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Invariance of Plurigenera

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In this paper we give a proof of the following long conjectured result on the invariance of the plurigenera.

Main Theorem. Let $\pi : X \to \Delta$ be a projective family of compact complex manifolds parametrized by the open unit 1-disk Δ. Assume that the family $\pi : X \to \Delta$ is of general type. Then for every positive integer m the plurigenus $\dim_{\mathbf{C}} \Gamma(X_t, mK_{X_t})$ is independent of $t \in \Delta$, where $X_t = \pi^{-1}(t)$ and K_{X_t} is the canonical line bundle of X_t.

Notations and Terminology. The canonical line bundle of a complex manifold Y is denoted by K_Y. The coordinate of the open unit 1-disk Δ is denoted by t. Let n be the complex dimension of each X_t for $t \in \Delta$. In the assumption of the Main Theorem the property of the family $\pi : X \to \Delta$ being of general type means that for every $t \in \Delta$ there exist a positive integer m_t and a point $P_t \in X_t$ with the property that one can find elements $s_0, s_1, \ldots, s_{n+1} \in \Gamma(X, m_t K_X)$ such that s_0 is nonzero at P_t and $\frac{s_1}{s_0}, \ldots, \frac{s_{n+1}}{s_0}$ form a local coordinate system of X at P_t. By the family $\pi : X \to \Delta$ being projective we mean that there exists a positive holomorphic line bundle on X.

Let $K_{X,\pi}$ be the line bundle on X whose restriction to X_t is K_{X_t} for each $t \in \Delta$. Since the normal bundle of X_t in X is trivial, the two line bundles K_X and $K_{X,\pi}$ are naturally isomorphic. Under this natural isomorphism a local section s of $K_{X,\pi}$ corresponds to the local section $s \wedge \pi^*(dt)$ of K_X. Unless there is some risk of confusion, in this paper we will, without any further explicit mention, always identify $K_{X,\pi}$ with K_X by this natural isomorphism. Under this identification the Main Theorem is equivalent to the statement that for every $t \in \Delta$ and every integer m every element of $\Gamma(X_t, mK_{X_t})$ can be extended to an element of $\Gamma(X, mK_X)$.

The Hermitian metrics of holomorphic line bundles in this paper are allowed to have singularities and may not be smooth. For a Hermitian metric

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$h = e^{-\varphi}$ of a holomorphic line bundle L over X_0 we denote by $\mathcal{I}_h$ or by $\mathcal{I}_\varphi$ its multiplier ideal sheaf on X_0 which by definition means the ideal sheaf on X_0 of all local holomorphic function germs f on X_0 such that $|f|^2 e^{-\varphi}$ is integrable. For the proof of the Main Theorem, only multiplier ideal sheaves on X_0 are considered and no multiplier ideal sheaves on X are used. In the case of a Hermitian metric $\tilde{h} = e^{-\tilde{\varphi}}$ of a holomorphic line bundle $\tilde{L}$ over X, for notational simplicity we simply use the notation $\mathcal{I}_{\tilde{h}}$ or $\mathcal{I}_{\tilde{\varphi}}$ to mean the multiplier ideal sheaf for the Hermitian metric $\tilde{h}|X_0 = e^{-\tilde{\varphi}}|X_0$ of the holomorphic line bundle $\tilde{L}|X_0$ over X_0 and suppress the notation for restriction to X_0.

The stalk of a sheaf $\mathcal{F}$ at a point P is denoted by $\mathcal{F}_P$. The structure sheaf of a complex manifold Y is denoted by $\mathcal{O}_Y$. If s is a global holomorphic section of a holomorphic line bundle L over a complex manifold Y and if $\mathcal{I}$ is an ideal sheaf on Y, we say that the germ of s at a point P belongs to $\mathcal{I}_P$ if the holomorphic function germ at P which corresponds to the germ of s at P with respect to some local trivialization of L belongs to $\mathcal{I}_P$. We say that s is everywhere locally contained in $\mathcal{I}$ if at every point $P \in Y$ the germ of s at P belongs to $\mathcal{I}_P$, or equivalently, s belongs to the subset $\Gamma(Y, \mathcal{I} \otimes L)$ of $\Gamma(Y, L)$. If E is another holomorphic line bundle over Y with a Hermitian metric $e^{-\chi}$, we say that $|s|^2 e^{-\chi}$ is locally uniformly bounded (respectively locally integrable) on Y if at every point $P \in Y$ with respect to some local trivializations of L and E on some open neighborhood U of P the function on U corresponding to $|s|^2 e^{-\chi}$ is uniformly bounded (respectively integrable) on U. We say that s is locally L^2 with respective to $e^{-\chi}$ on Y if $|s|^2 e^{-\chi}$ is locally integrable on Y.

