AN EXTENSION OF LOMONOSOV'S TECHNIQUES TO NON–COMPACT OPERATORS

By

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To Adriana and the Stones.

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Abstract

The existence of invariant subspaces for bounded linear operators acting on an infinite–dimensional Hilbert space appears to be one of the most difficult questions in the theory of linear transformations. The question is known as the *invariant subspace problem*. Very few affirmative answers are known regarding this problem. One of the most prominent ones is the theorem on the existence of hyper–invariant subspaces for compact operators due to V.I. Lomonosov.

The aim of this work is to generalize Lomonosov's techniques in order to apply them to a wider class of not necessarily compact operators. We start by establishing a connection between the existence of invariant subspaces and density of what we define as the associated Lomonosov space in a certain function space. On a Hilbert space approximation with Lomonosov functions results in an extended version of Burnside's Theorem. An application of this theorem to the algebra generated by an essentially self–adjoint operator A yields the existence of vector states on the space of all polynomials restricted to the essential spectrum of A. Finally, the invariant subspace problem for compact perturbations of self–adjoint operators is translated into an extreme problem and the solution is obtained upon differentiating certain real–valued functions at their extreme.

The invariant subspace theorem for essentially self–adjoint operators acting on an infinite–dimensional real Hilbert space is the main result of this work and represents an extension of the known techniques in the theory of invariant subspaces.

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Halifax, Nova Scotia — Aleksander Simonič
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Introduction

The question

> *Does every bounded linear operator on a Banach space have a non–trivial closed invariant subspace?*

is known as the *invariant subspace problem.*

The examples due to Enflo [**?**] and Read [**?**] show that the answer to the invariant subspace problem is in general negative. However, there are no known examples of operators without invariant subspaces acting on a reflexive Banach space and in particular, on a Hilbert space. Furthermore, there seems to be no evidence what should be an expected answer for the operators acting on a Hilbert space, and the experts in the field have different opinions on it.

It is therefore not surprising that there are relatively few special cases for which the existence of invariant subspaces have been established. One of the most prominent results is the one on the existence of hyperinvariant subspaces for compact operators due to 2Lomonosov [**?**, **?**]. Another class of operators that is well understood in terms of invariant subspaces are normal, and in particular, self–adjoint operators for which there is the powerful spectral theorem. However, it is not known whether a compact perturbation of a self–adjoint operator has a non–trivial invariant subspace.

This work focuses on the existence of invariant subspaces for essentially self–adjoint operators and culminates in an affirmative answer in the case where the underlying Hilbert space is assumed to be *real*.

When dealing with the existence of invariant subspaces it is a common practice [?, ?, ?] to study the space of certain continuous functions associated with the algebra generated by an operator rather than the operator itself. We follow this approach and establish a connection between the existence of invariant subspaces for an operator algebra and density of certain associated spaces of continuous functions called *Lomonosov spaces*. The construction of these functions is based on the idea of the *partition of unity* subordinate to an open cover, which is a standard tool in approximation theory [?] and differential geometry [?, ?]. In [?] the partition of unity is also used to prove the Arzela–Ascoli Theorem. It should be observed that similar argument was employed by V.I. Lomonosov in the proof of his celebrated result [?].

On a Hilbert space differentiability of the norm yields a numerical criterion for the construction of Lomonosov functions with certain properties. This results in an extension of the Burnside Theorem and implies the solution of what we define as the "essentially–transitive algebra problem".

An application of the extended Burnside Theorem to the algebra generated by an essentially self–adjoint operator yields the existence of vector states on the space of polynomials restricted to the essential spectrum of such an operator. The invariant subspace problem for compact perturbations of self–adjoint operators is translated into an extreme problem and the solution (in the case where the underlying Hilbert space is real) is obtained upon differentiating certain real–valued functions at their extreme.

Although the above–described techniques do not immediately extend to the complex Hilbert spaces, it is very likely that further analysis of the space of vector states will reduce the complex case to the real one and thus provide the affirmative answer to one of the most difficult questions in the theory of invariant subspaces [**?**].

Chapter 1

The Space of Lomonosov Functions

In this chapter we give a constructive proof of an abstract approximation theorem, inspired by the celebrated result of V.I. Lomonosov [?]. This theorem is applied to obtain an alternative proof of some recent characterizations of the invariant subspace problem, given in [?]. We also establish density of non–cyclic vectors for certain convex sets of compact quasinilpotent operators, and conclude with a related open question. In Chapter 2 we extend the techniques introduced in this chapter to non–compact operators acting on a Hilbert space.

1.1 Introduction

V.I. Lomonosov in his paper [?] conjectured that the adjoint of a bounded operator on a Banach space has a non–trivial closed invariant subspace. In view of the known examples of operators without an invariant subspace [?, ?], this is the strongest version of the invariant subspace problem that can possibly have an affirmative answer. In particular, if the Lomonosov conjecture is true, then every operator on a reflexive Banach space has a non–trivial invariant subspace.

Considering the strong influence of Lomonosov's results on the theory of invariant subspaces, it is not surprising that both the conjecture and the techniques developed in the interesting paper [?] received further attention. L. de Branges used this result to obtain a characterization of the invariant subspace problem in terms of density of certain functions. This stimulated another characterization of the invariant subspace problem given by Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw in [?]. Section 1.4 presents a more detailed account of this work.

We take a slightly different approach. First we give a constructive proof of the approximation theorem, inspired by the well known Lomonosov construction used in [?, ?]. This theorem is then applied to give an alternative proof of the main result in [?]. Our proof applies to both real and complex Banach spaces, while the original result was established for complex Banach spaces only. The alternative proof somehow explains the role of compact operators that appear in the characterizations of the invariant subspace problem [?].

One may notice that the weak*–compactness of the unit ball in dual Banach spaces plays an important role in [?, ?, ?, ?], as well as in the applications given in this chapter. In other words, if the Lomonosov conjecture is true, then the compactness of the unit ball, with respect to the weak* topology, is likely to be an important ingredient of its proof.

In the last section we put this observation to the test. A straightforward application of the approximation theorem obtained in Section 1.3, together with the Schauder–Tychonoff Fixed Point Theorem, yields density of non–cyclic vectors for the dual of a convex set of compact quasinilpotent operators. We end with the open problem of obtaining a similar result for the original set, rather than its dual.

This work is more or less self–contained and the notation and terminology used in it is (supposed to be) standard. However, here are a few conventions that hold throughout this chapter:

By an operator we always mean a bounded linear operator acting on a real or complex Banach space. If $\mathcal{A}$ is a set of operators and K is a fixed operator then $\mathcal{A}K$ stands for the set $\{AK \mid A \in \mathcal{A}\}$. Saying that a set of operators $\mathcal{A}$, acting on a Banach space X, admits an invariant subspace, means that there exists a non–trivial closed subspace of X that is invariant under all operators in $\mathcal{A}$. The space of all linear operators on a Banach space X is denoted by $\mathbf{B}(X)$, while $C(S, X)$ stands for the set of all continuous functions $f\colon S \longrightarrow X$. If S is a subset of a Banach space X, then in saying that a linear operator A is in $C(S, X)$, we actually refer to the restriction of the operator A to the set S.

1.2 Reflexive Topological Spaces and Continuous Indicator Functions

This section introduces some topological preliminaries that lead to a fairly general treatment of the approximation theory in the next section, where an important role is played by the partition of unity and the "continuous indicator functions" associated with a basis for the topology on a compact domain of certain functions. The existence of continuous indicator functions can be characterized by a purely topological property of the underlying space, which is defined as "reflexivity" of the topological space. In this section we introduce both concepts and establish the connection between them.

Definition 1.2.1. Let $S = (S, \tau)$ be a topological space and denote by $C(S, \mathbb{R})$ the space of all continuous real–valued functions on S. A topological space S is called *reflexive* if the topology τ coincides with the weakest topology τ_w on S for which all the functions in $C(S, \mathbb{R})$ are continuous.

Remark 1.2.1. The reflexivity of topological spaces is not to be confused with the corresponding concept of the reflexivity of Banach spaces. Indeed, we conclude this section by showing that every subset of a locally convex space is reflexive.

Proposition 1.2.1. *Reflexivity is a hereditary property; i.e. a subspace S of a reflexive topological space X is reflexive with the relative topology.*

Proof. Consider the restrictions of the functions in $C(X, \mathbb{R})$ to the subset S, and observe that they induce the relative topology on S, whenever X is reflexive. □

Definition 1.2.2. Suppose U is an open subset of a topological space S. A continuous function $\Gamma \colon S \longrightarrow [0, \infty)$ is called a *continuous indicator function* of U in S if

$$U = \{s \in S \mid \Gamma(s) > 0\} .$$

Remark 1.2.2. If X is a metric space then every open ball

$$U = U(x_0, r) = \{x \in X \mid d(x, x_0) < r\} ,$$

admits a continuous indicator function $\Gamma_U \colon X \longrightarrow [0, \infty)$, defined by

$$\Gamma_U(x) = \max\{0,\, r - d(x, x_0)\} .$$

Furthermore, suppose $f \in C(S, X)$. Then the open set $V = f^{-1}(U) \subset S$ "inherits" an indicator function from U by setting: $\Gamma_V(s) = \Gamma_U(f(s))$.

This yields the following characterization of reflexivity.

Proposition 1.2.2. *A topological space $S = (S, \tau)$ is reflexive if and only if there exists an open basis $\mathcal{B}$ for the topology τ such that each set $V \in \mathcal{B}$ admits a continuous indicator function $\Gamma_V : S \longrightarrow [0, \infty)$.*

Proof. By definition of reflexivity, the family

$$\mathcal{B}_0 = \left\{ f^{-1}(U) \mid f \in C(S, \mathbb{R}) \text{ and } U = (a, b) \subset \mathbb{R} \right\}$$

is a sub–basis for the topology τ on a reflexive space S. Clearly,

$$\Gamma_U(t) = \max \left\{ 0,\ \tfrac{b-a}{2} - \left| \tfrac{a+b}{2} - t \right| \right\}$$

is a continuous indicator function of the open interval $U = (a, b)$ in $\mathbb{R}$. Consequently, $\Gamma_V(s) = \Gamma_U\big(f(s)\big)$ is a continuous indicator function for the set $V = f^{-1}(U)$ in S. Let $V = V_1 \cap \ldots \cap V_n$ for $V_k \in \mathcal{B}_0$. A continuous indicator function of V can be defined by

$$\Gamma_V(s) = \prod_{k=1}^{n} \Gamma_{V_k}(s).$$

Therefore, each set in a basis

$$\mathcal{B} = \{ V_1 \cap \ldots \cap V_n \mid V_k \in \mathcal{B}_0;\ n < \infty \},$$

admits a continuous indicator function.

The other direction is trivial, because the continuous indicator functions form a subset of $C(S, \mathbb{R})$. □

Remark 1.2.3. The argument in the proof of Proposition 1.2.2 shows that the space $\mathbb{R}$ can be replaced by any metric vector space over $\mathbb{R}$ in the definition of reflexivity. In particular, considering the complex valued functions would not change the definition of reflexivity.

Remark 1.2.4. While an open set U is uniquely determined by any of its continuous indicator functions, the converse is of course not true. However, Proposition 1.2.2 allows us to choose a basis $\mathcal{B}$, and a corresponding family

$$\Gamma_{\mathcal{B}} = \{\Gamma_U \colon S \longrightarrow [0,\infty) \mid U \in \mathcal{B}\}$$

of continuous indicator functions associated with the basis $\mathcal{B}$ for the topology on a reflexive topological space S. In that sense, the correspondence between the elements of $\mathcal{B}$ and an associated family of continuous indicator functions $\Gamma_{\mathcal{B}}$ can be established.

