Delta\psi_t + J(\psi, \Delta\psi + 2\mu) = -\sigma\Delta\psi + \nu(-\Delta)^{s+1}\psi + f \tag{1}$$

written in the geographical coordinates (λ, μ), where λ is the longitude, $\mu = \sin\phi$ (ϕ is the latitude), $\Delta\psi(x,t)$ is the relative vorticity, $\psi(x,t)$ is the streamfunction, $\Delta\psi + 2\mu$ is the absolute vorticity, $f(x,t)$ is the forcing,

$$J(\psi, h) = (\vec{\mathbf{n}} \times \nabla\psi)\cdot\nabla h = \psi_\lambda h_\mu - \psi_\mu h_\lambda \tag{2}$$

is the Jacobian, $\vec{\mathbf{n}}$ is the unit vector normal to the surface of the sphere S, and

$$\nabla h = \left(\frac{1}{\sqrt{1-\mu^2}} h_\lambda, \sqrt{1-\mu^2} h_\mu\right) \tag{3}$$

is the gradient of the function h. The terms $J(\psi, 2\mu) = 2\psi_\lambda$, $\sigma\Delta\psi$ and $\nu(-\Delta)^{s+1}\psi$ describe the sphere rotation, linear drag (Rayleigh friction) and turbulent viscosity, respectively ($\sigma > 0$, $\nu > 0$). The order $s = 1$ corresponds to the viscosity term in the Navier–Stokes equations, while the order $s = 2$ was used in [4,6,24–26]. The natural numbers s were introduced by Lions [27]. We consider here the general case of a real order s: $s \geqslant 1$.

Hereinafter we assume that both the forcing $f(x,t)$ and the solution $\psi(x,t)$ of problem (1) are orthogonal to a constant function on the sphere S [6]:

$$\int_S f(x)\,dx = 0, \qquad \int_S \psi(x)\,dx = 0. \tag{4}$$

Let $\psi(x,t) = \Psi(x) + \phi(x,t)$, where $\Psi(x)$ is the streamfunction of a basic flow and $\phi(x,t)$ is its perturbation. The dynamics of infinitesimal perturbation is described by the linearized equation

$$\zeta_t = \mathcal{L}\zeta, \tag{5}$$

where

$$\mathcal{L}\zeta \equiv J(\Omega, \Delta^{-1}\zeta) - J(\Psi, \zeta) - \left[\sigma + \nu(-\Delta)^s\right]\zeta \tag{6}$$

is the linear operator defined on sufficiently smooth complex-valued functions, $\Omega = \Delta\Psi + 2\mu$ is the absolute vorticity of the basic flow, and $\zeta(x,t) \equiv \Delta\phi(x,t)$ is the relative vorticity of the perturbation.

The normal mode method suggests for (5) a solution of the form

$$\zeta(x,t) = G(x)\exp(\omega t), \tag{7}$$

where ω and $G(x)$ are the eigenvalue and eigenfunction of the spectral problem

$$\mathcal{L}G(x) = \omega G(x) \tag{8}$$

for the operator (6). The mode (7) is said to be stable if $\operatorname{Re}\omega \leqslant 0$, unstable if $\operatorname{Re}\omega > 0$, and neutral if $\operatorname{Re}\omega = 0$.

3. Hilbert spaces of functions on a sphere

Let $C_0^\infty(S)$ denote the set of infinitely differentiable functions $h(x)$ on the unit sphere S such that

$$\int_S h(x)\,dx = 0. \tag{9}$$

The inner product and the norm of these functions are defined as

$$\langle \zeta, h \rangle = \int_S \zeta(x)\bar{h}(x)\,dx \tag{10}$$

and

$$\|h\| = \langle h, h \rangle^{1/2}, \tag{11}$$

respectively, where $\bar{h}(x)$ is the complex conjugate of $h(x)$. The Hilbert space obtained by closing the set $C_0^\infty(S)$ in this norm denote as H^0.