History and Sketch of the Proof of the Main Theorem. Iitaka [I69-71] proved the special case of the invariance of the plurigenera in a family of surfaces. His method works only for surfaces because it uses much of the information from the classification of surfaces. Levine [L83,L85] proved that for every positive integer m every element of $\Gamma(X_0, mK_{X_0})$ can be extended to the fiber of X over the double point of Δ at $t = 0$. So far there is no way to continue the process to get an extension to the fiber of X over a point of Δ at $t = 0$ of any finite order. Nakayama [Nak86] pointed out that if the relative case of the minimal model program can be carried out for a certain dimension, the conjecture of the invariance of the plurigenera for that dimension would follow directly from it. Thus the invariance of the plurigenera for threefolds is a consequence of the completion of the relative case of the minimal model program for the case of threefolds by Kollar and Mori [KM92].

For the proof of the Main Theorem here we use a strategy completely different from those used by the others in the past. We now sketch our strategy and leave out the less essential technical details. There are some unavoidable technical inaccuracies in the sketch due to the suppression of precise details. There are three ingredients in our proof: Nadel's multiplier ideal sheaves [Nad89], Skoda's result on the generation of ideals with L^2 estimates with respect to a plurisubharmonic weight [Sk72], and the extension theorem of Ohsawa-Takegoshi-Manivel for holomorphic top-degree forms which are L^2 with respect to a plurisubharmonic weight [OT87,M93]. The extension theorem of Ohsawa-Takegoshi-Manivel is for the setting of a Stein domain or manifold and a global plurisubharmonic function as weight. Here we adapt it to the case of a projective family of compact complex manifolds and a Hermitian metric of a line bundle with nonnegative curvature current. The adaptation is done by restricting to a Stein Zariski open subset on which the line bundle is globally trivial, because L^2 bounds automatically extend the domain of definition from the Zariski open subset to the family of compact manifolds.

We take the m^{th} roots of basis elements of $\Gamma(X_0, mK_{X_0})$ for every positive integer m to use them in an infinite series to construct a Hermitian metric $e^{-\varphi}$ for K_{X_0}. We also take the m^{th} roots of basis elements of $\Gamma(X, mK_X)|X_0$ for every positive integer m to use them in an infinite series to construct a Hermitian metric $e^{-\tilde{\varphi}}$ for K_{X_0}. If for an infinite number of integers ℓ_ν the singularity of $e^{-\ell_\nu\tilde{\varphi}}$ are only worse than that of $e^{-\ell_\nu\varphi}$ by some fixed amount independent of ν, then the extension theorem of Ohsawa-Takegoshi-Manivel can be applied to yield the Main Theorem.

If the contrary holds, then for some appropriate positive integer m we can construct some Hermitian metric for mK_{X_0} whose singularity is suitably chosen to be between those of $e^{-m\tilde{\varphi}}$ and $e^{-m\varphi}$ so that, by Skoda's result, we can use this Hermitian metric to produce an element s of $\Gamma(X_0, mK_{X_0})$ which is L^2 with respect to $e^{-(m-1)\tilde{\varphi}}$ but not locally L^2 with respect to $e^{-m\tilde{\varphi}}$ everywhere on X_0. On the other hand, by the extension theorem of Ohsawa-Takegoshi-Manivel we can regard s as an $(m-1)K_{X_0}$-valued top-degree form on X_0 which is L^2 with respect to $e^{-(m-1)\tilde{\varphi}}$ and can therefore extend it to an element of $\Gamma(X, mK_X)$. The definition of $e^{-\tilde{\varphi}}$ implies that $|s|^2 e^{-m\tilde{\varphi}}$ is locally uniformly bounded on X_0 and consequently s is L^2 with respect to $e^{-m\tilde{\varphi}}$ everywhere on X_0, which is a contradiction.

One of the technical details is that the Hermitian metric $e^{-m\tilde{\varphi}}$ has to be slightly modified to make sure that its curvature current dominates a smooth positive $(1,1)$-form in order to apply the extension theorem of Ohsawa-Takegoshi-Manivel. For that modification the Kodaira technique of writing some high multiple of a big line bundle as an effective divisor plus an ample line bundle is used.

