

On The Eneström-Kakeya Theorem

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Abstract

In this paper, we prove some generalizations of results concerning the Eneström-Kakeya theorem. The results obtained considerably improve the bounds by relaxing the hypothesis in some cases.

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1. Introduction and Statement of Results

The following result due to Eneström and Kakeya [1] is well known in the theory of distribution of the zeros of polynomials.

Theorem A. If $P(z) := \sum_{j=0}^n a_j z^j$ is a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \dots \geq a_1 \geq a_0 \geq 0,$$

then $P(z)$ does not vanish in $|z| > 1$

In the literature, [2-8], there exist extensions and generalizations of Eneström-Kakeya theorem. Joyal, Labelle and Rahman [9] extended this theorem to a polynomial whose coefficients are monotonic but not necessarily non negative by proving the following result.

Theorem B. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \dots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$|z| \leq \frac{1}{|a_n|} \{a_n - a_0 + |a_0|\}.$$

Dewan and Bidkham [10] generalized Theorem B and proved the following:

Theorem C. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that for some $t > 0$ and $0 < \lambda \leq n$,

$$a_n t^n \leq a_{n-1} t^{n-1} \leq \dots \leq a_\lambda t^\lambda \geq a_{\lambda-1} t^{\lambda-1} \geq \dots \geq a_1 \geq a_0,$$

then $P(z)$ has all the zeros in the circle

$$|z| \leq \frac{t}{|a_n|} \left\{ \left(\frac{2a_\lambda}{t^{n-\lambda}} - a_n \right) + \frac{1}{t^n} (|a_0| - a_0) \right\}.$$

By using Schwarz's Lemma, Aziz and Mohammad [11] generalized Eneström-Kakeya theorem in a different way and proved the following:

Theorem D. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with real positive coefficients. If $t_1 > t_2 \geq 0$ can be found such that

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \geq 0, \text{ for } r = 1, 2, \dots, n+1$$

$$(a_{-1} = a_{n+1} = 0),$$

then all the zeros of $P(z)$ lie in $|z| \leq t_1$

Aziz and Zargar [12] also relaxed the hypothesis of Eneström-Kakeya theorem in a different way and proved the following result.

Theorem E. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that for some $K \geq 1$,

$$K a_n \geq a_{n-1} \geq a_{n-2} \geq \dots \geq a_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$|z + K - 1| \leq K.$$

While studying Theorem E, a natural question arises that what happens if we relax the hypothesis of Theorem D in a similar way and only assume that

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \geq 0, \text{ for } r = 2, 3, \dots, n$$

In this paper, we study such a case and prove a more general result from which many known results follow on a fairly uniform procedure. Infact we prove:

Theorem 1. Let $P(z) := \sum_{j=0}^n \alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + i b_j$ where a_j and b_j , $j = 0, 1, \dots, n$ are real numbers and if $t_1 > t_2 \geq 0$ can be found such that for $r = 2, 3, \dots, n$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0,$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \geq 0,$$

and for some $K \geq 1$,

$$K a_n(t_1 - t_2) - a_{n-1} \geq 0,$$

$$K b_n(t_1 - t_2) - b_{n-1} \geq 0,$$

then all the zeros of $P(z)$ lie in $|z + (K-1)(t_1 - t_2)| \leq R$,

where

$$R = \frac{1}{|\alpha_n|} \left\{ K(a_n + b_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1) \frac{t_2}{t_1^{n-1}} - (a_0 + b_0) \frac{1}{t_1^{n-1}} + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} + (|a_0| + |b_0|) \frac{t_2}{t_1^n} \right\}.$$

The following interesting result immediately follows from Theorem 1, if we assume that all the coefficients of the polynomial $P(z)$ are real.