Although not all topological spaces are reflexive (consider for example the topology of finite complements on any infinite set) the next proposition shows that convex balanced neighborhoods in a locally convex space admit continuous indicator functions, and consequently, all locally convex spaces are reflexive.

Proposition 1.2.3. *Every locally convex vector space X is reflexive (as a topological space).*

Proof. Suppose $\mathcal{B}$ is a base for the topology on X consisting of open convex balanced sets. Then for each $U \in \mathcal{B}$:

$$U = \{x \in X \mid \mu_U(x) < 1\},$$

where μ_U is the Minkowski functional of U. The function

$$\Gamma_U(x) = \max\{0,\, 1 - \mu_U(x)\}$$

is a continuous indicator function for U. By Proposition 1.2.2, X is reflexive. □

1.3 Lomonosov Functions

The proof of the celebrated result of V.I. Lomonosov [**?**, **?**] was based on the ingenious idea of defining a continuous function with compact domain in a Banach space, assuming that certain local conditions are met. In this section we generalize this idea in the form of an approximation theorem. Since our construction was greatly inspired by the proof of Lomonosov's Lemma [**?**, **?**], we suggest the following definition.

Definition 1.3.1. Let $\mathcal{A} \subset C(S, X)$ be a subset of the space of continuous functions from a topological space S to a locally convex space X. The convex subset $\mathcal{L}(\mathcal{A}) \subset C(S, X)$, defined by

$$\mathcal{L}(\mathcal{A}) = \left\{ \sum_{k=1}^{n} \alpha_k A_k \mid A_k \in \mathcal{A}, \alpha_k \in C(S, [0,1]) \text{ and } \sum_{k=1}^{n} \alpha_k \equiv 1; \; n < \infty \right\}.$$

is called the *Lomonosov space* associated with the set $\mathcal{A}$, and a function $\Lambda \in \mathcal{L}(\mathcal{A})$ is called a *Lomonosov function.*

Recall that the *uniform topology* on $C(S, X)$ is induced by the topology on a linear space X. If $\mathcal{B}$ is a local basis for the topology on X then the sets

$$\widehat{U} = \{ f \in C(S, X) \mid f(S) \subset U \in \mathcal{B} \}$$

define a local basis for the uniform topology on $C(S, X)$. If X is a locally convex space then so is $C(S, X)$. In particular, if X is a Banach space then $C(S, X)$ with the uniform topology is a Banach space, as well.

We are now ready to give a construction of the Lomonosov function that uniformly approximates a continuous function within a given neighborhood.

Lemma 1.3.1. *Let $\mathcal{A} \subset C(S,X)$ be a subset of continuous functions from a reflexive compact topological space S to a locally convex space X. Fix an open convex neighborhood U of 0 in X. Suppose $f\colon S \longrightarrow X$ is a continuous function that at each point of S can be approximated within U by some element of $\mathcal{A}$; i.e. for every point $s \in S$ there exists a function $A_s \in \mathcal{A}$ such that $A_s(s) - f(s) \in U$. Then there exists a finite subset $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$, together with continuous nonnegative functions $\alpha_k\colon S \longrightarrow [0,1]$, such that $\sum_{k=1}^{n} \alpha_k \equiv 1$, and the Lomonosov function $\Lambda \in \mathcal{L}(\mathcal{A})$, defined by*

$$\Lambda(s) = \sum_{k=1}^{n} \alpha_k(s) A_k(s),$$

lies in the prescribed neighborhood $\widehat{U}$ of f in $C(S,X)$; i.e. $\Lambda(s) - f(s) \in U$ for every $s \in S$.

Proof. By the hypothesis, for every point $s \in S$ there exists a function $A_s \in \mathcal{A}$ such that $A_s(s) - f(s) \in U$. Continuity of the functions f and A_s implies the existence of a (basic) neighborhood W_s of s in S such that $A_s(w) - f(w) \in U$ for every $w \in W_s$. In this way we obtain an open cover for S with the sets W_s. Compactness of S yields a finite subcover: $W_{s_1} \cup \ldots \cup W_{s_n} \supset S$.

Each set W_s is associated with a continuous indicator function $\Gamma_{W_s}\colon S \longrightarrow [0,\infty)$. Every point in S lies in at least one neighborhood W_{s_k}; therefore the sum $\sum_{j=1}^{n} \Gamma_{W_{s_j}}(s)$ is strictly positive for all elements $s \in S$. Consequently, the functions $\alpha_k\colon S \longrightarrow [0,1]$, defined by

$$\alpha_k(s) = \frac{\Gamma_{W_{s_k}}(s)}{\sum_{j=1}^{n} \Gamma_{W_{s_j}}(s)} \qquad (k = 1, \ldots, n),$$

are well defined and continuous on S. Also, $\sum_{k=1}^{n} \alpha_k(s) = 1$ for every $s \in S$, and $\alpha_k(s) > 0$ if and only if $s \in W_{s_k}$. Therefore, $\alpha_k(s) > 0$ implies that $A_{s_k}(s) - f(s) \in U$.

Set $A_k = A_{s_k}$ $(k = 1, \ldots, n)$. Continuity of the functions $\alpha_k \colon S \longrightarrow [0,1]$ and $A_k \colon S \longrightarrow X$ implies that the Lomonosov function $\Lambda \in \mathcal{L}(\mathcal{A})$, defined by

$$\Lambda(s) = \sum_{k=1}^{n} \alpha_k(s) A_k(s),$$

is continuous. Observe that

$$\Lambda(s) - f(s) = \sum_{k=1}^{n} \alpha_k(s)\big(A_k(s) - f(s)\big)$$

is a convex combination of the elements in U, because only those coefficients $\alpha_k(s)$ for which $A_k(s) - f(s) \in U$ are nonzero. Since U is a convex set, it follows that the image of $\Lambda - f$ is contained in U. In other words, Λ lies in the prescribed neighborhood $\widehat{U}$ of f in $C(S, X)$. □

Remark 1.3.1. The proof of Lomonosov's Lemma [**?**, **?**] introduces a special case of the above construction: S is a compact set in a Banach space X, defined as the closure of the image of the unit ball around a fixed vector x_0, under a given nonzero compact operator K. Furthermore, the vector x_0 is chosen so that the set S doesn't contain the zero vector; $\mathcal{A}$ is the restriction to S of an algebra of bounded linear operators on X that admits no invariant subspaces. Under the stated hypothesis a construction of the function $\Lambda \colon S \longrightarrow X$ is given such that $\Lambda \in \mathcal{L}(\mathcal{A}K)$ maps S into the unit ball around x_0; or equivalently, the constant function $f \equiv x_0$ can be approximated on S within 1 by the elements of $\mathcal{L}(\mathcal{A}K)$. It is clear from the original construction as well as from Theorem 1.3.2 that in that case the set S can be mapped into an arbitrary small neighborhood of x_0; or equivalently, the function $f \equiv x_0$ is in the closure of the space $\mathcal{L}(\mathcal{A}K)$.

The following approximation theorem follows immediately from Lemma 1.3.1.

Theorem 1.3.2. *Let $\mathcal{A} \subset C(S, X)$ be a subset of continuous functions from a reflexive compact topological space S to a locally convex space X. Suppose that $f : S \longrightarrow X$ is a continuous function that at each point of S can be approximated by some element of $\mathcal{A}$; i.e. for every $s \in S$ and every neighborhood U of 0 in X there exists a function $A_s \in \mathcal{A}$ such that $A_s(s) - f(s) \in U$. Then the function f can be approximated uniformly on S by the elements of the associated Lomonosov space $\mathcal{L}(\mathcal{A})$.*

In the next section we employ Theorem 1.3.2 to obtain an alternative proof of a characterization of the existence of invariant subspaces for algebras of bounded linear operators acting on a real or complex Banach space. The complex version of this theorem was first established in [**?**], using rather different techniques built on the result of L. de Branges [**?**].

1.4 A Characterization of the Invariant Subspace Problem

We introduce some basic concepts and notation that is consistent with [**?**]. However, for more details and further references on the *invariant subspace problem*, the reader is advised to consult the nicely written and comprehensible original [**?**].

In this section X stands for a real or complex Banach space of dimension greater than one and X' for its norm dual. The algebra of all bounded linear operators on X is denoted by $\mathbf{B}(X)$. If $\mathcal{A}$ is any subset of $\mathbf{B}(X)$, then the adjoint set $\mathcal{A}'$ of $\mathcal{A}$ is defined by $\mathcal{A}' = \{A' \mid A \in \mathcal{A}\}$, where A' is the Banach adjoint of A.

The set $S = \{x \in X' \mid \|x\| \leq 1\}$ denotes the unit ball in the dual space X', equipped with its weak* topology.

Definition 1.4.1. The vector space of all continuous functions from S to X', where both spaces are equipped with the weak* topology, is denoted by $C(S, X')$. As usual, $C(S)$ denotes the commutative Banach algebra of all continuous complex valued functions on S with the uniform norm.

Note that for each $T \in \mathbf{B}(X)$ the restriction of the adjoint operator $T' \colon S \longrightarrow X'$ is a member of $C(S, X')$. The vector space $C(S, X')$, equipped with the norm

$$\|f\| = \sup_{s \in S} \|f(s)\| ,$$

is a Banach space.

The Banach space $C(S, X')$ played the central role in [**?**, **?**, **?**]. Lomonosov [**?**] based his proof of an interesting extension of Burnside's Theorem on the characterization of the extreme points of the unit ball in the norm dual of $C(S, X')$ using the argument of the celebrated de Branges' proof of the Stone–Weierstrass Theorem [**?**]. Louis de Branges [**?**] performed a deep analysis of the behaviour of these extreme points that yielded a vector generalization of the Weierstrass approximation theorem, similar to the approximation theorem in the previous section. This approach resulted in a characterization of the existence of a nontrivial invariant subspace for the algebra $\mathcal{A}'$ in terms of density of the linear span of the set

$$\{\alpha A' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\} ,$$

in the space of restrictions of the adjoint operators to S, with respect to a topology in $C(S, X')$, introduced by L. de Branges.

Building upon this work, Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw in [**?**], obtained the following characterizations of the existence of a non–trivial invariant subspace for an algebra $\mathcal{A}$ of bounded linear operators acting on a complex Banach space X:

Theorem 1.4.1 (Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw). *There is a non–trivial closed $\mathcal{A}$–invariant subspace of X if and only if there exists an operator $T \in \mathbf{B}(X)$ and a compact operator $K \in \mathbf{B}(X)$ such that $K'T'$ does not belong to the norm closure of the vector subspace of $C(S, X')$ generated by the collection*

$$\{\alpha K'A' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\}.$$

Theorem 1.4.2 (Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw). *There is a non–trivial closed $\mathcal{A}'$–invariant subspace of X' if and only if there exists an operator $T \in \mathbf{B}(X)$ and a compact operator $K \in \mathbf{B}(X)$ such that $T'K'$ does not belong to the norm closure of the vector subspace of $C(S, X')$ generated by the collection*

$$\{\alpha A'K' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\}.$$

We will give a short proof of both theorems as an application of Theorem 1.3.2. Our proof applies to real or complex Banach spaces, where in the case of a real Banach space, $C(S)$ stands for the Banach algebra of all real–valued continuous functions on the set S.