Lemma 1. Let L be a holomorphic line bundle over an n-dimensional compact complex manifold Y with a Hermitian metric which is locally of the form $e^{-\xi}$ with ξ plurisubharmonic. Let $\mathcal{I}_\xi$ be the multiplier ideal sheaf of the Hermitian metric $e^{-\xi}$. Let E be an ample holomorphic line bundle over Y such that for every point P of Y there are a finite number of elements of $\Gamma(Y, E)$ which all vanish to order at least $n+1$ at P and which do not simultaneously vanish outside P. Then $\Gamma(Y, \mathcal{I}_\xi \otimes (L + E + K_Y))$ generates $\mathcal{I}_\xi \otimes (L + E + K_Y)$ at every point of Y.

Proof. The key ingredient is the following result of Skoda [Sk72, Th.1, pp.555-556].

Let Ω be a pseudoconvex domain in $\mathbf{C}^n$ and ψ be a plurisubharmonic function on Ω. Let $g_1, \ldots, g_p$ be holomorphic functions on Ω. Let $\alpha > 1$ and $q = \inf(n, p-1)$. Then for every holomorphic function f on Ω such that

$$\int_\Omega |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda < \infty,$$

there exist holomorphic functions $h_1, \ldots, h_p$ on Ω such that

$$f = \sum_{j=1}^{p} g_j h_j$$

and

$$\int_\Omega |h|^2 |g|^{-2\alpha q} e^{-\psi} d\lambda \leq \frac{\alpha}{\alpha - 1} \int_\Omega |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda,$$

where

$$|g| = \left(\sum_{j=1}^{p} |g_j|^2 \right)^{\frac{1}{2}}, \qquad |h| = \left(\sum_{j=1}^{p} |h_j|^2 \right)^{\frac{1}{2}},$$

and $d\lambda$ is the Euclidean volume element of $\mathbf{C}^n$.

Fix arbitrarily $P_0 \in Y$. Take an arbitrary element s of $(\mathcal{I}_\xi)_{P_0}$. Let $z = (z_1, \ldots, z_n)$ be a local coordinate system on some open neighborhood U of P_0 with $z(P_0) = 0$ such that $L|U$ is trivial. Let ρ be a cut-off function centered at P_0 so that ρ is a smooth nonnegative-valued function with compact support in U which is identically 1 on some Stein open neighborhood Ω of P_0. Choose $u_1, \ldots, u_N \in \Gamma(Y, E)$ whose common zero-set consists of the single point P_0 and which all vanish to order at least $n+1$ at P_0. Let h_E be a smooth Hermitian metric of E whose curvature form is strictly positive at every point of Y. Let $0 < \eta < \frac{1}{n+1}$. By the standard techniques of L^2 estimates of $\bar\partial$, we can solve the equation

$$\bar\partial\sigma = \rho\bar\partial s$$

for a smooth section σ of $L + E + K_Y$ over Y which is L^2 with respect to the Hermitian metric

$$\frac{e^{-\xi}\,(h_E)^\eta}{\left(\sum_{j=1}^N |u_j|^2\right)^{1-\eta}}$$

of $L + E$. Then $\rho s - \sigma$ is an element of $\Gamma(Y, \mathcal{I}_\xi \otimes (L + E + K_Y))$. Since $\rho\bar\partial s$ is identically zero on Ω, it follows that σ is holomorphic on Ω. We now apply Skoda's result to the case $g_j = z_j$ $(1 \le j \le n)$ with $q = n-1$ and $\alpha = \frac{(1-\eta)(n+1)-1}{n-1} > 1$ and $\psi = \xi$. (For the case $n = 1$ we simply choose α be any number greater than 1, because in that case αq is always zero.) Let $|z| = \left(\sum_{j=1}^n |z_j|^2\right)^{\frac{1}{2}}$. Since $u_1, \ldots, u_N$ all vanish to order at least $n+1$ at P_0, it follows that

$$\int_\Omega |\sigma|^2 e^{-\xi}|z|^{-2\alpha q-2} = \int_\Omega |\sigma|^2 e^{-\xi}|z|^{-2(1-\eta)(n+1)} < \infty.$$

By Skoda's reuslt

$$\sigma = \sum_{j=1}^n \tau_j z_j$$

locally at P_0 for some $\tau_1, \ldots, \tau_n \in (\mathcal{I}_\xi)_{P_0}$.