Corollary 1. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with real coefficients. If $t_1 > t_2 \geq 0$ can be found such that

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \text{ for } r = 2, 3, \dots, n$$

and for some $K \geq 1$,

$$K a_n(t_1 - t_2) - a_{n-1} \geq 0,$$

then all the zeros of $P(z)$ lie in $|z + (K-1)(t_1 - t_2)| \leq R^*$,

where

$$R^* = \frac{1}{|\alpha_n|} \left\{ K a_n(t_1 - t_2) + a_n t_2 - a_1 \frac{t_2}{t_1^{n-1}} - a_0 \frac{1}{t_1^{n-1}} + |a_1 t_1 t_2 + a_0(t_1 - t_2)| \frac{1}{t_1^n} + |a_0| \frac{t_2}{t_1^n} \right\}.$$

Remark 1. If we assume that all the coefficients of $P(z)$ are real and positive, then for $K = 1$, Corollary 1 satisfies the statement of Theorem D and a simple calculation shows that in this case also all the zeros of $P(z)$ lie in $|z| \leq t_1$.

Next, if in the Theorem 1, we take $t_2 = 0$ and assume that coefficients to be real, we get the following:

Corollary 2. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with real coefficients. If for some $t > 0$ and $K \geq 1$,

$$K a_n t^n \geq a_{n-1} t^{n-1} \geq a_{n-2} t^{n-2} \geq \dots \geq a_1 t > a_0,$$

then all the zeros of $P(z)$ lie in

$$|z + (K-1)t| \leq \frac{t}{|a_n|} \left(K a_n + \frac{|a_0|}{t^n} - \frac{a_0}{t^n} \right).$$

Remark 2. If we put $t = 1$ in Corollary 2, we get the result due to Aziz and Zargar [2] and for $t = 1, K = 1$, Corollary 2 reduces to Theorem B.

We next prove the following more general result which is of independent interest.

Theorem 2. Let $P(z) := \sum_{j=0}^n \alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + i b_j$ where a_j and $b_j, j = 0, 1, 2, \dots, n$ are real numbers. If $t_1 > t_2 \geq 0$ can be found such that for $r = 2, 3, \dots, n+1$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0,$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \geq 0,$$

and for some real numbers u and $v, u \geq 1, v \geq 1$

$$u a_n(t_1 - t_2) - a_{n-1} \geq 0,$$

$$v b_n(t_1 - t_2) - b_{n-1} \geq 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z + (t_1 - t_2) \left(\frac{u a_n + i v b_n}{\alpha_n} - 1 \right) \right| \leq R_1,$$

where

$$R_1 = \frac{1}{|\alpha_n|} \left\{ (u a_n + v b_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1) \frac{t_2}{t_1^{n-1}} - (a_0 + b_0) \frac{1}{t_1^{n-1}} + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} + (|a_0| + |b_0|) \frac{t_2}{t_1^n} \right\}.$$

If in Theorem 2, we take

$$u = \frac{a_{n-1}}{a_n(t_1 - t_2)} \text{ and } v = \frac{b_{n-1}}{b_n(t_1 - t_2)},$$

so that $u \geq 1, v \geq 1$, we get the following:

Corollary 3. Let $P(z) := \sum_{j=0}^n \alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + i b_j$ where a_j and $b_j, j = 0, 1, 2, \dots, n$ are real numbers. If $t_1 > t_2 \geq 0$ can be found such that

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \text{ for } r = 2, 3, \dots, n$$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \leq 0, \text{ for } r = n+1$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \geq 0, \text{ for } r = 2, 3, \dots, n$$

$$b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2} \leq 0, \text{ for } r = n+1,$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{\alpha_{n-1}}{\alpha_n} - (t_1 - t_2) \right| \leq R_1^*,$$

where

$$R_1^* = \frac{1}{|\alpha_n|} \left\{ (a_n + b_n)t_2 + (a_{n-1} + b_{n-1}) - (a_1 + b_1) \frac{t_2}{t_1^{n-1}} \right. \\ \left. - (a_0 + b_0) \frac{1}{t_1^{n-1}} + |a_1 t_1 t_2 + a_0 (t_1 - t_2)| \frac{1}{t_1^n} + |b_1 t_1 t_2 + b_0 (t_1 - t_2)| \right. \\ \left. \frac{1}{t_1^n} + (|a_0| + |b_0|) \frac{t_2}{t_1^n} \right\}.$$