For any real s and each infinitely differentiable function $h(x) \in C_0^\infty(S)$ define a multiplicative operator Λ^s as [6,25]

$$\Lambda^s h(x) = \sum_{n=1}^{\infty} \chi_n^{s/2} Y_n(h; x), \tag{12}$$

where

$$\chi_n = n(n+1) \tag{13}$$

and

$$Y_n(h; x) = (2n+1)(h * P_n)(x)$$

is the orthogonal projection of $h(x)$ on the subspace H_n of the homogeneous spherical polynomials of the degree n. Here $P_n(\mu)$ is the Legendre polynomial of the degree n, and $(h * P_n)(x)$ is the convolution of $h(x)$ and $P_n(\mu)$ [28]. It is well known that subspace H_n is generated by $2n+1$ spherical harmonics $Y_n^m(x) = P_n^m(\mu)e^{im\lambda}$ $(-n \leqslant m \leqslant n)$; that is,

$$Y_n(h; x) = \sum_{m=-n}^{n} h_n^m Y_n^m(x), \tag{14}$$

where h_n^m is the Fourier coefficient, and $P_n^m(\mu)$ is the associated Legendre function of the degree n and zonal wavenumber m [29]. We assume here that the spherical harmonics are normalized so that $\|Y_n^m(\lambda, \mu)\| = 1$, and hence,

$$\|Y_n(h; x)\|^2 = \sum_{m=-n}^{n} |h_n^m|^2. \tag{15}$$

According to definition (12),

$$\Lambda^{2k} h = (-\Delta)^k h$$

for any integer k. In particular, $\Lambda^2 = -\Delta$ and $\Lambda^{-2} = (-\Delta)^{-1}$; that is, operator Λ can be interpreted as the square root of the positive and symmetric Laplace operator: $\Lambda = (-\Delta)^{1/2}$. Since the Laplace operator is in essence the only differential operator on a sphere invariant under isometries [30], it is natural to use operator Λ as the derivative to characterize differential properties of a smooth function on the sphere S [31–35].

For any real s, operator $\Lambda^s : C_0^\infty(S) \to C_0^\infty(S)$ is symmetric, and hence, closable.

Definition 1. Let s be a real. The closure of set $C_0^\infty(S)$ in the norm

$$\|h\|_s \equiv \|\Lambda^s h\| \tag{16}$$

is denoted as H^s. It is a Hilbert space with the inner product defined as

$$\langle \zeta, h\rangle_s = \langle \Lambda^s \zeta, \Lambda^s h\rangle.$$

Definition 2. Let s and r be real. An element $z \in H^s$ is denoted $\Lambda^r \psi$ and called rth derivative of $\psi \in H^s$ if

$$\langle z, h\rangle_s = \langle \psi, \Lambda^r h\rangle_s$$

for any $h \in C_0^\infty(S)$.

The extension operator $\Lambda^r : H^s \to H^s$ is closed.

Lemma 1. *Let s be a real, $r > 0$ and $h \in H^{s+r}$. Then*

$$\|h\|_{s+r} = \|\Lambda^r h\|_s \tag{17}$$

and

$$\|h\|_s \leqslant 2^{-r/2}\|h\|_{s+r}. \tag{18}$$

Proof. Let first $h \in C_0^\infty(S)$ and $\psi = \Lambda^s h$. Obviously, (17) is fulfilled. Further, using (16), (12), (15) and (13) we get

$$\begin{aligned}\|h\|_{s+r}^2 &= \|\Lambda^{s+r} h\|^2 = \|\Lambda^r \psi\|^2 = \sum_{n=1}^{\infty} \chi_n^r \sum_{m=-n}^{n} \left|\psi_n^m\right|^2 \\ &\geqslant 2^r \|\psi\|^2 = 2^r \|h\|_s^2;\end{aligned}$$

that is, (18) is also fulfilled for any $h \in C_0^\infty(S)$. Let now $h \in H^{s+r}$. Then there exists a Cauchy sequence $\{h_n\} \in C_0^\infty(S)$ such that $\|h_n - h\|_{s+r} \to 0$. Note that both $\{h_n\}$ and $\{\Lambda^r h_n\}$ are Cauchy sequences in H^s, besides (17) and (18) are valid for their elements. Therefore, the application of $\lim_{n\to\infty}$ to $\|h_n\|_{s+r} = \|\Lambda^r h_n\|_s$ and $\|h_n\|_s \leqslant 2^{-r/2}\|h_n\|_{s+r}$ proves the lemma. □

Lemma 2. *Let $h \in H^1$. Then $\|\nabla h\| = \|\Lambda h\| = \|h\|_1$.*

Proof. Let first $h \in C_0^\infty(S)$. Applying the partition of unity [36] subordinate to a covering of the sphere S, one can easily get [6]

$$\|\nabla h\|^2 = \langle \nabla h, \nabla h\rangle = \langle -\Delta h, h\rangle = \langle \Lambda^2 h, h\rangle = \|\Lambda h\|^2 = \|h\|_1^2. \tag{19}$$