Observe that the Lomonosov spaces $\mathcal{L}(K'\mathcal{A}')$ and $\mathcal{L}(\mathcal{A}'K')$, as defined in the previous section, are subsets of the linear manifolds introduced in Theorems 1.4.1 and 1.4.2.

Definition 1.4.2. The vector x in a Banach space X is *cyclic* for the set of operators $\mathcal{A} \subset \mathbf{B}(X)$ whenever the orbit

$$\mathcal{A}x = \{Ax \mid A \in \mathcal{A}\}$$

is a dense subset of X. If every nonzero vector is cyclic for $\mathcal{A}$, we say that $\mathcal{A}$ acts *transitively* on X. The terms τ*–cyclic* and τ*–transitive* are defined in the same way, by considering the space X equipped with a topology τ, instead of the norm.

The following well known characterizations of the existence of a non–trivial invariant subspace for an algebra $\mathcal{A} \subset \mathbf{B}(X)$ follow immediately from the definition.

Proposition 1.4.3. *Suppose $\mathcal{A} \subset \mathbf{B}(X)$ is a subalgebra of bounded linear operators on X. The following are equivalent:*

1. *$\mathcal{A}$ admits no nontrivial closed invariant subspace.*
2. *$\mathcal{A}$ acts weak–transitively on X.*
3. *$\mathcal{A}$ acts transitively on X.*
4. *$\mathcal{A}'$ admits no nontrivial weak*–closed invariant subspace.*
5. *$\mathcal{A}'$ acts weak*–transitively on X'.*

As in [**?**] we introduce the subspace of completely continuous functions in $C(S, X')$.

Definition 1.4.3. A function $f \in C(S, X')$ is said to be *completely continuous* if it is continuous with respect to the weak* topology on S and the norm topology on X'. The subspace of all completely continuous functions is denoted by $\mathcal{K}(S, X')$.

Note that $K'\colon S \longrightarrow X'$ is completely continuous whenever $K \in \mathbf{B}(X)$ is a compact operator on X (Theorem 6 [**?**, p. 486]).

We are now ready to give a short proof of Theorems 1.4.1 and 1.4.2.

Proof of Theorems 1.4.1 and 1.4.2.

We start with Theorem 1.4.2, which is an almost straightforward consequence of Proposition 1.4.3 and Theorem 1.3.2, applied to the space $\mathcal{K}(S, X')$.

Suppose $\mathcal{A}'$ has a non–trivial closed invariant subspace. Then by Proposition 1.4.3, there exists a pair of nonzero vectors $x', y' \in S$ such that $\|A'x' - y'\| \geq \varepsilon > 0$ for all $A' \in \mathcal{A}'$. Choose any vector $x \in X$ such that $\langle x', x\rangle = 1$, and define the rank–one operators $K = x \otimes x'$ and $T = x \otimes y'$. Clearly $T'K'x' = y'$, and since $T'K'$ cannot be approximated by the operators $A'K'$ at the point x', it follows that $T'K'$ is not in the norm closure of the linear space generated by $\{\alpha A'K' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\}$.

Conversely, suppose $\mathcal{A}'$ admits no non–trivial closed invariant subspaces. Therefore, $\mathcal{A}'$ acts transitively on X', and consequently, every operator $T'K'$ can be approximated by $A'K'$ at each point of S. Furthermore, since K is a compact operator in $\mathbf{B}(X)$, it follows that $T'K' \in \mathcal{K}(S, X')$. Theorem 1.3.2 implies that $T'K'$ is in the norm closure of the Lomonosov space $\mathcal{L}(\mathcal{A}'K')$ and thus completes the proof.

The proof of Theorem 1.4.1 is just slightly more complicated.

Suppose the algebra $\mathcal{A}$ admits a nontrivial closed invariant subspace $\mathcal{M}$. Then $\mathcal{M}^{\perp}$ is an invariant subspace for $\mathcal{A}'$. Fix a nonzero vector $x \in \mathcal{M}$ and a nonzero functional $y' \in M^{\perp}$, and choose a vector $y \in X$ such that $\langle y', y\rangle = 1$ and a functional $x' \in X'$, with $\langle x', x\rangle = 1$. Define the rank–one operators $K = x \otimes y'$ and $T = y \otimes x'$. Then $K'T'y' = y' \neq 0$, while $K'A'y' = 0$ for every $A' \in \mathcal{A}'$. Consequently, the operator

$K'T'$ is not in the norm closure of the linear span of the completely continuous functions $\{\alpha K'A' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\}$.

Conversely, suppose that there exists a compact operator K and an operator T such that $K'T'$ is not in the closure of the linear subspace generated by the completely continuous functions $\{\alpha K'A' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\}$. Theorem 1.3.2 implies that there exists a nonzero vector $x' \in S$ such that the orbit $\mathcal{M} = \{K'A'x' \mid A \in \mathcal{A}\}$ is not a norm–dense manifold in the closure of the subspace $\mathcal{N} = \{K'T'x' \mid T \in \mathbf{B}(X)\}$. By the Hahn–Banach Theorem there exists a functional $y'' \in X''$ such that $\langle y'', K'A'x'\rangle = 0$ for every $A' \in \mathcal{A}'$, and $\langle y'', K'T'x'\rangle = 1$ for some $T \in \mathbf{B}(X)$. Consequently, $K''y'' \neq 0$. Compactness of K implies that $y = K''y'' \in X$, where X is considered naturally embedded in its second dual X'' (Theorem 5.5 [**?**, p. 185] or Theorem 2 [**?**, p. 482]). From $\langle x', Ay\rangle = 0$ for all $A \in \mathcal{A}$, it follows that the algebra $\mathcal{A}$ admits a non–trivial closed invariant subspace. □

It is possible to obtain similar characterizations that do not involve compact operators, by considering some other topology on $C(S, X')$. Theorem 3.1 in [**?**] and Theorem 6 in [**?**] are examples of results in that direction. We conclude this section by giving yet another characterization of transitivity for an algebra $\mathcal{A}$ in terms of the closure of the Lomonosov space $\mathcal{L}(\mathcal{A}')$ with respect to the *uniform* topology τ_{w^*}, induced on $C(S, X')$ by the weak* topology on the dual Banach space X'.

Theorem 1.4.4. *Suppose $\mathcal{A} \subset \mathbf{B}(X)$ is a set of bounded linear operators on X. Then the dual set $\mathcal{A}' = \{A' \mid A \in \mathcal{A}\}$ acts weak*–transitively on S if and only if the τ_{w^*}–closure of the Lomonosov space $\mathcal{L}(\mathcal{A}')$ is equal to the subspace*

$$C_0(S, X') = \{f \in C(S, X') \mid f(0) = 0\}.$$

Proof. The proof is almost identical to those of Theorems 1.4.1 and 1.4.2 except that Theorem 1.3.2 is now applied to the space $C(S, X')$ equipped with the topology τ_{w^*}, instead of $\mathcal{K}(S, X')$ with the norm topology.

If the set $\mathcal{A}'$ does not act weak*–transitively on X' then there exists a nonzero vector $x' \in S$ together with a weak* neighborhood W of y' in S such that $A'x' \notin W$ for all $A' \in \mathcal{A}'$. Choose a vector $x \in X$ such that $\langle x', x\rangle = 1$ and let $T = x \otimes y'$. Then $T'x' = y'$, and since $T' \in C_0(S, X')$ cannot be approximated by the operators in $\mathcal{A}'$ at the point x', it follows that T' is not in the τ_{w^*}–closure of the associated Lomonosov space $\mathcal{L}(\mathcal{A}')$.

Conversely, if the set $\mathcal{A}'$ acts weak*–transitively on S it follows from Theorem 1.3.2 that every function $f \in C_0(S, X')$ can be uniformly approximated by the elements of $\mathcal{L}(\mathcal{A}')$, and thus f is in the τ_{w^*}–closure of the Lomonosov space $\mathcal{L}(\mathcal{A}')$. □

Corollary 1.4.5. *The algebra $\mathcal{A}$ admits no non–trivial closed invariant subspace if and only if the τ_{w^*}–closure of the Lomonosov space $\mathcal{L}(\mathcal{A}')$ is equal to the subspace*

$$C_0(S, X') = \{f \in C(S, X') \mid f(0) = 0\}\,.$$

Proof. By Proposition 1.4.3, the fact that $\mathcal{A}$ admits no non–trivial invariant subspace is equivalent to $\mathcal{A}'$ acting weak*–transitively on S. The result now follows from Theorem 1.4.4. □

Note that the τ_{w^*}–closure of the Lomonosov space $\mathcal{L}(\mathbf{B}(X)')$ is always equal to $C_0(S, X')$. This observation yields a few alternative formulations of Corollary 1.4.5, which are left to the reader.

1.5 On Convex Sets of Compact Quasinilpotent Operators

In this section we combine Lemma 1.3.1 with the Schauder–Tychonoff Fixed Point Theorem, to establish a density result for non–cyclic vectors for the dual of a convex set of compact quasinilpotent operators. We discuss in what sense this result generalizes the celebrated Lomonosov Lemma [**?**], and conclude with a problem of establishing a similar result for the original set, rather than its dual.

Recall that an operator is called *quasinilpotent* if 0 is the only point in its spectrum.

Theorem 1.5.1. *Suppose $\mathcal{A}$ is a convex set of compact quasinilpotent operators acting on a real or complex Banach space X, and let $\mathcal{A}' = \{A' \mid A \in \mathcal{A}\}$ be its dual in $\mathbf{B}(X')$. Then the set of non–cyclic vectors for $\mathcal{A}'$ is dense in X'.*

Proof. Suppose not; then there exists a functional $x_0 \in X'$ and a positive number $r > 0$ such that all vectors in the ball $S = \{x \in X' \mid \|x - x_0\| \leq r\}$ are cyclic for $\mathcal{A}'$. In particular, for every functional $x \in S$ there exists an operator $A' \in \mathcal{A}'$ such that $\|A'x - x_0\| < r$. By Lemma 1.3.1 it follows that there exists a Lomonosov function $\Lambda \in \mathcal{L}(\mathcal{A}')$ such that $\|\Lambda(x) - x_0\| < r$ for all $x \in S$. Consequently, Λ maps S into itself (weak*–continuously).

The Schauder–Tychonoff Fixed Point Theorem [**?**, p. 456] implies that Λ has a fixed point $x_1 = \Lambda(x_1)$ in S. By the definition of the Lomonosov space

$$\Lambda = \sum_{k=1}^{n} \alpha_k A'_k, \quad \text{where } A_k \in \mathcal{A},\ \alpha_k \in C(S, [0,1]) \text{ and } \sum_{k=1}^{n} \alpha_k \equiv 1;\ n < \infty.$$

Therefore $A' = \sum_{k=1}^{n} \alpha_k(x_1) A'_k$ is an operator in the convex set $\mathcal{A}'$. From $\Lambda(x_1) = x_1$, we conclude that $A'x_1 = x_1$. Since $x_1 \neq 0$, it follows that 1 is an eigenvalue for A', contradicting the assumption that A' is a quasinilpotent operator. □

Remark 1.5.1. Note that (unless $\mathcal{A}$ is assumed to be an algebra) it is not enough to require that the operators in $\mathcal{A}'$ have no common invariant subspace, in order to ensure that $\mathcal{A}'$ acts transitively on X'. It is indeed possible to give examples of manifolds of nilpotent operators without a non–trivial closed common invariant subspace. For such examples on finite–dimensional vector spaces see [**?**]. By Theorem 1.5.1 a manifold of such operators cannot act transitively on the underlying space.