In particular, if

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \geq 0, \text{ for } r = 1, 2, \dots, n$$

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \leq 0, \text{ for } r = n+1,$$

$$b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2} \geq 0, \text{ for } r = 1, 2, \dots, n$$

$$b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2} \leq 0, \text{ for } r = n+1,$$

then

$$a_1 t_1 t_2 + a_0 (t_1 - t_2) \geq 0,$$

$$b_1 t_1 t_2 + b_0 (t_1 - t_2) \geq 0$$

and we get in this case all the zeros of $P(z)$ lie in

$$f(z) = (t_2 + z)(t_1 - z)P(z) \\ = -\alpha_n z^{n+2} + (\alpha_n (t_1 - t_2) - \alpha_{n-1}) z^{n+1} + (\alpha_n t_1 t_2 + \alpha_{n-1} (t_1 - t_2) - \alpha_{n-2}) z^n + \dots \\ + (\alpha_2 t_1 t_2 + \alpha_1 (t_1 - t_2) - \alpha_0) z^2 + (\alpha_1 t_1 t_2 + \alpha_0 (t_1 - t_2)) z + \alpha_0 t_1 t_2 \tag{1}$$

$$= -\alpha_n z^{n+2} - (K-1)\alpha_n (t_1 - t_2) z^{n+1} + (K\alpha_n (t_1 - t_2) - \alpha_{n-1}) z^{n+1} + (\alpha_n t_1 t_2 + \alpha_{n-1} (t_1 - t_2) - \alpha_{n-2}) z^n + \dots \\ + (\alpha_2 t_1 t_2 + \alpha_1 (t_1 - t_2) - \alpha_0) z^2 + (\alpha_1 t_1 t_2 + \alpha_0 (t_1 - t_2)) z + \alpha_0 t_1 t_2 \tag{2}$$

$$= -\alpha_n z^{n+2} - (K-1)\alpha_n (t_1 - t_2) z^{n+1} + (Ka_n (t_1 - t_2) - a_{n-1}) z^{n+1} + (a_n t_1 t_2 + a_{n-1} (t_1 - t_2) - a_{n-2}) z^n + \dots \\ + (a_2 t_1 t_2 + a_1 (t_1 - t_2) - a_0) z^2 + (a_1 t_1 t_2 + a_0 (t_1 - t_2)) z + a_0 t_1 t_2 + i \left[(Kb_n (t_1 - t_2) - b_{n-1}) z^{n+1} + \right. \\ \left. (b_n t_1 t_2 + b_{n-1} (t_1 - t_2) - b_{n-2}) z^n + \dots + (b_2 t_1 t_2 + b_1 (t_1 - t_2) - b_0) z^2 + (b_1 t_1 t_2 + b_0 (t_1 - t_2)) z + b_0 t_1 t_2 \right].$$

This gives

$$|f(z)| \geq |\alpha_n| |z|^{n+1} \left| z + (K-1)(t_1 - t_2) \right| - |Ka_n (t_1 - t_2) - a_{n-1}| |z|^{n+1} - |a_n t_1 t_2 + a_{n-1} (t_1 - t_2) - a_{n-2}| |z|^n - \dots \\ - |a_2 t_1 t_2 + a_1 (t_1 - t_2) - a_0| |z|^2 - |a_1 t_1 t_2 + a_0 (t_1 - t_2)| |z| - |a_0 t_1 t_2| - \left[|Kb_n (t_1 - t_2) - b_{n-1}| |z|^{n+1} \right. \\ \left. + |b_n t_1 t_2 + b_{n-1} (t_1 - t_2) - b_{n-2}| |z|^n + \dots + |b_2 t_1 t_2 + b_1 (t_1 - t_2) - b_0| |z|^2 + |b_1 t_1 t_2 + b_0 (t_1 - t_2)| |z| + |b_0 t_1 t_2| \right].$$

$$\left| z + \frac{\alpha_{n-1}}{\alpha_n} - (t_1 - t_2) \right| \leq \frac{1}{|\alpha_n|} \{ (a_n + b_n)t_2 + (a_{n-1} + b_{n-1}) \}.$$

Remark 3. A result of Shah and Liman [7, Theorem 1] is a special case of Corollary 3, if we assume that all the coefficients of $P(z)$ are real.