Let J be the ideal at P_0 generated by elements of $\Gamma(Y, \mathcal{I}_\xi \otimes (L + E + K_Y))$ over $(\mathcal{O}_Y)_{P_0}$. It follows from

$$\rho s - \sigma \in \Gamma(Y, \mathcal{I}_\xi \otimes (L + E + K_Y))$$

that

$$s \in J + \mathbf{m}_{P_0} \left(\mathcal{I}_\xi\right)_{P_0},$$

where $\mathbf{m}_{P_0}$ is the maximum ideal of Y at P_0. Since s is an arbitrary element of $\left(\mathcal{I}_\xi\right)_{P_0}$, it follows that

$$\left(\mathcal{I}_\xi\right)_{P_0} \subset J + \mathbf{m}_{P_0} \left(\mathcal{I}_\xi\right)_{P_0}.$$

Clearly we have $J \subset \left(\mathcal{I}_\xi\right)_{P_0}$. Thus

$$\left(\mathcal{I}_\xi\right)_{P_0} / J \subset \mathbf{m}_{P_0} \left(\left(\mathcal{I}_\xi\right)_{P_0} / J\right).$$

By Nakayama's lemma,

$$\left(\mathcal{I}_\xi\right)_{P_0} / J = 0$$

and $J = \left(\mathcal{I}_\varphi\right)_{P_0}$. Q.E.D.

For every positive integer m we choose a basis

$$s_1^{(m)}, \cdots, s_{q_m}^{(m)} \in \Gamma\left(X_0, mK_{X_0}\right)$$

and we choose

$$\tilde{s}_1^{(m)}, \cdots, \tilde{s}_{\tilde{q}_m}^{(m)} \in \Gamma\left(X, mK_X\right)$$

so that

$$\tilde{s}_1^{(m)} | X_0, \cdots, \tilde{s}_{\tilde{q}_m}^{(m)} | X_0$$

is a basis of $\Gamma\left(X, mK_X\right)|_{X_0}$ and $\tilde{s}_\nu^{(m)} = s_\nu^{(m)}$ for $1 \leq \nu \leq \tilde{q}_m$. We can choose a sequence of positive numbers θ_m so that

$$\sum_{m=1}^\infty \theta_m \left(\sum_{\nu=1}^{q_m} \left|s_\nu^{(m)}\right|^{\frac{2}{m}}\right)$$

converges uniformly on compact subsets of X_0 to a Hermitian metric of $-K_{X_0}$ and

$$\sum_{m=1}^\infty \theta_m \left(\sum_{\nu=1}^{\tilde{q}_m} \left|\tilde{s}_\nu^{(m)}\right|^{\frac{2}{m}}\right)$$

converges uniformly on compact subsets of X to a Hermitian metric of $-K_X$. Locally on X_0 we define

$$\varphi = \log \sum_{m=1}^\infty \theta_m \left(\sum_{\nu=1}^{q_m} \left|s_\nu^{(m)}\right|^{\frac{2}{m}}\right)$$

so that $e^{-\varphi}$ is a Hermitian metric of K_{X_0}. Locally on X we define

$$\tilde{\varphi} = \log \sum_{m=1}^{\infty} \theta_m \left(\sum_{\nu=1}^{\tilde{q}_m} \left| \tilde{s}_\nu^{(m)} \right|^{\frac{2}{m}} \right)$$

so that $e^{-\tilde{\varphi}}$ is a Hermitian metric of K_X.

Since the family $\pi : X \to \Delta$ is of general type, we can choose an integer $m_0 \geq 2$ such that $m_0 K_X = D + F$, where D is an effective divisor on X not containing X_0 and F is such a high multiple of a positive line bundle on X that

(i) for every point $P \in X_0$ there exist a finite number of elements of $\Gamma(X, F - 2K_X)|X_0$ whose common zero-set consists only of the single point P and which all vanish to order at least $n+1$ at P and

(ii) a basis of $\Gamma(X, F)|X_0$ embeds X_0 as a complex submanifold of some complex projective space.

Let s_D be the canonical section of the holomorphic line bundle D so that the divisor of s_D is D. Let

$$u_1, \ldots, u_N \in \Gamma(X, F)$$

such that

$$u_1 \mid X_0, \ldots, u_N \mid X_0$$

form a basis of $\Gamma(X, F)|X_0$. From $s_D u_j \in \Gamma(X, m_0 K_X)$ $(1 \leq j \leq N)$ and the non simultaneous vanishing of $u_1, \ldots, u_N$ at any point of X_0 it follows from the definition of $\tilde{\varphi}$ that $|s_D|^2 e^{-m_0 \tilde{\varphi}}|_{X_0}$ is locally uniformly bounded on X_0. Let

$$h_F = \frac{1}{\sum_{j=1}^N |u_j|^2}$$

and we introduce the Hermitian metric

$$e^{-\psi} = \left(\frac{h_F}{|s_D|^2} \right)^{\frac{1}{m_0}}$$

for the line bundle K_X. Choose $0 < \epsilon < 1$ such that $e^{-\epsilon\psi}|X_0$ is locally integrable on X_0. For any positive integer ℓ we introduce the Hermitian metric

$$h_\ell = e^{-(\ell-\epsilon)\varphi - (m_0+\epsilon)\psi}$$

for the line bundle $(\ell + m_0)K_{X_0}$.