Theorem 1.5.1 does not follow from the original work of V.I. Lomonosov [**?**]. On the other hand, Lomonosov's Lemma [**?**] easily follows from Theorem 1.5.1, in the case when the underlying Banach space is reflexive. In that sense Theorem 1.5.1 is a generalization of the Lomonosov Lemma.

This discussion suggests the following question, which we have not been able to resolve:

> *Does there exist a convex set $\mathcal{A}$ of compact quasinilpotent operators acting on a real or complex Banach space X such that the set of non–cyclic vectors for $\mathcal{A}$ is* not *dense in X?*

By Theorem 1.5.1 the underlying Banach space in such an example (if it exists) cannot be reflexive. Furthermore, Lomonosov's Lemma implies that the set $\mathcal{A}$ cannot be of the form $\mathcal{A}K$ or $K\mathcal{A}$, where K is a fixed compact operator. In particular, the set $\mathcal{A}$ in such an example can never be an algebra.

Since, according to Theorems 1.4.1 and 1.4.2, compact operators are closely related to the existence of invariant subspaces for algebras of operators, the answer to the above question might be of some interest to the theory of invariant subspaces.

Chapter 2

An Extension of Burnside's Theorem

In this chapter we combine differentiability of the Hilbert norm with the Schauder–Tychonoff Fixed Point Theorem to show that for every weakly closed subalgebra $\mathcal{A} \neq \mathbf{B}(\mathcal{H})$, acting on a real or complex Hilbert space $\mathcal{H}$, there exist nonzero vectors $f, g \in \mathcal{H}$ such that for every $A \in \mathcal{A}$:

$$|\mathrm{Re}\,\langle Af, g\rangle| \leq \|\mathrm{Re}\,A\|_{\mathrm{ess}}\,\langle f, g\rangle\,.$$

This result generalizes an extension of Burnside's Theorem, recently obtained by V.I. Lomonosov, using rather different techniques. The theory developed in this chapter has an interesting application to the invariant subspace problem for essentially self–adjoint operators which is given in the last chapter.

2.1 Introduction

In the first chapter we defined the *Lomonosov space* and gave a constructive proof of the approximation theorem inspired by the well known result of V.I. Lomonosov [**?**].

This theorem was then applied to obtain a connection between the existence of invariant subspaces for the norm dual of an algebra of bounded operators on a Banach space, and density of the associated Lomonosov space in certain function spaces. These results cover recent characterizations of the invariant subspace problem by Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw in [**?**], who obtained their results using the techniques introduced in [**?**] and further exploited in [**?**].

In this chapter we combine differentiability of the Hilbert norm with a construction of the Lomonosov functions and the Schauder–Tychonoff Fixed Point Theorem to establish a connection between the Lomonosov space and the transitive algebra problem [**?**].

We start by briefly introducing a simplified Hilbert space terminology, that is consistent with the first chapter, where the corresponding terms are defined in more general, Banach space setting.

Definition 2.1.1. Suppose $\mathcal{S}$ is a bounded closed convex subset of a real or complex Hilbert space $\mathcal{H}$, equipped with the relative weak topology. The set of all continuous functions $f\colon \mathcal{S} \longrightarrow \mathcal{H}$, where both spaces are equipped with the weak topology, is denoted by $C(\mathcal{S}, \mathcal{H})$. Similarly, $C(\mathcal{S}, [0,1])$ stands for the set of all weakly–continuous functions $f\colon \mathcal{S} \longrightarrow [0,1]$.

Remark 2.1.1. Recall that a bounded closed convex subset $\mathcal{S}$ in a Hilbert space is weakly compact. Observe also, that a bounded linear operator $A \in \mathbf{B}(\mathcal{H})$ is in $C(\mathcal{S}, \mathcal{H})$. Whenever we say that A is in $C(\mathcal{S}, \mathcal{H})$, we actually refer to the restriction of the operator $A \in \mathbf{B}(\mathcal{H})$ to the subset $\mathcal{S} \subset \mathcal{H}$.

Definition 2.1.2. Let $\mathcal{A}$ be a subset of $C(\mathcal{S},\mathcal{H})$. The convex set $\mathcal{L}(\mathcal{A}) \subset C(\mathcal{S},\mathcal{H})$, defined by

$$\mathcal{L}(\mathcal{A}) = \left\{ \sum_{k=1}^{n} \alpha_k A_k \;\middle|\; A_k \in \mathcal{A},\ \alpha_k \in C(\mathcal{S},[0,1]) \text{ and } \sum_{k=1}^{n} \alpha_k \equiv 1;\ n < \infty \right\}$$

is called the *Lomonosov space* associated with the set $\mathcal{A}$, and a function $\Lambda \in \mathcal{L}(\mathcal{A})$ is called a *Lomonosov function.*

Definition 2.1.3. Let $\mathcal{W}$ be a basic weak neighborhood of a vector $f \in \mathcal{H}$:

$$\mathcal{W} = \{ g \in \mathcal{H} \mid \ |\langle f - g, h_k \rangle| < 1, \quad h_k \in \mathcal{H}, \qquad k = 1, \ldots, n;\ n < \infty \}. \tag{2.1.1}$$

A continuous nonnegative function $\Gamma_{\mathcal{W}} \colon \mathcal{H} \longrightarrow [0,1]$, defined by

$$\Gamma_{\mathcal{W}}(g) = \prod_{k=1}^{n} \max \{0, 1 - |\langle f - g, h_k \rangle| \}, \tag{2.1.2}$$

is called a *continuous indicator function* of $\mathcal{W}$.

Remark 2.1.2. Clearly, $\Gamma_{\mathcal{W}}$ is a nonnegative weakly continuous function and

$$\mathcal{W} = \{ g \in \mathcal{H} \mid \ \Gamma_{\mathcal{W}}(g) > 0 \}.$$

The following proposition and its corollary introduce the idea that will lead to the main result of this chapter.

Proposition 2.1.1. *Let $\mathcal{S}$ be a closed bounded and convex subset of $\mathcal{H}$. Suppose the set $\mathcal{A} \subset C(\mathcal{S}, \mathcal{H})$ satisfies the following property:*

> *For every $s \in \mathcal{S}$ there exists a function $A_s \in \mathcal{A}$ together with a weak neighborhood $\mathcal{W}_s$ of s such that $A_s(\mathcal{W}_s) \subset \mathcal{S}$.*

Then there exists a Lomonosov function $\Lambda \in \mathcal{L}(\mathcal{A})$ that maps the set $\mathcal{S}$ into itself.

Proof. By the hypothesis for every point $s \in \mathcal{S}$ there exists a function A_s together with a basic weak neighborhood $\mathcal{W}_s$ of s such that $A_s(\mathcal{W}_s) \subset \mathcal{S}$. In this way we obtain an open cover of $\mathcal{S}$. Since $\mathcal{S}$ is a weakly compact set there exists a finite subcover $\mathcal{W}_1, \ldots, \mathcal{W}_n$, together with functions $A_1, \ldots, A_n$, with the property that $A_k(\mathcal{W}_k) \subset \mathcal{S}$ for $k = 1, \ldots, n$.

Let $\Gamma_k : \mathcal{S} \longrightarrow [0, 1]$ denote the continuous indicator function of $\mathcal{W}_k$ as defined by (2.1.2). Each point $s \in \mathcal{S}$ lies at least in one neighborhood $\mathcal{W}_k$ ($k = 1, \ldots, n$), therefore the sum $\sum_{j=1}^{n} \Gamma_j(s)$ is strictly positive for all vectors $s \in \mathcal{S}$. Hence, the functions $\alpha_k \colon \mathcal{S} \longrightarrow [0, 1]$, defined by

$$\alpha_k(s) = \frac{\Gamma_k(s)}{\sum_{j=1}^{n} \Gamma_j(s)} \qquad (k = 1, \ldots, n),$$

are well defined and weakly continuous on $\mathcal{S}$. Also, $\sum_{k=1}^{n} \alpha_k(s) = 1$ for every $s \in \mathcal{S}$, and $\alpha_k(s) > 0$ if and only if $s \in \mathcal{W}_k$.

The Lomonosov function $\Lambda \colon \mathcal{S} \longrightarrow \mathcal{S}$, in the Lomonosov space $\mathcal{L}(\mathcal{A})$, associated with the set of functions $\mathcal{A} \subset C(\mathcal{S}, \mathcal{H})$, is defined by

$$\Lambda(s) = \sum_{k=1}^{n} \alpha_k(s) A_k(s).$$

Observe that $\Lambda(s)$ is a convex combination of the elements in $\mathcal{S}$, and consequently, Λ maps the set $\mathcal{S}$ into itself (weak–continuously). □

Corollary 2.1.2. *Suppose $\mathcal{A}$ is a convex subset of $C(\mathcal{S}, \mathcal{H})$ satisfying the condition of Proposition 2.1.1. Then there exists an element $A \in \mathcal{A}$ with a fixed point $s \in \mathcal{S}$.*

Proof. By the Schauder–Tychonoff Fixed Point Theorem the Lomonosov function $\Lambda : \mathcal{S} \longrightarrow \mathcal{S}$, constructed in the proof of Proposition 2.1.1, has a fixed point $s \in \mathcal{S}$. Let $A = \sum_{k=1}^{n} \alpha_k(s) A_k$. Convexity of the set $\mathcal{A}$ implies that $A \in \mathcal{A}$. Furthermore, from $\Lambda(s) = s$ it follows that $A(s) = s$. □

Remark 2.1.3. In our applications we will consider the situations when $\mathcal{S}$ is a closed ball of radius $r \in (0, 1)$ around a fixed unit vector $f_0 \in \mathcal{H}$, and $\mathcal{A}$ is a convex subset of $\mathbf{B}(\mathcal{H})$. If the set $\mathcal{A}$ satisfies the condition of Proposition 2.1.1 then Corollary 2.1.2 implies that the set $\mathcal{A}$ contains an operator A with an eigenvalue 1.

This gives rise to the following two questions:

1. *When does the set $\mathcal{A}$ satisfy the condition of Proposition 2.1.1?*
2. *When is the operator A in Corollary 2.1.2 different from the identity operator?*

Complete continuity of compact operators, restricted to $\mathcal{S}$, yields an affirmative answer to the first question whenever $\mathcal{A}$ is a set of compact operators with the property that for every $s \in \mathcal{S}$ there exists an operator $A_s \in \mathcal{S}$ such that $\|A_s s - f_0\| < r$. Furthermore, if the space $\mathcal{H}$ is assumed to be infinite–dimensional then an affirmative answer to the second question follows from the fact that the identity is not a compact operator. However, compactness of the operators in $\mathcal{A}$ is much too strong an assumption. In the next two sections we develop conditions based on the properties of the essential spectrum and differentiability of the Hilbert norm that will replace the condition of Proposition 2.1.1.

2.2 On the Essential Spectrum

In this section we state some well known properties of the essential spectrum in the form applicable to the situations arising later. We start with a few definitions and introduce notation and terminology that is consistent throughout this chapter.

Definition 2.2.1. Suppose $\mathcal{H}$ is a real or complex Hilbert space. The algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathbf{B}(\mathcal{H})$, while $\mathcal{K}(\mathcal{H})$ stands for the ideal of compact operators in $\mathbf{B}(\mathcal{H})$. The *spectral radius* of the operator $A \in \mathbf{B}(\mathcal{H})$ is denoted by $r(A)$ and its *essential norm*, i.e. the norm of A in the *Calkin algebra* $\mathbf{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, is denoted by $\|A\|_{\text{ess}}$.