The following result also follows from Theorem 2, if we assume that $t_2 = 0$ and $t_1 = 1$.]

Corollary 4. Let $P(z) := \sum_{j=0}^n \alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j , $j = 0, 1, 2, \dots, n$ are real numbers. If for some $u \geq 1$ and $v \geq 1$,

$$ua_n \geq a_{n-1} \geq \dots \geq a_0 \geq 0,$$

$$vb_n \geq b_{n-1} \geq \dots \geq b_0 \geq 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{ua_n + ivb_n}{\alpha_n} - 1 \right| \leq \frac{ua_n + vb_n}{|\alpha_n|}$$

Many other known results and generalizations similarly follows from Theorem 2 with suitable substitutions. We leave this to the readers.

2. Proofs of the Theorems

Proof of Theorem 1. Consider the polynomial

$$= |z|^{n+1} \left\{ |z + (K-1)(t_1 - t_2)| |\alpha_n| - (|Ka_n(t_1 - t_2) - a_{n-1}| + |Kb_n(t_1 - t_2) - b_{n-1}|) - (|a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}| \right. \\ \left. + |b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2}|) \frac{1}{|z|} - \dots - (|a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0| + |b_2 t_1 t_2 + b_1(t_1 - t_2) - b_0|) \frac{1}{|z|^{n-1}} \right. \\ \left. - (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{|z|^n} - (|a_0 t_1 t_2| + |b_0 t_1 t_2|) \frac{1}{|z|^{n+1}} \right\}.$$

For $|z| > t_1$, we have by using hypothesis

$$|f(z)| \geq |z|^{n+1} \left\{ |z + (K-1)(t_1 - t_2)| |\alpha_n| - (|Ka_n(t_1 - t_2) - a_{n-1}| + |Kb_n(t_1 - t_2) - b_{n-1}|) \right. \\ \left. - (|a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}| + |b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2}|) \frac{1}{t_1} - \dots \right. \\ \left. - (|a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0| + |b_2 t_1 t_2 + b_1(t_1 - t_2) - b_0|) \frac{1}{t_1^{n-1}} \right. \\ \left. - (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} - (|a_0 t_1 t_2| + |b_0 t_1 t_2|) \frac{1}{t_1^{n+1}} \right\} > 0,$$

if

$$|z + (K-1)(t_1 - t_2)| |\alpha_n| > Ka_n(t_1 - t_2) + Kb_n(t_1 - t_2) + a_n t_2 + b_n t_2 - a_1 \frac{t_2}{t_1^{n-1}} - b_1 \frac{t_2}{t_1^{n-1}} - \frac{a_0}{t_1^{n-1}} - \frac{b_0}{t_1^{n-1}} \\ + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} + |a_0| \frac{t_2}{t_1^n} + |b_0| \frac{t_2}{t_1^n}.$$

Therefore, for $|z| \geq t_1$, $|f(z)| > 0$, if

$$|z + (K-1)(t_1 - t_2)| > \frac{1}{|\alpha_n|} \left\{ K(a_n + b_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1) \frac{t_2}{t_1^{n-1}} - (a_0 + b_0) \frac{1}{t_1^{n-1}} \right. \\ \left. + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} + (|a_0| + |b_0|) \frac{t_2}{t_1^n} \right\}.$$

Hence all the zeros of $f(z)$ whose modulus is greater than t_1 lie in the circle

$$|z + (K-1)(t_1 - t_2)| \leq \frac{1}{|\alpha_n|} \left\{ K(a_n + b_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1) \frac{t_2}{t_1^{n-1}} - (a_0 + b_0) \frac{1}{t_1^{n-1}} \right. \\ \left. + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} + (|a_0| + |b_0|) \frac{t_2}{t_1^n} \right\}.$$