As stated at the beginning of the paper, in the statement of Lemma 2 below and for the rest of the paper the notation $\mathcal{I}_{(\ell_\nu+m_0-1-\epsilon)\tilde{\varphi}+\epsilon\psi}$ denotes an ideal sheaf on X_0 and not an ideal sheaf on X and it is the multiplier ideal sheaf for the metric $e^{-(\ell_\nu+m_0-1-\epsilon)\tilde{\varphi}-\epsilon\psi}|X_0$ of the holomorphic line bundle $(\ell_\nu + m_0 - 1)K_{X_0}$.

Lemma 2. Let ℓ_0 be a positive integer. Suppose there exists a sequence of positive integers $\ell_\nu \nearrow \infty$ $(1 \leq \nu < \infty)$ such that

$$\mathcal{I}_{h_{\ell_\nu}} \subset \mathcal{I}_{(\ell_\nu+m_0-1-\epsilon)\tilde{\varphi}+\epsilon\psi}.$$

Then any element s of $\Gamma(X_0, \ell_0 K_{X_0})$ is everywhere locally contained in $\mathcal{I}_{\ell_0\tilde{\varphi}}$.

Proof. Without loss of generality we can assume after reindexing the sequence $\{\ell_\nu\}_{1\leq\nu<\infty}$ that $\ell_\nu > 2\ell_0$ for $1 \leq \nu < \infty$. Take an arbitrary $P_0 \in X_0$. Let $\ell_\nu = q_\nu \ell_0 + r_\nu$ with $0 \leq r_\nu < \ell_0$ $(1 \leq \nu < \infty)$. Take a non-identically-zero $\sigma \in (\mathcal{O}_X)_{P_0}$ such that $|\sigma|^2 e^{-\ell_0\tilde{\varphi}}$ is bounded in the supremum norm on some open neighborhood U of P_0 in X_0 (for example, we can take σ to be the germ at P_0 of some nonzero element of $\Gamma(X, \ell K_X)|X_0$ for some $\ell \geq \ell_0$). Since $|s^{q_\nu}|^2 e^{-q_\nu \ell_0 \varphi}$ is uniformly bounded on X_0 from the definition of φ, it follows from $0 \leq r_\nu < \ell_0$ and $\tilde{\varphi} \leq \varphi$ and the integrability of $e^{-\epsilon\psi}$ that the germ of $s^{q_\nu}\sigma s_D$ at P_0 belongs to $\left(\mathcal{I}_{h_{\ell_\nu}}\right)_{P_0}$. By the assumption of the Lemma, the germ of $s^{q_\nu}\sigma s_D$ at P_0 belongs to $\left(\mathcal{I}_{(\ell_\nu+m_0-1-\epsilon)\tilde{\varphi}+\epsilon\psi}\right)_{P_0}$. There exists some relatively compact open neighborhood W of P_0 in U with $K_{X_0}|W$ trivial such that

$$\int_W |s^{q_\nu}\sigma s_D|^2 e^{-(\ell_\nu+m_0-1-\epsilon)\tilde{\varphi}-\epsilon\psi} < \infty.$$

Let $\frac{1}{q_\nu} + \frac{1}{q'_\nu} = 1$. Then $q'_\nu = \frac{q_\nu}{q_\nu - 1}$ and $\frac{q'_\nu}{q_\nu} = \frac{1}{q_\nu - 1}$ and we have by Hölder's inequality

$$\int_W |s|^2 e^{-\ell_0\tilde{\varphi}} = \int_W \left| s\sigma^{\frac{1}{q_\nu}} s_D^{\frac{1}{q_\nu}} \sigma^{\frac{-1}{q_\nu}} s_D^{\frac{-1}{q_\nu}} \right|^2 e^{-\ell_0\tilde{\varphi}} e^{\frac{-\epsilon\psi}{q_\nu}} e^{\frac{\epsilon\psi}{q_\nu}}$$