Definition 2.2.2. If $\lambda \in \mathbb{C}$ is a complex number then $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ denote its real and imaginary parts respectively, i.e. $\lambda = \operatorname{Re}\lambda + i\operatorname{Im}\lambda$. On the other hand, for a bounded linear operator $A \in \mathbf{B}(\mathcal{H})$, $\operatorname{Re} A$ and $\operatorname{Im} A$ stand for its real and imaginary parts:

$$\operatorname{Re} A = \frac{A + A^*}{2} \quad \text{and} \quad \operatorname{Im} A = \frac{A - A^*}{2},$$

where A^* is the Hilbert adjoint of A in $\mathbf{B}(\mathcal{H})$.

Clearly, for every $A \in \mathbf{B}(\mathcal{H})$ we have $A = \operatorname{Re} A + \operatorname{Im} A$. Furthermore, this decomposition makes sense on a real or complex Hilbert space, and

$$\|\operatorname{Re} A\|_{\text{ess}} \leq \|A\|_{\text{ess}} \leq \|A\|. \tag{2.2.1}$$

Proposition 2.2.1. *Suppose δ and M are positive numbers, and A is a fixed operator in $\mathbf{B}(\mathcal{H})$. Then there exists a weak neighborhood $\mathcal{W}$ of 0 in $\mathcal{H}$ such that every vector $f \in \mathcal{W}$ with $\|f\| \leq M$, satisfies the inequality*

$$|\operatorname{Re}\langle Af, f\rangle| < \|\operatorname{Re} A\|_{\text{ess}} \|f\|^2 + \delta.$$

Proof. From $\operatorname{Re}\langle Af, f\rangle = \langle(\operatorname{Re} A)f, f\rangle$ and $(\operatorname{Re} A) = (\operatorname{Re} A)^*$ it follows that

$$\|\operatorname{Re} A\| = \sup_{f\neq 0} |\operatorname{Re}\langle Af, f\rangle| \, \|f\|^{-2}.$$

By definition of the essential norm, we have

$$\|\operatorname{Re} A\|_{\text{ess}} = \inf_{K\in\mathcal{K}(\mathcal{H})} \|(\operatorname{Re} A) + K\| = \inf_{K\in\mathcal{K}(\mathcal{H})} \|\operatorname{Re}(A+K)\|.$$

Hence, there exists a compact operator K such that

$$\|\operatorname{Re} A\|_{\text{ess}} > \|\operatorname{Re}(A+K)\| - \tfrac{1}{2}\delta M^{-2} \geq |\operatorname{Re}\langle (A+K)f, f\rangle| \, \|f\|^{-2} - \tfrac{1}{2}\delta M^{-2}.$$

The proposition now follows by the mixed (weak–to–norm) continuity of compact operators on bounded sets. □

The following proposition plays an important role in the subsequent sections.

Proposition 2.2.2. *Suppose $\mathcal{H}$ is a real or complex Hilbert space, and $\lambda \in \mathbb{C}$ is a point in the spectrum of the operator $A \in \mathbf{B}(\mathcal{H})$, such that*

$$|\operatorname{Re}\lambda| > \|\operatorname{Re} A\|_{\text{ess}}. \tag{2.2.2}$$

Then the norm closure of the algebra generated by A contains a nonzero finite–rank operator.

Proof. We may assume that the Hilbert space $\mathcal{H}$ is complex, as long as we can construct a finite–rank operator in the closure of the *real* algebra generated by A.

Clearly, (2.2.2) implies that λ is not in the essential spectrum of A. From the well known properties of the essential spectrum (for example, Theorem 6.8 and Proposition 6.9 in [**?**, p. 366]), it follows that every point in the spectrum of the operator A, satisfying the condition (2.2.2) is an isolated eigenvalue of A, and the corresponding Riesz projection has finite rank.

After first replacing the operator A by $-A$ in the case when $\operatorname{Re}\lambda < 0$, and then substituting the translation $A - \max\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}$ for A, we may assume that $\max_{\lambda\in\sigma(A)} \operatorname{Re}\lambda = 0$. The condition (2.2.2) implies that

$$\sigma_0(A) = \{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda = 0\},$$

is a nonempty finite set, consisting of eigenvalues of A with finite multiplicity. By the Riesz Decomposition Theorem [?, p. 31], the space $\mathcal{H}$ can be decomposed as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\dim(\mathcal{H}_1) < \infty$, and the operator A is similar to the operator $A_1 \oplus A_2$, corresponding to this decomposition. Furthermore, the spectrum of A_1 is $\sigma_0(A)$, and the spectrum of A_2 lies in the open left complex half–plane. Therefore $r(e^{tA_2}) < 1$ for $t > 0$, while $r(e^{tA_1}) = 1$ for any real argument t. By Rota's Theorem [?, p. 136], the operator e^{A_2} is similar to a strict contraction. Consequently,

$$\lim_{n\longrightarrow\infty} \frac{e^{nA_2}}{\|e^{nA_1}\|} = 0.$$

On the other hand, finite–dimensionality of $\mathcal{H}_1$ implies that the sequence

$$T_n = \frac{e^{nA_1}}{\|e^{nA_1}\|}, \qquad n = 0, 1, \ldots,$$

has a subsequence converging in norm to a nonzero finite–rank operator. □

Remark 2.2.1. Recall that the exponential function e^A admits the power series:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Hence, the finite–rank operator constructed in the proof of Proposition 2.2.2 is indeed contained in the norm closure of the *real* algebra, generated by A.

2.3 Preliminary Geometric Results

This section contains preliminary results that are needed in the constructive proof of the main theorem, given in the next section. The results presented here are mostly easy observations and the proofs are somehow tedious and straightforward calculations, involving the standard "ε, δ" arguments.

Throughout this section we make the following conventions:

$\mathcal{H}$ is a real or complex Hilbert space. Fix a unit vector $f_0 \in \mathcal{H}$ and choose a positive number $r \in (0, 1)$. The set $\mathcal{S}$ is defined as follows:

$$\mathcal{S} = \{f \in \mathcal{H} \mid \ \|f_0 - f\| \leq r\}.$$

Lemma 2.3.1. *Let $\mathcal{W}$ be a subset of $\mathcal{S}$ and let A be a bounded linear operator on $\mathcal{H}$. Suppose that*

$$\operatorname{Re}\langle Af, f_0 - f\rangle \geq \delta > 0, \quad \text{for all } f \in \mathcal{W}.$$

Then there exists a positive number $\mu > 0$, such that for any $\varepsilon \in (0, \mu)$:

$$\|f_0 - (I + \varepsilon A)f\| < \|f_0 - f\|, \quad \text{for all } f \in \mathcal{W}.$$

Proof. Note that $f \in \mathcal{S}$ implies $\|f\| \leq 1 + r < 2$. Therefore for all $f \in \mathcal{S}$:

$$\|Af\| \leq \|A\| \, \|f\| \leq (1 + r) \|A\| < 2 \|A\|.$$

Set $\mu = \frac{\delta}{2\|A\|^2}$. For any $\varepsilon \in (0, \mu)$ and $f \in \mathcal{W}$ we have:

$$\begin{aligned}
\|f_0 - (I + \varepsilon A)f\|^2 &= \|f_0 - f - \varepsilon Af\|^2 \\
&= \|f_0 - f\|^2 - 2\varepsilon \operatorname{Re}\langle Af, f_0 - f\rangle + \varepsilon^2 \|Af\|^2 \\
&< \|f_0 - f\|^2 - 2\varepsilon\delta + \varepsilon^2 4 \|A\|^2 \\
&= \|f_0 - f\|^2 - 2\varepsilon(\delta - \varepsilon 2 \|A\|^2) < \|f_0 - f\|^2.
\end{aligned}$$

Hence μ is the required positive number. $\square$

Remark 2.3.1. Let $\psi'(0)$ denote the derivative of the function

$$\psi(t) = \|(I + tA)f - f_0\|^2 ,$$

with respect to t, at the point $t = 0$. A straightforward calculation yields

$$\psi'(0) = -2\,\mathrm{Re}\,\langle Af, f_0 - f\rangle .$$

Therefore the statement of Lemma 2.3.1 corresponds to the well known fact that a real function with (strictly) negative derivative is (strictly) decreasing. Note, however, that positivity of $\mathrm{Re}\,\langle Af, f_0 - f\rangle$ does not imply that the mapping $\Psi(f) = (I + \varepsilon A)f$ is a contraction, as a function from $\mathcal{W}$ to $\mathcal{S}$.

Lemma 2.3.1 gives a numerical criterion for the subset $\mathcal{W} \subset \mathcal{S}$ to be mapped into $\mathcal{S}$, namely positivity of the function $\Phi(f) = \mathrm{Re}\,\langle Af, f_0 - f\rangle$ on $\mathcal{W}$. Since $\Phi(f_0) = 0$, this criterion cannot be employed at the point f_0. However, the problem of constructing a function $\Lambda\colon \mathcal{S} \longrightarrow \mathcal{S}$ can be easily reduced to the subset of $\mathcal{S}$ not containing the point f_0. A simple observation in $\mathbb{R}^2$ suggests the following definition.

Definition 2.3.1. For a fixed ball $\mathcal{S} = \{f \in \mathcal{H} \mid \ \|f_0 - f\| \le r\}$ around the unit vector $f_0 \in \mathcal{H}$, the *polar hyperplane* $\mathcal{P}_{\mathcal{S}}$, of the origin with respect to $\mathcal{S}$, is defined by the following set:

$$\mathcal{P}_{\mathcal{S}} = \left\{f \in \mathcal{H} \mid \ \langle f, f_0\rangle = 1 - r^2\right\} .$$

Remark 2.3.2. Every vector f in $\mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$ can be decomposed as $f = (1 - r^2)f_0 + g$, where $g \perp f_0$ and $\|g\|^2 \le r^2(1 - r^2)$. In particular, the boundary of $\mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$:

$$\mathcal{P}_{\mathcal{S}} \cap \partial\mathcal{S} = \left\{(1 - r^2)f_0 + g \;\middle|\; g \perp f_0 \quad \text{and} \quad \|g\|^2 = r^2(1 - r^2)\right\} ,$$

contains exactly the points where the tangents from the origin to the ball $\mathcal{S}$ intersect the set $\mathcal{S}$. Recall that in $\mathbb{R}^2$ such a line is called a *polar*, and our definition is just a straightforward generalization of this geometric term to the higher dimensional Hilbert spaces.

The following lemma will reduce the problem of constructing a Lomonosov function $\Lambda\colon \mathcal{S} \longrightarrow \mathcal{S}$ to the polar hyperplane.

Lemma 2.3.2. *The function* $\Lambda_0\colon \mathcal{S} \longrightarrow \mathcal{S}$*, defined by*

$$\Lambda_0(f) = \frac{1}{r^2 + \langle f, f_0\rangle} f,$$

maps the set $\mathcal{S} = \{f \in \mathcal{H} \mid \; \|f - f_0\| \leq r\}$ *weak–continuously into itself. Furthermore, the set of all fixed points for* Λ_0 *is equal to* $\mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$*.*

Proof. Since $\operatorname{Re}\langle f, f_0\rangle > 0$ for $f \in \mathcal{S}$, it follows that Λ_0 is well defined and weakly continuous on $\mathcal{S}$. Clearly, $f \in \mathcal{S}$ is a fixed point for Λ_0 if and only if $r^2 + \langle f, f_0\rangle = 1$. By the definition of the polar hyperplane, that is equivalent to $f \in \mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$.

We have to prove that $\|\Lambda_0(f) - f_0\| \leq r$ for all $f \in \mathcal{S}$.