Let now $h \in H^1$ and $\{h_n\} \in C_0^\infty(S)$ such that $\|h_n - h\|_1 \to 0$. Then the application of $\lim_{n\to\infty}$ to both parts of $\|\nabla h_n\|^2 = \|h_n\|_1^2$ proves the lemma. □

Corollary 1. *Let $s = 0$, $r = 1$ and $h \in H^1$. Then* (18) *is the Poincaré–Steklov inequality.*

Theorem 1. *Let s be a real. Then $\Lambda^{-r} : H^s \to H^s$ is the compact operator for $r > 0$, the Hilbert–Schmidt operator for $r > 1$, and the trace class operator for $r > 2$.*

Proof. Let $r > 0$. By (12), the multipliers $\chi_n^{-r/2} = [n(n+1)]^{-r/2}$ of the operator Λ^{-r} tend to zero as n increases. Therefore Λ^{-r} is compact due to the compactness criterium [36, Theorem of Section 12.2]. Let $r > 1$. Since $\|Y_n^m\| = 1$, we get

$$\sum_{n=1}^{\infty} \sum_{m=-n}^{n} \|\Lambda^{-r} Y_n^m\|^2 = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \chi_n^{-r} = \sum_{n=1}^{\infty} \frac{2n+1}{[n(n+1)]^r}. \tag{20}$$

This series is the square of the Schmidt norm of the operator Λ^{-r} [37]. Using inequality

$$\frac{n+(n+1)}{[n(n+1)]^r} = \frac{1}{n^{r-1}(n+1)^r} + \frac{1}{n^r(n+1)^{r-1}} \leqslant \frac{2}{n^{2r-1}}$$

and comparing (20) with the series $\sum_{n=1}^{\infty} n^{-r}$, we obtain that series (20) converges if $r > 1$. Therefore, the Schmidt norm of operator Λ^{-r} and its trace norm

$$\sum_{n=1}^{\infty} \sum_{m=-n}^{n} \chi_n^{-r/2}$$

are finite if $r > 1$ and $r > 2$, respectively. Thus Λ^{-r} is the Hilbert–Schmidt operator for $r > 1$, and the trace class operator for $r > 2$ [36, Lemma of Section 13.2]. The theorem is proved. □

Corollary 2. *Let s be a real and $r > 0$. Then a bounded set in H^{s+r} is compact in H^s.*

4. Operator spectrum for a viscous fluid

We now show that for a viscous fluid, the resolvent of operator (6) of the spectral problem (8) is compact.

Theorem 2. *Let $s \geqslant 1$, $\nu > 0$, and let $\Psi(x)$ be a basic flow on the sphere S such that*

$$\max_{x\in S}\left|\nabla\Psi(x)\right| \leqslant C_1, \qquad \max_{x\in S}\left|\nabla\Omega(x)\right| \leqslant C_2. \tag{21}$$

Then $\mathcal{L}: H^0 \to H^0$ with the domain H^{2s} is the operator with a compact resolvent, and hence, its spectrum consists entirely of isolated eigenvalues with finite multiplicity. The only accumulation point, if it exists, lies at infinity.

Proof. The spectral problem (8) can be written as

$$\nu \Lambda^{2s} G + \sigma G + J(\Psi, G) - J(\Omega, \Delta^{-1} G) = -\omega G. \tag{22}$$

Its generalized formulation is to find a nonzero solution $\zeta \in H^s$ such that

$$\nu \langle \Lambda^s \zeta, \Lambda^s h \rangle + M(\zeta, h) = -(\omega + \sigma)\langle \zeta, h \rangle \tag{23}$$

is fulfilled for any function $h \in H^s$ where

$$M(\zeta, h) = \big\langle J(\Psi, \zeta), h \big\rangle - \big\langle J(\Omega, \Delta^{-1}\zeta), h \big\rangle. \tag{24}$$

A sufficiently smooth solution of (23) is also a solution of problem (22).