$$\leq \left(\int_W |s^{q_\nu}\sigma s_D|^2 e^{-q_\nu\ell_0\tilde{\varphi}-\epsilon\psi}\right)^{\frac{1}{q_\nu}} \left(\int_W \left|\sigma^{\frac{-q'_\nu}{q_\nu}} s_D^{\frac{-q'_\nu}{q_\nu}}\right|^2 e^{\frac{\epsilon\psi q'_\nu}{q_\nu}}\right)^{\frac{1}{q'_\nu}}$$

$$= \left(\int_W |s^{q_\nu}\sigma s_D|^2 e^{-q_\nu\ell_0\varphi-\epsilon\psi}\right)^{\frac{1}{q_\nu}} \left(\int_W \left|\sigma^{\frac{-1}{q_\nu-1}} s_D^{\frac{-1}{q_\nu-1}}\right|^2 e^{\frac{\epsilon\psi}{q_\nu-1}}\right)^{\frac{q_\nu}{q_\nu-1}}$$

$$\leq C \left(\int_W \left| s^{q_\nu} \sigma s_D \right|^2 e^{-(\ell_\nu + m_0 - 1 - \epsilon)\tilde\varphi - \epsilon\psi} \right)^{\frac{1}{q_\nu}} \left(\int_W \left| \sigma^{\frac{-1}{q_\nu - 1}} s_D^{\frac{-1}{q_\nu - 1}} \right|^2 e^{\frac{\epsilon\psi}{q_\nu - 1}} \right)^{\frac{q_\nu}{q_\nu - 1}},$$

where C is the supremum of $e^{\frac{(m_0 + r_\nu - 1)\tilde\varphi}{q_\nu}}$ on W. For q_ν sufficiently large,

$$\int_W \left| \sigma^{\frac{-1}{q_\nu - 1}} s_D^{\frac{-1}{q_\nu - 1}} \right|^2 e^{\frac{\epsilon\psi}{q_\nu - 1}} < \infty.$$

Hence

$$\int_W |s|^2 e^{-\ell_0 \tilde\varphi} < \infty$$

and the germ of s at P_0 belongs to $\left(\mathcal{I}_{\ell_0\tilde\varphi}\right)_{P_0}$. Q.E.D.

For the next step in our proof of the Main Theorem we need the following extension statement which is an adaptation of the extension theorem of Ohsawa-Takegoshi-Manivel.

Proposition 3. Let $\gamma : Y \to \Delta$ be a projective family of compact complex manifolds parametrized by the open unit 1-disk Δ. Let $Y_0 = \gamma^{-1}(0)$ and let n be the complex dimension of Y_0. Let L be a holomorphic line bundle with a Hermitian metric which locally is represented by $e^{-\chi}$ such that $\sqrt{-1}\partial\bar\partial\chi \geq \omega$ in the sense of currents for some smooth positive $(1,1)$-form ω on Y. Let $0 < r < 1$ and $\Delta_r = \{\, t \in \Delta \mid |t| < r \,\}$. Then there exists a positive constant A_r with the following property. For any holomorphic L-valued n-form f on Y_0 with

$$\int_{Y_0} |f|^2 e^{-\chi} < \infty,$$

there exists a holomorphic L-valued $(n+1)$-form $\tilde f$ on $\gamma^{-1}(\Delta_r)$ such that $\tilde f|_{Y_0} = f \wedge \gamma^*(dt)$ at points of Y_0 and

$$\int_Y |\tilde f|^2 e^{-\chi} \leq A_r \int_{Y_0} |f|^2 e^{-\chi}.$$

Here no metrics of the tangent bundles of Y_0 and Y are needed to define the integrals of the absolute-value squares of top-degree holomorphic forms f and $\tilde f$ respectively on Y_0 and Y.

Proof. The proof can be easily adapted in the following way from the techniques given in [Si96] for the alternative proof there of the theorem of Ohsawa-Takegoshi. (Proofs can also be obtained by modifying those in [OT83, M93].)

Let v be a meromorphic section of L over Y so that neither the pole-set nor the zero-set of v contains Y_0. Choose a complex hypersurface Z in Y containing the zero-set and the pole-set of v such that Z does not contain Y_0 and $Y - Z$ is Stein. For every positive integer ν let Ω_ν be a relatively compact Stein open subset of $X - Z$ with smooth strictly pseudoconvex boundary such that $\cup_{\nu=1}^{\infty}\Omega_\nu = X - Z$ and Ω_ν is relatively compact in $\Omega_{\nu+1}$. On $X - Z$ under the isomorphism defined by division by v the line bundle $L|X - Z$ is globally trivial. We let $\tilde{\chi}$ be the plurisubharmonic function $-\log\left(|v|^2 e^{-\chi}\right)$ on $X - Z$.