Every vector $f \in \mathcal{S}$ can be decomposed as $f = \langle f, f_0\rangle f_0 + g$, where $g \perp f_0$ and $\|g\|^2 \leq r^2 - |1 - \langle f, f_0\rangle|^2$.

A straightforward calculation, using this decomposition, yields:

$$\begin{aligned}\|\Lambda_0(f) - f_0\|^2 &= \left\| \frac{1}{r^2 + \langle f, f_0\rangle} f - f_0 \right\|^2 = \frac{1}{|r^2 + \langle f, f_0\rangle|^2}\left(r^4 + \|g\|^2\right) \\ &\leq \frac{1}{|r^2 + \langle f, f_0\rangle|^2}\left(r^4 + r^2 - |1 - \langle f, f_0\rangle|^2\right).\end{aligned}$$

The conclusion follows if we can establish the following inequality:

$$r^4 + r^2 - |1 - \langle f, f_0\rangle|^2 \leq r^2 \left|r^2 + \langle f, f_0\rangle\right|^2.$$

Setting $\langle f, f_0\rangle = x + iy$, this can be translated to

$$r^4 + r^2 - (1 - x)^2 - y^2 \leq r^2(r^2 + x)^2 + r^2 y^2,$$

or equivalently,

$$\begin{aligned}(1+r^2)y^2 &\geq r^4 + r^2 - (1-x)^2 - r^2(r^2+x)^2\\ &= (r^4 + r^2 - 1 - r^6) + 2(1-r^4)x - (1+r^2)x^2\\ &= -(1+r^2)\big((1-r^2)-x\big)^2.\end{aligned}$$

The last inequality is obviously always satisfied, with the strict inequality holding everywhere, except in the polar hyperplane $\mathcal{P}_{\mathcal{S}}$. □

Definition 2.3.2. For every operator A in $\mathbf{B}(\mathcal{H})$ define a real function $\Delta_A \colon \mathcal{S} \longrightarrow \mathbb{R}$ as follows:

$$\Delta_A(f) = \frac{1}{r^2(1-r^2)} \operatorname{Re} \langle Af, f_0 - f\rangle .$$

Remark 2.3.3. Note that Δ_A is a "normalization" of the function $\operatorname{Re}\langle Af, f_0 - f\rangle$ in Lemma 2.3.1.

From the definition of the set $\mathcal{P}_{\mathcal{S}}$ it follows that every vector f in $\mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$ can be decomposed as $f = (1-r^2)f_0 + r\sqrt{1-r^2}g$, where $g \perp f_0$ and $\|g\| \leq 1$. Consequently,

$$\begin{aligned}\Delta_A(f) &= \operatorname{Re}\left\langle A\left(f_0 + \tfrac{r}{\sqrt{1-r^2}}g\right), f_0 - \tfrac{\sqrt{1-r^2}}{r}g\right\rangle\\ &= \operatorname{Re}\langle Af_0, f_0\rangle - \operatorname{Re}\langle Ag, g\rangle - \operatorname{Re}\left\langle \left(\tfrac{\sqrt{1-r^2}}{r}A - \tfrac{r}{\sqrt{1-r^2}}A^*\right) f_0, g\right\rangle .\end{aligned} \tag{2.3.1}$$

In particular, for the identity operator I on $\mathcal{H}$, we have $\Delta_I(f) = 1 - \|g\|^2$. Therefore, $\Delta_I \geq 0$ on $\mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$, with the equality $\Delta_I(f) = 0$ holding if an only if $f \in \mathcal{P}_{\mathcal{S}} \cap \partial\mathcal{S}$.

Observe that the function $\Delta_A \colon \mathcal{S} \longrightarrow \mathbb{R}$ is norm continuous, but it is in general not weakly continuous, due to the presence of the quadratic form $\operatorname{Re}\langle Ag, g\rangle = \langle(\operatorname{Re} A)g, g\rangle$ in (2.3.1).

The next lemma imposes an additional condition on the operator A that guarantees the existence of a weak neighborhood of f in $\mathcal{S}$ on which Δ_A is positive.

Lemma 2.3.3. *Suppose f is a vector in the polar hyperplane $\mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$, satisfying the following* strict *inequality for some $A \in \mathbf{B}(\mathcal{H})$:*

$$\Delta_A(f) > \|\mathrm{Re}\, A\|_{\mathrm{ess}}\, \Delta_I(f).$$

Then there exists a positive number $\delta > 0$, together with a weak neighborhood $\mathcal{W}$ of f, such that for every $h \in \mathcal{W} \cap \mathcal{S}$:

$$\Delta_A(h) > \|\mathrm{Re}\, A\|_{\mathrm{ess}}\, |\Delta_I(h)| + \delta.$$

Proof. By the hypothesis, there exists a positive number $\delta > 0$ such that:

$$\Delta_A(f) > \|\mathrm{Re}\, A\|_{\mathrm{ess}}\, \Delta_I(f) + 5\delta. \tag{2.3.2}$$

For any positive number $0 < \varepsilon < r^2$, define a weak neighborhood $\mathcal{W}_\varepsilon$ of $\mathcal{P}_{\mathcal{S}}$:

$$\mathcal{W}_\varepsilon = \left\{ h \in \mathcal{H} \mid \ \left| 1 - r^2 - \langle h, f_0 \rangle \right| < \varepsilon \right\}.$$

Every vector $h \in \mathcal{W}_\varepsilon \cap \mathcal{S}$ can be decomposed as $h = \langle h, f_0 \rangle f_0 + g$, where $g \perp f_0$ and

$$\|g\|^2 \leq r^2 - |1 - \langle h, f_0 \rangle|^2 < r^2 - (r^2 - \varepsilon)^2.$$

Estimating roughly, we conclude:

$$\begin{aligned} \Delta_I(h) &= \frac{\mathrm{Re}\left(\langle h, f_0 \rangle (1 - \overline{\langle h, f_0 \rangle}) \right) - \|g\|^2}{r^2(1 - r^2)} \\ &> \frac{(1 - r^2 - \varepsilon)(r^2 - \varepsilon) - r^2 + (r^2 - \varepsilon)^2}{r^2(1 - r^2)} > -\frac{3\varepsilon}{r^2(1 - r^2)}. \end{aligned}$$

Therefore, a weak neighborhood $\mathcal{W}_\varepsilon$ of $\mathcal{P}_{\mathcal{S}}$, such that $\|\mathrm{Re}\, A\|_{\mathrm{ess}}\, \Delta_I(h) > -\delta$, for every vector $h \in \mathcal{W}_\varepsilon \cap \mathcal{S}$, can be obtained by setting

$$\varepsilon = \frac{\delta r^2 (1 - r^2)}{1 + 3\, \|\mathrm{Re}\, A\|_{\mathrm{ess}}}.$$

A straightforward calculation yields:

$$\Delta_A(f+g) = \Delta_A(f) + \tfrac{1}{r^2(1-r^2)}\left(\operatorname{Re}\langle Ag, f_0 - f\rangle - \operatorname{Re}\langle Af, g\rangle - \operatorname{Re}\langle Ag, g\rangle\right). \quad (2.3.3)$$

Proposition 2.2.1 implies the existence of a weak neighborhood $\mathcal{W}_1$ of 0, such that for every vector $g \in \mathcal{W}_1$, with $\|g\| \le 2$:

$$\operatorname{Re}\langle Ag, g\rangle < \|\operatorname{Re} A\|_{\mathrm{ess}} \|g\|^2 + r^2(1-r^2)\delta. \quad (2.3.4)$$

Clearly, by the weak–continuity of both sides of the inequality, there exists a weak neighborhood $\mathcal{W}_2$ of 0, such that for $g \in \mathcal{W}_2$:

$$\begin{aligned} &\operatorname{Re}\langle Ag, f_0 - f\rangle - \operatorname{Re}\langle Af, g\rangle > \\ &\qquad \|\operatorname{Re} A\|_{\mathrm{ess}} \big(\operatorname{Re}\langle g, f_0 - f\rangle - \operatorname{Re}\langle f, g\rangle\big) - r^2(1-r^2)\delta. \end{aligned} \quad (2.3.5)$$

Let $\mathcal{W} = \mathcal{W}_\varepsilon \cap \big(f + \mathcal{W}_1 \cap \mathcal{W}_2\big)$ be a weak neighborhood of f. Every vector h in $\mathcal{W} \cap \mathcal{S}$ can be written as $h = f + g$, where $g \in \mathcal{W}_1 \cap \mathcal{W}_2$ and $\|g\| < 2$. Putting the inequalities (2.3.2−2.3.5) together, and using $\|\operatorname{Re} A\|_{\mathrm{ess}} \Delta_I(h) > -\delta$, implies:

$$\begin{aligned} \Delta_A(h) &= \Delta_A(f+g) \\ &= \Delta_A(f) + \tfrac{1}{r^2(1-r^2)}\left(\operatorname{Re}\langle Ag, f_0 - f\rangle - \operatorname{Re}\langle Af, g\rangle - \operatorname{Re}\langle Ag, g\rangle\right) \\ &> \|\operatorname{Re} A\|_{\mathrm{ess}} \big(\Delta_I(f) + \tfrac{1}{r^2(1-r^2)}(\operatorname{Re}\langle g, f_0 - f\rangle - \operatorname{Re}\langle f, g\rangle - \operatorname{Re}\langle g, g\rangle)\big) + 3\delta \\ &= \|\operatorname{Re} A\|_{\mathrm{ess}} \Delta_I(f+g) + 3\delta \\ &= \|\operatorname{Re} A\|_{\mathrm{ess}} \Delta_I(h) + 3\delta \\ &> \|\operatorname{Re} A\|_{\mathrm{ess}} |\Delta_I(h)| + \delta. \end{aligned}$$

Consequently, $\mathcal{W}$ is a weak neighborhood of f, with the required property. □

2.4 The Main Result

We are now ready to give the main result of this chapter, which is quite technical, but applicable to several situations discussed later.

Proposition 2.4.1. *Let $\mathcal{A} \subset \mathbf{B}(\mathcal{H})$ be a convex subset of bounded linear operators acting on a real or complex Hilbert space $\mathcal{H}$. Fix a unit vector $f_0 \in \mathcal{H}$ and choose a positive number $r \in (0,1)$. Suppose that for every vector $g \perp f_0$ and $\|g\| \le 1$, there exists an operator $A \in \mathcal{A}$, satisfying the following strict inequality:*

$$\operatorname{Re}\left\langle A\left(f_0 + \tfrac{r}{\sqrt{1-r^2}}g\right), f_0 - \tfrac{\sqrt{1-r^2}}{r}g\right\rangle > \|\operatorname{Re} A\|_{\mathrm{ess}}\,(1-\|g\|^2). \tag{2.4.1}$$

Then $\mathcal{A}$ contains an operator A_0, with an eigenvector in the set

$$\mathcal{S} = \{f \in \mathcal{H} \mid \ \|f_0 - f\| \le r\},$$

and the corresponding eigenvalue λ satisfies the condition: $|\operatorname{Re}\lambda| > \|\operatorname{Re} A_0\|_{\mathrm{ess}}$.