Since all the zeros whose modulus is less than t_1 already lie in this circle, we conclude that all the zeros of $f(z)$ and therefore $P(z)$ lies in

$$|z + (K-1)(t_1 - t_2)| \leq \frac{1}{|\alpha_n|} \left\{ K(a_n + b_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1) \frac{t_2}{t_1^{n-1}} - (a_0 + b_0) \frac{1}{t_1^{n-1}} \right. \\ \left. + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} + (|a_0| + |b_0|) \frac{t_2}{t_1^n} \right\}.$$

This completes the proof of the Theorem 1.

Proof of Theorem 2. Consider the polynomial

$$\begin{aligned}
 f(z) &= (t_2 + z)(t_1 - z)P(z) \\
 &= -\alpha_n z^{n+2} + (\alpha_n(t_1 - t_2) - \alpha_{n-1})z^{n+1} + (\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2})z^n + \dots \\
 &\quad + (\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0)z^2 + (\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2))z + \alpha_0 t_1 t_2 \\
 &= -\alpha_n z^{n+2} + (a_n(t_1 - t_2) - a_{n-1})z^{n+1} + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2})z^n + \dots + (a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0)z^2 \\
 &\quad + (a_1 t_1 t_2 + a_0(t_1 - t_2))z + a_0 t_1 t_2 + i \left[(b_n(t_1 - t_2) - b_{n-1})z^{n+1} + (b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2})z^n + \dots \right. \\
 &\quad \left. + (b_2 t_1 t_2 + b_1(t_1 - t_2) - b_0)z^2 + (b_1 t_1 t_2 + b_0(t_1 - t_2))z + b_0 t_1 t_2 \right] \\
 &= -\alpha_n z^{n+2} - (u-1)a_n(t_1 - t_2)z^{n+1} + (ua_n(t_1 - t_2) - a_{n-1})z^{n+1} + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2})z^n + \dots \\
 &\quad + (a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0)z^2 + (a_1 t_1 t_2 + a_0(t_1 - t_2))z + a_0 t_1 t_2 + i \left[-(v-1)b_n(t_1 - t_2)z^{n+1} + (vb_n(t_1 - t_2) - b_{n-1})z^{n+1} \right. \\
 &\quad \left. + (b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2})z^n + \dots + (b_2 t_1 t_2 + b_1(t_1 - t_2) - b_0)z^2 + (b_1 t_1 t_2 + b_0(t_1 - t_2))z + b_0 t_1 t_2 \right] \\
 &= -\alpha_n z^{n+2} + \{-(ua_n + ivb_n) + \alpha_n\}(t_1 - t_2)z^{n+1} + (ua_n(t_1 - t_2) - a_{n-1})z^{n+1} + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2})z^n + \dots \\
 &\quad + (a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0)z^2 + (a_1 t_1 t_2 + a_0(t_1 - t_2))z + a_0 t_1 t_2 + i \left[(vb_n(t_1 - t_2) - b_{n-1})z^{n+1} + \right. \\
 &\quad \left. (b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2})z^n + \dots + (b_2 t_1 t_2 + b_1(t_1 - t_2) - b_0)z^2 + (b_1 t_1 t_2 + b_0(t_1 - t_2))z + b_0 t_1 t_2 \right].
 \end{aligned}$$