We now apply the techniques in [Si96] of the alternative proof of the theorem of Ohsawa-Takegoshi to extend, after multiplication by $\gamma^*(dt)$, the top-degree holomorphic form $\frac{f}{v}$ on $\Omega_\nu \cap Y_0$ which is L^2 on $\Omega_\nu \cap Y_0$ with respect to $e^{-\tilde{\chi}}$ to a top-degree holomorphic form G_ν on $\gamma^{-1}(\Delta_r) \cap \Omega_\nu$ whose L^2 norm on $\gamma^{-1}(\Delta_r) \cap \Omega_\nu$ with respect to $e^{-\tilde{\chi}}$ is bounded by a finite constant independent of ν. When we apply the techniques of the alternative proof of the theorem of Ohsawa-Takegoshi, we have to use holomorphic tangent vector fields of the Stein manifold $\Omega_{\nu+1}$ to get a sequence of smooth plurisubharmonic functions on Ω_ν which approach the plurisubharmonic function $\tilde{\chi}$ on Ω_ν. The extension $\tilde{f}$ is obtained as the limit of $G_\nu v$ as ν goes to infinity. The smooth positive $(1,1)$-form ω in the assumption is needed for the ν-independent *a priori* estimates for the solution of the modified $\bar{\partial}$ equation on $\gamma^{-1}(\Delta_r) \cap \Omega_\nu$ in the techniques of the alternative proof of the theorem of Ohsawa-Takegoshi. Q.E.D.

Lemma 4. If m is an integer ≥ 2 and if $s \in \Gamma(X_0, mK_{X_0})$ is everywhere locally contained in $\mathcal{I}_{(m-1-\epsilon)\tilde{\varphi}+\epsilon\psi}$, then s can be extened to an element of $\Gamma(X, mK_X)$.

Proof. We apply Proposition 3 to the case $L = (m-1)K_X$, $\chi = (m-1-\epsilon)\tilde{\varphi} + \epsilon\psi$, and $f = s$ to extend s to an $(m-1)K_X$-valued holomorphic $(n+1)$-form on $\pi^{-1}(\Delta_r)$ for some $0 < r < 1$, where Δ_r is the open 1-disk of radius r centered at the origin. Then we use the standard theory of coherent sheaves and Stein spaces to get the extension from $\pi^{-1}(\Delta_r)$ to all of X. Q.E.D.

Lemma 5. If m is an integer ≥ 2 and if $s \in \Gamma(X_0, mK_{X_0})$ is everywhere locally contained in $\mathcal{I}_{m\tilde{\varphi}}$, then s can be extended to an element of $\Gamma(X, mK_X)$.

Proof. Since s is everywhere locally contianed in $\mathcal{I}_{m\tilde{\varphi}}$, we can cover X_0 by a finite number of open subsets U_j $(1 \leq j \leq k)$ such that $K_{X_0}|U_j$ is trivial on

U_j and

$$\int_{U_j} |s|^2 e^{-m\tilde{\varphi}} < \infty$$

for $1 \leq j \leq k$. Take $0 < \eta < 1$ and consider the Hermitian metric $e^{-(m-1-\eta)\tilde{\varphi}-\eta\psi}$ for $(m-1)K_X$. We apply Hölder's inequality with $p = \frac{m}{m-1-\eta}$ and $p' = \frac{p}{p-1} = \frac{m}{1+\eta}$ to get

$$\int_{U_j} |s|^2 e^{-(m-1-\eta)\tilde{\varphi}-\eta\psi} \leq \left(\int_{U_J} |s|^2 e^{-m\tilde{\varphi}}\right)^{\frac{m-1-\eta}{m}} \left(\int_{U_j} |s|^2 e^{\frac{-m\eta\psi}{1+\eta}}\right)^{\frac{1+\eta}{m}}.$$

When η is sufficiently small,

$$e^{\frac{-m\eta\psi}{1+\eta}} = \left(\frac{h_F}{|s_D|^2}\right)^{\frac{-m\eta\psi}{(1+\eta)m_0}}$$

is locally integrable at every point of X_0. Hence s is L^2 as an $(m-1)K_{X_0}$-valued n-form on X_0 with respect to the Hermitian metric

$$e^{-(m-1-\eta)\tilde{\varphi}-\eta\psi}|X_0$$

whose curvature current, because of the factor h_F, is bounded from below by a smooth positive $(1,1)$-form on X_0. Now the Lemma follows from Proposition 3 (or Lemma 4). Q.E.D.