By Schwarz inequality,

$$\langle \Lambda^s \zeta, \Lambda^s h \rangle \leqslant \|\Lambda^s \zeta\| \|\Lambda^s h\| = \|\zeta\|_s \|h\|_s. \tag{25}$$

Thus, for a fixed $\zeta \in H^s$, the sesquilinear form $\langle \Lambda^s \zeta, \Lambda^s h \rangle$ is a bounded linear functional of $h \in H^s$, and by Riesz' theorem [37],

$$\langle \Lambda^s \zeta, \Lambda^s h \rangle = \langle \mathcal{T} \zeta, h \rangle_s, \tag{26}$$

where $\mathcal{T}: H^s \to H^s$ is the non-negative selfadjoint and bounded operator.

Taking into account (21), Lemmas 1 and 2, and the fact that $s \geqslant 1$, we get

$$\begin{aligned} \big|M(\zeta, h)\big| &\leqslant \big(C_1 \|\nabla \zeta\| \|h\| + C_2 \|\nabla \Delta^{-1} \zeta\|\big) \|h\| \\ &= \big(C_1 \|\Lambda \zeta\| + C_2 \|\Lambda^{-1} \zeta\|\big) \|h\| \\ &= \big(C_1 \|\zeta\|_1 + C_2 \|\zeta\|_{-1}\big) \|h\| \leqslant C \|\zeta\|_s \|h\| \leqslant K \|\zeta\|_s \|h\|_s. \end{aligned} \tag{27}$$

Thus for a fixed $\zeta \in H^s$, the sesquilinear form (24) is a bounded linear functional of h in H^s, and by Riesz' theorem,

$$M(\zeta, h) = \langle \mathcal{F} \zeta, h \rangle_s, \tag{28}$$

where operator $\mathcal{F}: H^s \to H^s$ is bounded.

We now show that operator $\mathcal{F}: H^s \to H^s$ is compact. Let $\{u_n\}$ be a sequence of elements in H^s which weakly converges to an element u. Since $\mathcal{F}: H^s \to H^s$ is the bounded operator, $\{\mathcal{F} u_n\}$ also weakly converges to an element $\mathcal{F} u$ in H^s. Due to Corollary 2, both sequences converge in H^0: $\|u_n - u\| \to 0$ and $\|\mathcal{F} u_n - \mathcal{F} u\| \to 0$ as $n \to \infty$. Using (27) and (28) we obtain

$$\|\mathcal{F} u_n - \mathcal{F} u_m\|_s^2 \leqslant C \|u_n - u_m\|_s \|\mathcal{F} u_n - \mathcal{F} u_m\|.$$

Thus $\{\mathcal{F} u_n\}$ is the Cauchy sequence in H^s; that is, $\mathcal{F}: H^s \to H^s$ is compact. Finally, since

$$\langle \zeta, h \rangle \leqslant \|\zeta\| \|h\| \leqslant 2^{-s/2} \|\zeta\| \|h\|_s,$$

the sesquilinear form $\langle\zeta, h\rangle$ is continuous with respect to $h \in H^s$ for each fixed $\zeta \in H^0$, and by Riesz' theorem,

$$\langle\zeta, h\rangle = \langle\mathcal{R}\zeta, h\rangle_s, \tag{29}$$

where operator $\mathcal{R}: H^0 \to H^s$ is bounded. Besides, its restriction $\mathcal{R}: H^s \to H^s$ is selfadjoint nonnegative (due to (29)) and compact as the product $\mathcal{R}\Lambda^{-s}$ of the compact operator $\Lambda^{-s}: H^s \to H^0$ (see Theorem 1) and the bounded operator $\mathcal{R}: H^0 \to H^s$ [37, Theorem 4.8].

Due to (26), (28) and (29), the identity (23) is equivalent to

$$\nu\langle\mathcal{T}\zeta, h\rangle_s + \langle\mathcal{F}\zeta, h\rangle_s = -(\omega+\sigma)\langle\mathcal{R}\zeta, h\rangle_s \tag{30}$$

or to operator equation

$$(\nu\mathcal{T} + \mathcal{F})\zeta = -(\omega+\sigma)\mathcal{R}\zeta. \tag{31}$$

