Simple C^*-algebras with locally finite decomposition rank

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Aim: Classify (separable, simple) nuclear C^*-algebras by K-theory data.

Kirchberg–Phillips classification (the purely infinite case) heavily uses characterization of pure infiniteness in terms of $\mathcal{O}_\infty$-stability.

In the stably finite case, there are only classification results known for $\mathrm{rr} \leq 1$ and $\mathrm{sr}\,1$. Moreover, all known results use topological dimension theories, such as $\dim_{\mathrm{AH}}$, $\dim_{\mathrm{ASH}}$, $\mathrm{sdg}_{\mathrm{AH}}$, $\mathrm{sdg}_{\mathrm{ASH}}$ or dr.
There are even counterexamples to the Elliott conjecture; these have infinite topological dimension.

However, all classes which have so far been classified consist entirely of $\mathcal{Z}$-stable C^*-algebras (Jiang–Su, Toms–W).

In many ways, the Jiang–Su algebra $\mathcal{Z}$ may be thought of as a stably finite analogue of $\mathcal{O}_\infty$.
In particular, both algebras are KK-equivalent to $\mathbb{C}$ and strongly self-absorbing (in the sense of Toms–W).

In the stably finite case of the Elliott program, we try to systematically use $\mathcal{Z}$-stability in place of dimension type conditions.

Idea: A is 'purely finite', if A is stably finite and $\mathcal{Z}$-stable.

Definition (Kirchberg–W.)

(i) A c.p. map $\varphi : F = M_{r_1} \oplus \ldots \oplus M_{r_s} \to A$ is n-decomposable, if there is a decomposition

$$\{1, \ldots, s\} = \coprod_{j=0}^{n} I_j$$

s.t. the restriction of φ to

$$\bigoplus_{i \in I_j} M_{r_i}$$

has order zero (i.e., preserves orthogonality) for each $j \in \{0, \ldots, n\}$.

(ii) A has decomposition rank n, $\operatorname{dr} A = n$, if n is the least integer such that the following holds:
Given $\mathcal{F} \subset A$ finite and $\varepsilon > 0$, there is a c.p. approximation $A \xrightarrow{\psi} F \xrightarrow{\varphi} A$ for $\mathcal{F}$ within ε such that F is finite-dimensional and φ is n-decomposable.

Definition

We say A has locally finite decomposition rank, if the following holds: Given $\mathcal{F} \subset A$ finite and $\varepsilon > 0$, there is a C^*-subalgebra $B \subset A$ such that $\mathrm{dist}(b, B) < \varepsilon \; \forall \, b \in \mathcal{F}$ and $\mathrm{dr}\, B < \infty$.

Note that we do not ask for a global bound on $\mathrm{dr}\, B$.

Proposition

(i) If A has locally finite decomposition rank, then A is nuclear and strongly quasidiagonal (thus stably finite).

(ii) Locally finite decomposition rank passes to inductive limits, quotients, tensor products and to hereditary subalgebras generated by projections.

Examples

(i) all C^*-algebras with finite decomposition rank, e.g.
- AF algebras
- rotation algebras
- the Jiang–Su algebra $\mathcal{Z}$

(ii) all (separable) commutative C^*-algebras

(ii) all (separable) AH C^*-algebras (Blackadar), including Villadsen's and Toms' examples (these have infinite topological dimension)

(iii) all (separable) ASH C^*-algebras (Ng–W)

Theorem (W)

Let A be a separable simple unital and $\mathcal{Z}$-stable C^*-algebra with real rank zero and locally finite decomposition rank.
Then, A has tracial rank zero in the sense of Lin.

Corollary (Lin)

Let A and B be separable simple unital C^*-algebras with real rank zero and locally finite decomposition rank; suppose A and B satisfy the UCT and are $\mathcal{Z}$-stable.
Then, A and B are isomorphic iff their Elliott invariants are.

Corollary (Using results of Dadarlat, Elliott, Gong et al.)

Let A be a separable simple unital C^*-algebra; suppose $A \otimes \mathcal{Z}$ has real rank zero and locally finite decomposition rank and satisfies the UCT. Then:

(i) $\dim_{\mathrm{AH}}(A \otimes \mathcal{Z}) \leq 3$

(ii) $\dim_{\mathrm{ASH}}(A \otimes \mathcal{Z}) \leq 2$

(iii) $\mathrm{dr}\,(A \otimes \mathcal{Z}) \leq 2$

(iv) $A \otimes \mathcal{Z}$ is approximately divisible

(v) A is $\mathcal{Z}$-stable iff A is approximately divisible.