Final Step of the Proof of the Main Theorem. From the definition of h_ℓ for $\ell = 1$ we have

$$\mathcal{I}_{h_1} = \mathcal{I}_{(1-\epsilon)\varphi+(m_0+\epsilon)\psi} \subset \mathcal{I}_{(m_0-\epsilon)\tilde{\varphi}+\epsilon\psi}, \tag{1}$$

because $|s_D|^2 e^{-m_0\tilde{\varphi}}|_{X_0}$ is locally uniformly bounded on X_0. Fix an arbitrary positive integer ℓ_0. To prove the Main Theorem, it suffices to show that every element of $\Gamma(X_0, \ell_0 K_{X_0})$ can be extended to an element of $\Gamma(X, \ell_0 K_X)$. Suppose the contrary and we are going to derive a contradiction. By Lemma 2 and Lemma 5 we can assume that there exists a positive integer $\ell_\#$ such that for $\ell \geq \ell_\#$ we have

$$\mathcal{I}_{h_\ell} \not\subset \mathcal{I}_{(\ell+m_0-1-\epsilon)\tilde{\varphi}+\epsilon\psi}.$$

By (1) we know that there is a smallest positive integer ℓ_* (which must be at least 2) such that

$$\mathcal{I}_{h_{\ell_*}} \not\subset \mathcal{I}_{(\ell_*+m_0-1-\epsilon)\tilde{\varphi}+\epsilon\psi}. \tag{2}$$

Then

$$\mathcal{I}_{h_{\ell_*-1}} \subset \mathcal{I}_{(\ell_*-1+m_0-1-\epsilon\psi)\tilde{\varphi}+\epsilon\psi}. \tag{3}$$

From the choice of F we know that the line bundle $(F|X_0) - K_{X_0}$ over X_0 is ample. We now apply Lemma 1 to the case $E = (F|X_0) - 2K_{X_0}$ and $L = \ell_* K_{X_0} + D$ with the Hermitian metric

$$e^{-\xi} = \frac{e^{-(\ell_*-\epsilon)\varphi-\epsilon\psi}}{|s_D|^2}.$$

Each of the two Hermitian metrics h_{ℓ_*} and $e^{-\xi}$ is locally bounded on X_0 by a positive constant times the other. Hence the two ideal sheaves $\mathcal{I}_\xi$ and $\mathcal{I}_{h_{\ell_*}}$ coincide everywhere on X_0. By Lemma 1 it follows from $(\ell_*+m_0-1)K_{X_0} = L + E + K_{X_0}$ that $\Gamma(X_0, \mathcal{I}_{h_{\ell_*}} \otimes (\ell_*+m_0-1)K_{X_0})$ locally generates $\mathcal{I}_{h_{\ell_*}}$ on X_0. From (2) it follows that there exists $s \in \Gamma(X_0, \mathcal{I}_{h_{\ell_*}} \otimes (\ell_*+m_0-1)K_{X_0})$ such that

(4) s is not everywhere locally contained in $\mathcal{I}_{(\ell_*+m_0-1-\epsilon)\tilde{\varphi}+\epsilon\psi}$.

From (3) and $\mathcal{I}_{h_{\ell_*}} \subset \mathcal{I}_{h_{\ell_*-1}}$ it follows that every element of $\Gamma(X_0, \mathcal{I}_{h_{\ell_*}} \otimes (\ell_*+m_0-1)K_{X_0})$ is locally contained in $\mathcal{I}_{(\ell_*-1+m_0-1-\epsilon)\tilde{\varphi}+\epsilon\psi}$ at every point of X_0. From Lemma 4 it follows that s can be extended to an element $\tilde{s}$ of $\Gamma(X, (\ell_*+m_0-1)K_X)$. Since from the definition of $\tilde{\varphi}$ we know that $|\tilde{s}|^2 e^{-(\ell_*+m_0-1)\tilde{\varphi}}$ is uniformly bounded on X_0, it follows from the integrability of $e^{-\epsilon\psi}$ that s is everywhere locally contained in $\mathcal{I}_{(\ell_*+m_0-1-\epsilon)\tilde{\varphi}+\epsilon\psi}$, which contradicts (4). This concludes the proof of the Main Theorem. Q.E.D.

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