We now show that the inverse to operator $\mathcal{B} = \nu\mathcal{T} + \mathcal{F} + \omega_0\mathcal{R}: H^s \to H^s$ is bounded if the positive number ω_0 is large enough. Indeed, let $\mathcal{B}v = w$. Taking into account that $\langle\mathcal{T}v, v\rangle_s$ and $\langle\mathcal{R}v, v\rangle_s$ are both nonnegative and using (31), (26), (27), (29) and ε-inequality, we obtain

$$\begin{aligned}
\|w\|_s\|v\|_s &\geqslant \left|\langle w, v\rangle_s\right| = \left|\langle\mathcal{B}v, v\rangle_s\right| \geqslant \nu\langle\mathcal{T}v, v\rangle_s + \omega_0\langle\mathcal{R}v, v\rangle_s - \left|\langle\mathcal{F}v, v\rangle_s\right| \\
&\geqslant \nu\|\Lambda^s v\|^2 + \omega_0\|v\|^2 - K\|v\|_s^2 \\
&\geqslant (\nu - K\varepsilon^2)\|v\|_s^2 + \left(\omega_0 - \frac{K}{4\varepsilon^2}\right)\|v\|^2.
\end{aligned}$$

Setting $\varepsilon^2 < \nu/K$ and then choosing ω_0 such that $\omega_0 > K/(4\varepsilon^2)$, we obtain the required estimate:

$$\|v\|_s \leqslant C\|w\|_s = C\|\mathcal{B}v\|_s.$$

Thus,

$$\mathcal{B}\zeta = (\omega_0 - \omega - \sigma)\mathcal{R}\zeta \tag{32}$$

or

$$\mathcal{A}\zeta = (\omega_0 - \omega - \sigma)^{-1}\zeta, \tag{33}$$

where the operator $\mathcal{A} = \mathcal{B}^{-1}\mathcal{R}: H^s \to H^s$ is compact being the product of the bounded operator $\mathcal{B}^{-1}$ and compact operator $\mathcal{R}$. Note that if $s > 1$, then both $\mathcal{R}: H^s \to H^s$ and $\mathcal{A}: H^s \to H^s$ are the Hilbert–Schmidt operators.

We conclude that ω is an eigenvalue of the problem (23) if and only if $(\omega_0 - \omega - \sigma)^{-1}$ is an eigenvalue of the compact operator $\mathcal{A}$. Therefore, there exists at most a countable set of isolated eigenvalues of the problem (23), and hence, of the problem (22), that is, the operator $\mathcal{L}$. Each bounded eigenvalue is of finite multiplicity. The only accumulation point of the eigenvalues may lie at infinity. The theorem is proved. □

For the case of a bounded domain on the plane and $s = 1$, this result was proved in [38]. The completeness of the system of eigenfunctions and generalized eigenfunctions of the operator $\mathcal{A}: H^s \to H^s$ follows from Naimark's [39] theorem (see also [40,41]).

Theorem 3. *Unstable manifold in a vicinity of a stationary flow $\Psi(x)$ is of finite dimension.*

Proof. Let $\mathcal{W} = -\mathcal{L}$. Then (22) and (27) lead to

$$\left|\langle \mathcal{W}\zeta, h\rangle\right| \leqslant C_0 \|\zeta\|_s \|h\|_s, \tag{34}$$

where $C_0 = \nu + \sigma 2^{-s} + K$. Taking into account (24), equation $\mathrm{Re}\langle J(\Psi, h), h\rangle = 0$ and estimates (27) and (18) we obtain

$$\mathrm{Re}\left\langle J(\Omega, \Delta^{-1}h), h\right\rangle \leqslant C_2 \|h\|_{-1} \|h\| \leqslant C_0 \|h\|^2$$

or

$$\begin{aligned} \mathrm{Re}\langle \mathcal{W}h, h\rangle &= \nu \|h\|_s^2 + \sigma \|h\|^2 - \mathrm{Re}\left\langle J(\Omega, \Delta^{-1}h), h\right\rangle \\ &\geqslant \nu \|h\|_s^2 - C_0 \|h\|^2. \end{aligned} \tag{35}$$

Thus, operator $\mathcal{W} = -\mathcal{L}$ satisfies all the conditions of Sattinger' assertion [42, Lemma 3.1], and all the eigenvalues of operator $\mathcal{L}$ (or spectral problem (8)) lie inside the parabolic domain

$$\mathrm{Re}\,\omega < -K_1 (\mathrm{Im}\,\omega)^2 + K_2, \tag{36}$$

where K_1 and K_2 are some fixed positive constants. It follows from (36) and Theorem 2 that operator $\mathcal{L}: H^0 \to H^0$ may have only a finite number of the eigenvalues with positive real part, that is, unstable manifold in a vicinity of the stationary flow $\Psi(x)$ is of finite dimension. The theorem is proved. □