In particular, if A is AH with real rank zero, then $A \otimes \mathcal{Z}$ has slow dimension growth as an AH algebra.
(That $A \otimes \mathcal{Z}$ has slow dimension growth as an ASH algebra is easy to see.)

Corollary

The class of separable simple unital $\mathcal{Z}$-stable ASH C^*-algebras with real rank zero satisfies the Elliott conjecture.

In the proof of the main result, $\mathcal{Z}$-stability enters in two ways:

- to ensure comparison (Rørdam)
- to circumvent trace space conditions (the proof becomes considerably easier in the unique trace case)

To prove the main theorem, we have to show:

For any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$ there is a finite-dimensional C^*-subalgebra $D \subset A$ such that

(i) $\|[\mathbf{1}_D, b]\| < \varepsilon \; \forall \, b \in \mathcal{F}$

(ii) $\mathrm{dist}(\mathbf{1}_D b \mathbf{1}_D, D) < \varepsilon \; \forall \, b \in \mathcal{F}$

(iii) $\tau(\mathbf{1}_A - \mathbf{1}_D) < \varepsilon \; \forall \, \tau \in T(A)$.

(At this point, we are already using $\mathcal{Z}$-stability.)

Since A has locally finite decomposition rank, we may assume that $\mathcal{F} \subset B$, where B is some (unital) C^*-subalgebra B of A such that $\operatorname{dr} B = n$ for some $n \in \mathbb{N}$.
Using $\operatorname{dr} B = n$ and $\operatorname{rr} A = 0$ we can find $n+1$ (not necessarily pairwise orthogonal) finite-dimensional C^*-algebras

$$\tilde{F}^{(0)}, \ldots, \tilde{F}^{(n)} \subset A$$

such that

$$\mathcal{F} \subset_\alpha \tilde{F}^{(0)} + \ldots + \tilde{F}^{(n)}.$$

One can then use $\mathcal{Z}$-stability of A to find a finite-dimensional C^*-subalgebra

$$D_1 := \bar{F}^{(0)} \oplus \ldots \oplus \bar{F}^{(n)} \subset A$$

satisfying (i) and (ii) above (with D_1 in place of D), if only α was chosen small enough.
This construction will not force D_1 to quite satisfy (iii) – but we can reach

$$\tau(\mathbf{1}_{D_1}) > \frac{1}{2(n+1)} =: \mu \; \forall \, \tau \in T(A).$$

We may try to repeat the above process with $C_1 := (\mathbf{1}_A - \mathbf{1}_{D_1})A(\mathbf{1}_A - \mathbf{1}_{D_1})$ in place of A to obtain a finite-dimensional $D_2 \supset D_1$ which not only satisfies (i) and (ii), but also

$$\tau(\mathbf{1}_{D_2}) > \mu(1-\mu) + \mu \;\forall\, \tau \in T(A).$$

Induction will then yield an increasing sequence $D_1 \subset D_2 \subset \ldots \subset A$ such that

$$\tau(\mathbf{1}_{D_k}) > \mu \sum_{i=0}^{k} (1-\mu)^i;$$

by the formula for the geometric series we have $\mu \sum_{i=0}^{\infty}(1-\mu)^i = 1$, whence

$$\tau(\mathbf{1}_{D_K}) > 1 - \varepsilon$$

for some large enough K.

If A has *finite* decomposition rank n, then all this works.

A problem arises in the case of only *locally finite* decomposition rank: The induction step might only work with some n' much larger than n – and this would destroy the final geometric series argument, since

$$\mu' = 1/2(n'+1) \ll \mu = 1/2(n+1).$$

The difficulty can be circumvented by choosing the compression with $(\mathbf{1}_A - \mathbf{1}_{D_k})$ to be almost multiplicative not only on $\mathcal{F}$ but on a *large* subset of B.

The procedure is complicated, since one has to carefully keep track of the approximation constants chosen along the way.

In fact, given $\mathcal{F}$ and ε, we first choose B and, at the same time, obtain n. This n determines how many induction steps will be needed ($\mu\sum_{i=0}^{K}(1-\mu)^i$ has to be larger than $1-\varepsilon$, and μ depends on n).

Next we choose α and the c.p. approximation (F,ψ,φ). The number α has to be so small that, even after K induction steps, the algebra D_K still satisfies (i) and (ii) above.

Only now we can let the induction process start, i.e., carry out the actual construction of the D_k for $k=1,\ldots,K$.