Proof. Introducing the polar hyperplane $\mathcal{P}_{\mathcal{S}}$ as before, observe that by (2.3.1) the condition (2.4.1) implies that every vector in $\mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$ satisfies the hypothesis of Lemma 2.3.3 for some operator $A \in \mathcal{A}$. Consequently, for every vector f in $\mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$ there exists an operator $A \in \mathcal{A}$, together with a (basic) weak neighborhood $\mathcal{W}$ of f, and a positive number δ, such that for every $h \in \mathcal{W} \cap \mathcal{S}$:

$$\Delta_A(h) > \|\operatorname{Re} A\|_{\mathrm{ess}}\,|\Delta_I(h)| + \delta.$$

By Lemma 2.3.1 there exists a positive number μ such that the operator $I + \varepsilon A$ maps the set $\mathcal{W} \cap \mathcal{S}$ into $\mathcal{S}$ whenever $\varepsilon \in (0, \mu)$.

In this way we obtain a weakly open cover of $\mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$ with basic neighborhoods. By the weak–compactness of the set $\mathcal{P}_{\mathcal{S}} \cap \mathcal{S}$, there exists a finite subcover $\mathcal{W}_1, \dots, \mathcal{W}_n$, together with the operators A_k in $\mathcal{A}$, and positive numbers $\mu_k > 0$, such that for $\varepsilon \in (0, \mu_k)$ the operator $I + \varepsilon A_k$ maps the set $\mathcal{W}_k \cap \mathcal{S}$ into $\mathcal{S}$, and

$$\Delta_{A_k}(h) > \|\operatorname{Re} A_k\|_{\text{ess}} |\Delta_I(h)|, \tag{2.4.2}$$

for every $h \in \mathcal{W}_k \cap \mathcal{S}$.

Define the weakly open set $\mathcal{W}_0 = \{f \in \mathcal{H} \mid \ |\langle f, f_0 \rangle - (1 - r^2)| > 0\}$. Associated with the set $\mathcal{W}_0$ is its continuous indicator function $\Gamma_0 \colon \mathcal{H} \longrightarrow [0, \infty)$:

$$\Gamma_0(f) = \left| \langle f - f_0 \rangle - (1 - r^2) \right|,$$

and the function $\Lambda_0 \colon \mathcal{S} \longrightarrow \mathcal{S}$, defined in Lemma 2.3.2: $\Lambda_0(f) = \big(r^2 + \langle f, f_0 \rangle\big)^{-1} f$.

Fix a positive number $\varepsilon \in (0, \min\{\mu_1, \dots, \mu_n\})$, and recall that every basic weak neighborhood $\mathcal{W}_k$ admits a continuous indicator function $\Gamma_k \colon \mathcal{S} \longrightarrow [0, 1]$, defined by (2.1.2). Each point $f \in \mathcal{S}$ lies at least in one neighborhood $\mathcal{W}_k$ ($k = 0, \dots, n$), therefore the sum $\sum_{j=0}^{n} \Gamma_j(f)$ is strictly positive for all vectors $f \in \mathcal{S}$. Hence, the functions $\alpha_k \colon \mathcal{S} \longrightarrow [0, 1]$,

$$\alpha_k(f) = \frac{\Gamma_k(f)}{\sum_{j=0}^{n} \Gamma_j(f)} \qquad (k = 0, \dots, n),$$

are well defined and weakly continuous on $\mathcal{S}$. Also, $\sum_{k=0}^{n} \alpha_k(f) = 1$ for every $f \in \mathcal{S}$, and $\alpha_k(f) > 0$ if and only if $f \in \mathcal{W}_k$.

The Lomonosov function $\Lambda \colon \mathcal{S} \longrightarrow \mathcal{S}$, in the Lomonosov space $\mathcal{L}(\mathcal{A} \cup \Lambda_0)$, associated with the set of functions $\mathcal{A} \cup \Lambda_0 \subset C(\mathcal{S}, \mathcal{H})$, is defined by

$$\Lambda(f) = \frac{\alpha_0(f)}{r^2 + \langle f, f_0 \rangle} f + \sum_{k=1}^{n} \alpha_k(f)(I + \varepsilon A_k) f.$$

Observe, that $\Lambda(f)$ is a convex combination of the elements in $\mathcal{S}$, and consequently, Λ maps the set $\mathcal{S}$ into itself (weak–continuously).

The Schauder–Tychonoff Fixed Point Theorem implies that the Lomonosov function $\Lambda\colon \mathcal{S} \longrightarrow \mathcal{S}$ has a fixed point $f_1 \in \mathcal{S}$. From $\Lambda(f_1) = f_1$, we conclude:

$$\begin{aligned}\varepsilon\big(\sum_{k=1}^{n} \alpha_k(f_1)A_k\big) f_1 &= \big(1 - \frac{\alpha_0(f_1)}{r^2 + \langle f_1, f_0\rangle} - \sum_{k=1}^{n} \alpha_k(f_1)\big) f_1 \\ &= \alpha_0(f_1)\big(1 - \frac{1}{r^2 + \langle f_1, f_0\rangle}\big) f_1.\end{aligned}$$

Outside the set $\mathcal{W}_1 \cup \ldots \cup \mathcal{W}_n$ the function Λ equals Λ_0 and has no fixed points. Consequently, $f_1 \in \mathcal{W}_k$ for at least one index $k \in \{1, \ldots, n\}$, and $\sum_{j=1}^{n} \alpha_j(f_1) > 0$. Set

$$\beta_k = \frac{\alpha_k(f_1)}{\sum_{j=1}^{n} \alpha_j(f_1)} = \frac{\alpha_k(f_1)}{1 - \alpha_0(f_1)}, \qquad (k = 1, \ldots, n).$$

Then $A_0 = \sum_{k=1}^{n} \beta_k A_k$ is an operator in the convex set $\mathcal{A}$. Clearly, $f_1 \in \mathcal{S}$ is an eigenvector for A_0, corresponding to the eigenvalue λ:

$$\lambda = \frac{\alpha_0(f_1)}{\varepsilon\big(1 - \alpha_0(f_1)\big)}\big(1 - \frac{1}{r^2 + \langle f_1, f_0\rangle}\big). \tag{2.4.3}$$

Recall that by (2.4.2) the strict inequality $\Delta_{A_k}(f_1) > \|\mathrm{Re}\, A_k\|_{\mathrm{ess}} |\Delta_I(f_1)|$ is satisfied whenever $\alpha_k(f_1) > 0$ (or equivalently $\beta_k > 0$). Therefore, nonnegativity of the coefficients β_k and subadditivity of the essential norm, imply

$$\Delta_{A_0}(f_1) = \sum_{k=1}^{n} \beta_k \Delta_{A_k}(f_1) > \sum_{k=1}^{n} \beta_k \|\mathrm{Re}\, A_k\|_{\mathrm{ess}} |\Delta_I(f_1)| \geq \|\mathrm{Re}\, A_0\|_{\mathrm{ess}} |\Delta_I(f_1)|.$$

By (2.4.3) the sign of $\mathrm{Im}\,\lambda$ is the same as the sign of $\mathrm{Im}\,\langle f_1, f_0\rangle = \mathrm{Im}\,\langle f_1, f_0 - f_1\rangle$. Hence, from $A_0 f_1 = \lambda f_1$ and $\Delta_{A_0}(f_1) > \|\mathrm{Re}\, A_0\|_{\mathrm{ess}} |\Delta_I(f_1)|$, we conclude:

$$\begin{aligned}|\mathrm{Re}\,\lambda|\, |\Delta_I(f_1)| &\geq (\mathrm{Re}\,\lambda)\Delta_I(f_1) = \tfrac{1}{r^2(1-r^2)} \mathrm{Re}\,\lambda\, \mathrm{Re}\,\langle f_1, f_0 - f_1\rangle \\ &\geq \tfrac{1}{r^2(1-r^2)}\big(\mathrm{Re}\,\lambda\,\mathrm{Re}\,\langle f_1, f_0 - f_1\rangle - \mathrm{Im}\,\lambda\,\mathrm{Im}\,\langle f_1, f_0 - f_1\rangle\big) \\ &= \tfrac{1}{r^2(1-r^2)} \mathrm{Re}\,\langle \lambda f_1, f_0 - f_1\rangle = \Delta_{A_0}(f_1) > \|\mathrm{Re}\, A_0\|_{\mathrm{ess}} |\Delta_I(f_1)|.\end{aligned}$$

The strict inequality implies that $\Delta_I(f_1) \neq 0$, and consequently λ satisfies the required condition: $|\mathrm{Re}\,\lambda| > \|\mathrm{Re}\, A_0\|_{\mathrm{ess}}$. □

2.5 Burnside's Theorem Revisited

V.I. Lomonosov [?] established the following extension of Burnside's Theorem to infinite–dimensional Banach spaces:

Theorem 2.5.1 (V.I. Lomonosov, 1991). *Suppose X is a complex Banach space and let $\mathcal{A}$ be a weakly closed proper subalgebra of $\mathbf{B}(X)$, $\mathcal{A} \neq \mathbf{B}(X)$. Then there exists $x \in X''$ and $y \in X'$, $x \neq 0$ and $y \neq 0$, such that for every $A \in \mathcal{A}$*

$$|\langle x, A'y \rangle| \leq \|A\|_{\text{ess}} . \tag{2.5.1}$$

The techniques introduced in the proof of this theorem, based on the argument of the celebrated de Branges' proof of the Stone–Weierstrass Theorem [?], received further attention in [?, ?]. Although in the Hilbert space case Theorem 2.5.1 is equivalent to another theorem, also given in [?], we take a different point of view and employ Proposition 2.4.1 to obtain a stronger extension of Burnside's Theorem to infinite–dimensional Hilbert spaces.

The condition (2.5.1) is equivalent to the existence of *unit* elements $x \in X''$ and $y \in X'$, and a nonnegative constant C (depending on $\mathcal{A}$), such that

$$|\langle x, A'y \rangle| \leq C \, \|A\|_{\text{ess}} , \qquad \text{for all } A \in \mathcal{A}. \tag{2.5.2}$$

In general, the constant C depends on the space X, and the algebra $\mathcal{A}$. It is not clear that on every Banach space there exists an upper bound for C, satisfying the condition (2.5.2), with respect to all proper weakly closed subalgebras of $\mathbf{B}(X)$. An example of such a space would certainly be of some interest. On the other hand, an affirmative answer to the Transitive Algebra Problem [?] is equivalent to $C = 0$.

At the moment we can provide no results concerning the estimates for the constant C in any infinite–dimensional Banach space, other than a Hilbert space. The next theorem implies that on a complex Hilbert space the constant C is at most one.

Theorem 2.5.2. *Suppose $\mathcal{H}$ is a complex Hilbert space and let $\mathcal{A}$ be a weakly closed subalgebra of $\mathbf{B}(\mathcal{H})$, $\mathcal{A} \neq \mathbf{B}(\mathcal{H})$. Then there exist nonzero vectors $f, h \in \mathcal{H}$, such that for all $A \in \mathcal{A}$:*

$$|\mathrm{Re}\,\langle Af, h\rangle| \leq \|\mathrm{Re}\,A\|_{\mathrm{ess}} \langle f, h\rangle\,. \tag{2.5.3}$$

Proof. Suppose not; then the hypothesis of Proposition 2.4.1 is satisfied for every unit vector f_0 and any positive number $r \in (0, 1)$. Consequently, the algebra $\mathcal{A}$ contains an operator A_0 with an eigenvalue λ, satisfying the condition:

$$|\mathrm{Re}\,\lambda| > \|\mathrm{Re}\,A\|_{\mathrm{ess}}\,.$$

Proposition 2.2.2 implies that the algebra $\mathcal{A}$ contains a nonzero finite–rank operator. Therefore [**?**, Theorem 8.2], the (transitive) algebra $\mathcal{A}$ is weakly dense in $\mathbf{B}(\mathcal{H})$, contradicting the assumption $\mathcal{A} \neq \mathbf{B}(\mathcal{H})$. □

Remark 2.5.1. Note that (after arbitrary choosing the unit vector f_0 and then letting $r \longrightarrow 0$) the argument in the proof of Theorem 2.5.2 shows that the set of all vectors $f \in \mathcal{H}$ for which there exists a nonzero vector $g \in \mathcal{H}$ satisfying the condition (2.5.3) is *dense* in $\mathcal{H}$.