5. Operator spectrum for an ideal fluid

We have showed that for a viscous fluid ($\nu > 0$), the spectrum of the operator $\mathcal{L}$ is discrete, besides the eigenvalues can cluster only at infinity. Consider now the normal mode stability of a flow of an ideal fluid on a rotating sphere ($\nu = 0$, $\sigma = 0$, $f = 0$). The fluid dynamics is then governed by [43],

$$\Delta\psi_t + J(\psi, \Delta\psi + 2\mu) = 0, \tag{37}$$

and unlike a viscous case, the operator $\mathcal{L}$ has a non-empty continuous spectrum as well [44]. Indeed, for a zonal flow $\Psi(\mu)$, this operator accepts the form

$$\mathcal{L}\zeta \equiv (\Psi_\mu - \Omega_\mu \Delta^{-1})\zeta_\lambda, \tag{38}$$

and solution of (5) can be found in the form of a normal mode

$$\zeta(t,\lambda,\mu) = G(\mu)\exp\bigl(im(\lambda-\omega t)\bigr) \tag{39}$$

or as streamfunctions

$$\phi(t,\lambda,\mu) = B(\mu)\exp\bigl(im(\lambda-\omega t)\bigr) = A(\lambda,\mu)\exp(-im\omega t).$$

The substitution of (39) in (5) leads to the spectral problem

$$(\mathcal{L}_1+\mathcal{L}_2)\zeta = \omega\zeta,$$
$$\mathcal{L}_1\zeta = -\Psi_\mu\zeta, \qquad \mathcal{L}_2\zeta = \Omega_\mu\Delta^{-1}\zeta. \tag{40}$$

Note that $\mathcal{L} = im(\mathcal{L}_1+\mathcal{L}_2)$ and

$$\mathcal{L}\zeta = \varrho\zeta, \quad \varrho = im\omega.$$

Also note that together with a solution ω, ζ, problem (40) has a complex conjugate solution $\overline{\omega}$, $\overline{\zeta}$. The operator $\mathcal{L}_1$ has a purely continuous spectrum

$$\Sigma(\mathcal{L}_1) = \Bigl[\min_{-1\leqslant\mu\leqslant 1}\{-\Psi_\mu\}, \max_{-1\leqslant\mu\leqslant 1}\{-\Psi_\mu\}\Bigr]. \tag{41}$$

By definition, the essential spectrum of an operator contains all its eigenvalues except for the isolated eigenvalues with finite multiplicity. Therefore, the essential spectrum $\Sigma_e(\mathcal{L}_1)$ of operator $\mathcal{L}_1$ coincides with $\Sigma(\mathcal{L}_1)$. On the other side, operator $\mathcal{L}_2$ is compact as the product of the bounded operator $\mathcal{L}_3 u = \Omega_\mu u$ and compact operator Δ^{-1}. Since the essential spectrum of an operator is conserved under a compact perturbation [37, Chapter IV, Theorem 5.35], we obtain that the essential spectrum $\Sigma_e(\mathcal{L}_1+\mathcal{L}_2)$ of operator $\mathcal{L}_1+\mathcal{L}_2$ coincides with $\Sigma(\mathcal{L}_1)$ and contains only real eigenvalues. Thus the following assertion is valid:

Theorem 4. *Let $\Psi(\mu)$ be a zonal flow, besides $|\Psi_\mu(\mu)| \leqslant C_1$ and $|\Omega_\mu(\mu)| \leqslant C_2$. Then for any eigenvalue from the essential spectrum of the operator* (38)*, the corresponding normal mode* (39) *is neutral. An unstable mode, if it exists, corresponds to an isolated eigenvalue ω with finite multiplicity.*

Example 1. Let us consider a superrotation flow

$$\Psi(\mu) = -C\mu \tag{42}$$

on the sphere (C is the rotation velocity). The essential spectrum of the operator $\mathcal{L}_1+\mathcal{L}_2$ consists of the single point $\omega = C$. According to Rayleigh–Kuo's linear instability condition [1,10], unstable normal modes can exist only if the absolute vorticity derivative $(d/d\mu)\Omega \equiv 2 + (d/d\mu)\Delta\Psi$ changes its sign in the interval $(-1,1)$. In our case, $(d/d\mu)\Omega = 2(C+1)$, and hence, flow (42) is linearly stable, besides all the modes (39) are neutral.