This gives

$$\begin{aligned}
 |f(z)| &\geq |\alpha_n| |z|^{n+1} \left| z + (t_1 - t_2) \left(\frac{ua_n + ivb_n}{\alpha_n} - 1 \right) \right| - |ua_n(t_1 - t_2) - a_{n-1}| |z|^{n+1} - |a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}| |z|^n - \dots \\
 &\quad - |a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0| |z|^2 - |a_1 t_1 t_2 + a_0(t_1 - t_2)| |z| - |a_0 t_1 t_2| - |vb_n(t_1 - t_2) - b_{n-1}| |z|^{n+1} \\
 &\quad - |b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2}| |z|^n - \dots - |b_2 t_1 t_2 + b_1(t_1 - t_2) - b_0| |z|^2 - |b_1 t_1 t_2 + b_0(t_1 - t_2)| |z| - |b_0 t_1 t_2|. \\
 &= |z|^{n+1} \left\{ |\alpha_n| \left| z + (t_1 - t_2) \left(\frac{ua_n + ivb_n}{\alpha_n} - 1 \right) \right| - (|ua_n(t_1 - t_2) - a_{n-1}| + |vb_n(t_1 - t_2) - b_{n-1}|) \right. \\
 &\quad \left. - (|a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}| + |b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2}|) \frac{1}{|z|} - \dots - (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{|z|^n} \right. \\
 &\quad \left. - (|a_0 t_1 t_2| + |b_0 t_1 t_2|) \frac{1}{|z|^{n+1}} \right\}
 \end{aligned}$$

For $|z| > t_1$, we have

$$\begin{aligned}
 |f(z)| &\geq |z|^{n+1} \left\{ \left| z + (t_1 - t_2) \left(\frac{ua_n + ivb_n}{\alpha_n} - 1 \right) \right| |\alpha_n| - (|ua_n(t_1 - t_2) - a_{n-1}| + |vb_n(t_1 - t_2) - b_{n-1}|) \right. \\
 &\quad \left. - (|a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}| + |b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2}|) \frac{1}{t_1} - \dots - (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} \right. \\
 &\quad \left. - (|a_0 t_1 t_2| + |b_0 t_1 t_2|) \frac{1}{t_1^{n+1}} \right\}.
 \end{aligned}$$

By using hypothesis, this gives

$$|f(z)| \geq |z|^{n+1} \left\{ \left| z + (t_1 - t_2) \left(\frac{ua_n + ivb_n}{\alpha_n} - 1 \right) \right| |\alpha_n| - \left[(ua_n + vb_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1) \frac{t_2}{t_1^{n-1}} - (a_0 + b_0) \frac{1}{t_1^{n-1}} \right] \right\}$$

$$-\left(|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|\right) \frac{1}{t_1^n} - (|a_0| + |b_0|) \frac{t_2}{t_1^n} \Big\} > 0,$$

if

$$\left| z + (t_1 - t_2) \left(\frac{ua_n + ivb_n}{\alpha_n} - 1 \right) \right| > \frac{1}{|\alpha_n|} \left\{ (ua_n + vb_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1) \frac{t_2}{t_1^{n-1}} - (a_0 + b_0) \frac{1}{t_1^{n-1}} \right. \\ \left. + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} + (|a_0| + |b_0|) \frac{t_2}{t_1^n} \right\}.$$

Hence all the zeros of $f(z)$ whose modulus is greater than t_1 lie in the circle

$$\left| z + (t_1 - t_2) \left(\frac{ua_n + ivb_n}{\alpha_n} - 1 \right) \right| \leq \frac{1}{|\alpha_n|} \left\{ (ua_n + vb_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1) \frac{t_2}{t_1^{n-1}} - (a_0 + b_0) \frac{1}{t_1^{n-1}} \right. \\ \left. + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} + (|a_0| + |b_0|) \frac{t_2}{t_1^n} \right\}.$$

Since all the zeros whose modulus is less than t_1 already lie in this circle, we conclude that all the zeros of $f(z)$ and therefore $P(z)$ lies in

$$\left| z + (t_1 - t_2) \left(\frac{ua_n + ivb_n}{\alpha_n} - 1 \right) \right| \leq R_1,$$

where

$$R_1 = \frac{1}{|\alpha_n|} \left\{ (ua_n + vb_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1) \frac{t_2}{t_1^{n-1}} - (a_0 + b_0) \frac{1}{t_1^{n-1}} + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|) \frac{1}{t_1^n} \right. \\ \left. + (|a_0| + |b_0|) \frac{t_2}{t_1^n} \right\}.$$

This proves Theorem 2 completely.

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4. References

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