The isolated eigenvalues and corresponding modes (39) can easily be obtained using the Rossby–Haurwitz wave [45]

$$\psi(\lambda,\mu,t) = -C\mu + \sum_{m=-n}^{n} a_m Y_n^m(\lambda - \omega_n t, \mu). \tag{43}$$

Indeed, wave (43) is the exact solution to vorticity equation (37) for arbitrary values C and a_m provided that

$$\omega_n = C - \frac{2(C+1)}{n(n+1)}, \quad n = 1, 2, 3, \ldots. \tag{44}$$

We can consider each term $a_m Y_n^m(\lambda - \omega_n t, \mu)$ in (43) as a perturbation of flow (42), besides this perturbation is also exact solution of the linearized equation (5) and the spectral problem (40) with ω equal to (44). Therefore, for each n, number (44) is the eigenvalue with multiplicity $2n+1$, and $2n+1$ linearly independent neutral normal modes (spherical harmonics)

$$\zeta(t,\lambda,\mu) = Y_n^m(\lambda - \omega_n t, \mu), \quad -n \leqslant m \leqslant n, \tag{45}$$

form the basis of the corresponding eigensubspace that coincides with the subspace H_n of the homogeneous spherical polynomials of the degree n (see (14)). In a vicinity of flow (42), the union of such normal modes for all numbers n form the basis for infinitesimal perturbations of the Hilbert space H^0. Thus, the discrete spectrum of $\mathcal{L}_1 + \mathcal{L}_2$ consists of the isolated real eigenvalues (44) belonging to the interval $[-1, C)$ for $C > -1$ and interval $(C, -1]$ for $C < -1$. The single point $\omega = C$ of the essential spectrum is the only accumulation point of the isolated eigenvalues.

Example 1 shows that, unlike a viscous fluid (Theorem 2), the operator spectrum for an ideal fluid can have bounded accumulation points.

Example 2. Consider now the zonal flow

$$\Psi(\mu) = -C\mu + aP_n(\mu), \tag{46}$$

where a is the flow amplitude, and $P_n(\mu)$ is the Legendre polynomial of degree n ($n = 1, 2, 3, \ldots$). This flow is linearly stable if $n = 1$ (Example 1) and $n = 2$ [12,15]. Let $n \geqslant 3$. According to the Rayleigh–Kuo instability condition [1,10], unstable normal modes (39) can exist only if the absolute vorticity derivative $(d/d\mu)\Omega = 2(C+1) - an(n+1)(d/d\mu)P_n$ of flow (46) changes its sign at least in one point of interval $(-1, 1)$. Thus, the instability can develop only if $|a| > a_{\mathrm{cr}}$; that is, when flow amplitude a exceeds a critical value

$$a_{\mathrm{cr}} = 2|C+1| \left\{ n(n+1) \max_{\mu} \left| \frac{d}{d\mu} P_n(\mu) \right| \right\}^{-1}. \tag{47}$$

The critical amplitude a_{cr} increases with $|C+1|$, decreases when degree n of the basic flow grows, and is absent when $C=-1$. By the other necessary instability condition [19,20,22], amplitude $A(\lambda,\mu)$ of each unstable mode must satisfy

$$\chi_A = n(n+1), \tag{48}$$

where $\sqrt{\chi_A}=\sqrt{\eta(A)/K(A)}$ is Fjörtoft's average spectral number of $A(\lambda,\mu)$ [46], and $K(A)=(1/2)\|\nabla A\|^2$ and $\eta(A)=(1/2)\|\Delta A\|^2$ are the kinetic energy and enstrophy of $A(\lambda,\mu)$, respectively. Due to (48), zonal wavenumber m of any unstable mode (39) must satisfy the inequality $0<|m|\leqslant n$ [6].

Substituting (46) in (40) and using again the wave (43) we obtain that point

$$\omega_n = C - \frac{2(C+1)}{n(n+1)}$$

is the eigenvalue of operator $\mathcal{L}_1+\mathcal{L}_2$ with multiplicity $2n+1$, which can be isolated or belong to its essential spectrum depending on the values of C, a and n. The corresponding $2n+1$ modes (45) are neutral and form orthogonal basis in the eigensubspace H_n (see (14)); that is, any disturbance of H_n is stable in spite of the fact that (48) is satisfied. Therefore, in addition to (48), the amplitude of each unstable mode must have nonzero projections on subspaces H_k with $k>n$ and $k<n$ at a time [19].

In both examples, essential spectrum of the operator $\mathcal{L}_1+\mathcal{L}_2$ is real, coincides with the spectrum $\Sigma(\mathcal{L}_1)$ and is bounded due to (41). We now show that all the eigenvalues corresponding to the unstable modes of the flow (46) are bounded as well. Indeed, in terms of the perturbation streamfunction, (40) accepts the form

$$-\Psi_\mu \Delta\phi + \Omega_\mu \phi = \omega\Delta\phi. \tag{49}$$

Taking the inner product (10) of each term of (49) with ϕ, and using (18) and instability condition (48), we obtain

$$|\omega| \leqslant \max_{-1\leqslant\mu\leqslant 1} |\Psi_\mu| + \frac{n(n+1)}{4} \max_{-1\leqslant\mu\leqslant 1} |\Omega_\mu|, \tag{50}$$

where $(d/d\mu)\Psi = a(d/d\mu)P_n - C$ and $(d/d\mu)\Omega$ is given in Example 2. Thus, the growth rate of unstable modes to flow (46) characterized by $\operatorname{Im}\omega$ is bounded. For $C=0$ this result was shown in [19]. It follows from (50) that an unstable manifold in a vicinity of flow (46) is of finite dimension provided that isolated eigenvalues with $\operatorname{Im}\omega>0$ have no accumulation points.

6. Stability matrix

We now briefly discuss the role of the linear drag, turbulent diffusion and sphere rotation in the normal mode stability study. Let M and N be triangular truncation numbers for the basic flow and disturbance, respectively. The stability matrix representing operator (6) in the subspace of spherical polynomials of degree $\leqslant N$ can be written as [6,15]

$$L_{\alpha\gamma} = \sum_{\beta}^{M} B_{\beta\alpha\gamma} + D_{\alpha\gamma}, \tag{51}$$

where

$$B_{\beta\alpha\gamma} = \left(\chi_\beta \chi_\gamma^{-1} - 1\right)\Psi_\beta \left\langle J(Y_\beta, Y_\gamma), Y_\alpha \right\rangle$$

is the coefficient of nonlinear interaction of the three spherical harmonics Y_α, Y_β and Y_γ, Ψ_β is the Fourier coefficient of the basic flow streamfunction $\Psi(x)$, and

$$D_{\alpha\gamma} = \left\{-\left(\sigma + \nu\chi_\gamma^s\right) + 2im_\gamma \chi_\gamma^{-1}\right\}\delta_{\alpha\gamma} \tag{52}$$

is the diagonal matrix. We have used here Platzman's [47] notations: a complex index $\alpha = (m_\alpha, n_\alpha)$, $\chi_\alpha = n_\alpha(n_\alpha + 1)$ and

$$\sum_{\beta}^{M} \equiv \sum_{n_\beta=1}^{M} \sum_{m_\beta=-n_\beta}^{n_\beta}.$$

The real parts $-(\sigma + \nu\chi_\gamma^s)$ of the diagonal entries of (52) represent the linear drag and turbulent diffusion. It follows from (52) that the linear drag results only in shifting all the eigenvalues of the matrix (51) along the real axis to the left by value σ. Thus, the stability of all the normal modes uniformly increases with σ. For a viscous fluid ($\nu > 0$), the diagonal terms $\nu\chi_\gamma^s = \nu[n_\gamma(n_\gamma + 1)]^s$ rise with ν and s and n_γ increasing the distance between eigenvalues due to Gerschgorin's theorem [48]. This fact is favorable for solving the spectral problem (8) with a higher accuracy.

Examples 1 and 2 show that the sphere rotation in general stabilizes a barotropic flow. For example, the critical amplitude a_{cr} for the zonal flow instability owes its origin to the sphere rotation. Indeed, according to Rayleigh–Kuo condition [1,10], a zonal flow is exponentially stable if its amplitude is small enough (or, that is the same, if the sphere rotation is sufficiently strong). In the stability matrix (51), the rotation is represented by the imaginary parts $2im_\gamma\chi_\gamma^{-1}$ of the diagonal entries of (52), which are small for small $|m_\gamma|$. Taking into account that the spherical harmonics $Y_\gamma(x)$ with a fix number m_γ form an invariant subspace of infinitesimal perturbations (see (39)), we can now explain the well-known numerical result, according to which zonal the wavenumber m of the most unstable modes (39) of a zonal flow is usually small ($1 \leqslant |m| \leqslant 3$) [12,18].

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