Corollary 2.5.3. *Suppose $\mathcal{H}$ is a complex Hilbert space and let $\mathcal{A}$ be a weakly closed subalgebra of $\mathbf{B}(\mathcal{H})$, $\mathcal{A} \neq \mathbf{B}(\mathcal{H})$. Then there exist* unit *vectors $f, h \in \mathcal{H}$, such that for all $A \in \mathcal{A}$:*

$$|\langle Af, h\rangle| \leq \|A\|_{\mathrm{ess}}\,. \tag{2.5.4}$$

Proof. By Theorem 2.5.2 there exist unit vectors $f, h \in \mathcal{H}$, such that for every $A \in \mathcal{A}$

$$|\mathrm{Re}\,\langle Af, h\rangle| \leq \|\mathrm{Re}\,A\|_{\mathrm{ess}}\,.$$

Set

$$\xi = \begin{cases} 1 & \text{if } \langle Af, h\rangle = 0, \\ \frac{\langle Af,h\rangle}{|\langle Af,h\rangle|} & \text{otherwise.} \end{cases}$$

Then

$$|\langle Af, h\rangle| = |\mathrm{Re}\,\langle \xi Af, h\rangle| \leq \|\mathrm{Re}(\xi A)\|_{\mathrm{ess}} \leq \|\xi A\|_{\mathrm{ess}} = \|A\|_{\mathrm{ess}}\,,$$

and consequently, the condition (2.5.4) is weaker than (2.5.3). □

The following definition yields an alternative formulation of the extended Burnside's Theorem.

Definition 2.5.1. A vector $f \in \mathcal{H}$ is called *essentially cyclic* for an algebra $\mathcal{A} \subset \mathbf{B}(\mathcal{H})$, if for every nonzero vector $h \in \mathcal{H}$ there exists an operator $A \in \mathcal{A}$ such that

$$\mathrm{Re}\,\langle Af, h\rangle > \|\mathrm{Re}\,A\|_{\mathrm{ess}}\,\|f\|\,\|h\|\,.$$

We say that a subalgebra $\mathcal{A}$ of $\mathbf{B}(\mathcal{H})$ is *essentially transitive* if every nonzero vector is essentially cyclic for $\mathcal{A}$.

Remark 2.5.2. Note that our definition of essentially transitive algebras does not coincide with the definition in [**?**]. In view of the discussion preceding Theorem 2.5.2, we required that C is at most one in the definition of essential transitivity, while the definition in [**?**] assumes no upper bound on C.

According to Definition 2.5.1, every essentially cyclic vector $f \in \mathcal{H}$ is also cyclic for $\mathcal{A}$, i.e. the orbit $\{Af \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}$. Consequently, an essentially transitive algebra is also transitive, as defined in [**?**].

Theorem 2.5.2 can be restated as the following solution of the "Essentially Transitive Algebra Problem".

Theorem 2.5.4. *An essentially transitive algebra of operators acting on a complex Hilbert space $\mathcal{H}$, is weakly dense in $\mathbf{B}(\mathcal{H})$.*

Remark 2.5.3. The reader may have noticed that, by Propositions 2.4.1 and 2.2.2, an essentially transitive algebra $\mathcal{A}$, acting on a *real* Hilbert space, still contains a finite–rank operator in its norm closure. However, in the case of a real Hilbert space this is not enough in order to conclude that $\mathcal{A}$ is weakly dense in $\mathbf{B}(\mathcal{H})$. A commutative algebra $\mathcal{J}$, generated by the matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

is an example of a proper essentially transitive algebra acting on $\mathbb{R}^2$. The tensor product $\mathbf{B}(\mathcal{H}) \otimes \mathcal{J}$ is an example of such an algebra acting on $\mathcal{H} \oplus \mathcal{H}$. However, the existence of a nonzero finite–rank operator in the closure of an essentially transitive algebra, implies the following commutative version of Theorem 2.5.4, which holds on real or complex infinite–dimensional Hilbert spaces.

Theorem 2.5.5. *A commutative algebra $\mathcal{A}$, of operators acting on a real or complex infinite–dimensional Hilbert space, is* never *essentially transitive.*

Proof. By Propositions 2.4.1 and 2.2.2 the (norm) closure of every essentially transitive algebra contains a nonzero finite–rank operator T. Since $TA = AT$ for every $A \in \mathcal{A}$, it follows that the range of T is a nontrivial (finite–dimensional) invariant subspace for $\mathcal{A}$, contradicting the (essential) transitivity of $\mathcal{A}$. □

Chapter 3

On Invariant Subspaces of Essentially Self–Adjoint Operators

An application of the main result of the previous chapter to the algebra generated by an essentially self–adjoint operator A yields the existence of nonzero vectors $x, y \in \mathcal{H}$ such that $\tau(p) = \langle p(A)x, y\rangle$ is a positive functional on the space of all polynomials on the essential spectrum of A. This result immediately implies the existence of real invariant subspaces for essentially self–adjoint operators acting on a complex Hilbert space. Elementary convex analysis techniques, applied to the space of certain vector states, yield the existence of invariant subspaces for essentially self–adjoint operators acting on an infinite–dimensional real Hilbert space.

3.1 Introduction

The existence of invariant subspaces for compact perturbations of self–adjoint operators appears to be one of the most difficult questions in the theory of invariant subspaces [**?**]. The positive results about the existence of the invariant subspaces for the Schatten–class perturbations of self–adjoint operators, acting on a complex

Hilbert space, date back to the late 1950's. For the facts concerning such operators see Chapter 6 in [?], where a brief history of the problem, together with the references to the related topics is given. The proofs of those results are based on the concept of the separation of spectra. However, Ljubič and Macaev [?] showed that there is no general spectral theory by constructing an example of an operator A such that $\sigma(A|\mathcal{M}) = [0,1]$ whenever $\mathcal{M}$ is a nonzero invariant subspace for A. This suggests that different techniques might be needed to establish the existence of invariant subspaces for essentially self–adjoint operators.

The fact that the right–hand side of the inequality (2.4.1) depends only on the essential norm of the real part of the operator A, suggests that Proposition 2.4.1 might have applications to the invariant subspace problem for compact perturbations of self–adjoint operators. In this chapter we apply Proposition 2.4.1 in order to construct positive functionals $\tau(p) = \langle p(A)x, y\rangle$ on the space of all polynomials restricted to the essential spectrum of A. Finally, in the case when the underlying Hilbert space is real, the existence of invariant subspaces for A is established after solving an extreme problem concerning certain convex subspaces of vector states.

Definition 3.1.1. Suppose $\mathcal{H}$ is a real or complex Hilbert space. An operator $A \in \mathbf{B}(\mathcal{H})$ is called *essentially self–adjoint*, if $\pi(A)$ is a self–adjoint element in the Calkin algebra $\mathbf{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\pi\colon \mathbf{B}(\mathcal{H}) \longrightarrow \mathbf{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the quotient mapping.

Remark 3.1.1. Clearly, by definition of the Calkin algebra, A is essentially self–adjoint if and only if $A = S + K$, where $S \in \mathbf{B}(\mathcal{H})$ is self–adjoint and K is a compact operator. Hence, saying that A is essentially self–adjoint, is the same as saying that A is a compact perturbation of a self–adjoint operator. Note, however, that this is *false* if we replace self–adjoint operators by normal ones.

3.2 On Real Invariant Subspaces

Recently V.I. Lomonosov [?] proved that every essentially self–adjoint operator acting on a complex Hilbert space has a nontrivial closed *real* invariant subspace. We give an alternative proof, based on Proposition 2.4.1, and thus introduce the idea that will be later generalized in order to yield the existence of proper invariant subspaces for essentially self–adjoint operators acting on a real Hilbert space.

Recall that a *real subspace* of a complex Hilbert space $\mathcal{H}$ is a subset that is closed under addition and multiplication by the *real* scalars. A real subspace $\mathcal{M} \subset \mathcal{H}$ is invariant for an operator $A \in \mathbf{B}(\mathcal{H})$ if and only if $\mathcal{M}$ is invariant under all operators in the *real* algebra generated by A, i.e. the algebra of all real polynomials in A.

Proposition 3.2.1. *Suppose $\mathcal{H}$ is an infinite–dimensional complex Hilbert space and let $\mathcal{A}$ be a convex set of commuting essentially self–adjoint operators. Then the set of non–cyclic vectors for $\mathcal{A}$ is dense in $\mathcal{H}$.*

Proof. Suppose not; then there exists a unit vector f_0 and a positive number $r \in (0,1)$ such that all vectors in the set

$$\mathcal{S} = \left\{ f \in \mathcal{H} \mid \ \|f_0 - f\| \leq \tfrac{r}{\sqrt{1-r^2}} \right\},$$

are cyclic for $\mathcal{A}$. In particular, for every vector $g \in \mathcal{H}$ and $\|g\| \leq 1$, there exists an operator $A \in \mathcal{A}$ such that

$$\mathrm{Re} \left\langle A \left(f_0 + \tfrac{r}{\sqrt{1-r^2}} g \right), -i \left(f_0 - \tfrac{\sqrt{1-r^2}}{r} g \right) \right\rangle > 0,$$

or equivalently,

$$\mathrm{Re} \left\langle iA \left(f_0 + \tfrac{r}{\sqrt{1-r^2}} g \right), f_0 - \tfrac{\sqrt{1-r^2}}{r} g \right\rangle > 0.$$

Since A is an essentially self–adjoint operator, it follows that

$$\|\operatorname{Im} A\|_{\text{ess}} = \|\operatorname{Re}(iA)\|_{\text{ess}} = 0,$$

and consequently the convex set $i\mathcal{A} = \{iA \mid A \in \mathcal{A}\}$, satisfies the hypothesis of Proposition 2.4.1. Therefore, there exists an element $A_0 \in \mathcal{A}$ ($A_0 \neq zI$), with an eigenvector $f_1 \in \mathcal{S}$. Since the operators in $\mathcal{A}$ commute, f_1 cannot be a cyclic vector for $\mathcal{A}$, contradicting the assumption that all vectors in $\mathcal{S}$ are cyclic for $\mathcal{A}$. □

Corollary 3.2.2 (V.I. Lomonosov, 1992). *Every essentially self–adjoint operator on an infinite–dimensional complex Hilbert space has a nontrivial closed real invariant subspace.*

Proof. The commutative algebra $\mathcal{A}_{\mathbb{R}}$ of all real polynomials in A consists of essentially self–adjoint operators whenever A is essentially self–adjoint. By Proposition 3.2.1 the set of non–cyclic vectors for $\mathcal{A}_{\mathbb{R}}$ is dense in $\mathcal{H}$. Since for every nonzero vector $f \in \mathcal{H}$ the closure of the orbit $\mathcal{A}_{\mathbb{R}} f = \{Tf \mid T \in \mathcal{A}_{\mathbb{R}}\}$ is a real invariant subspace for A, it follows that A has a nontrivial closed real invariant subspace. □

Remark 3.2.1. If A is a self–adjoint operator acting on a complex Hilbert space $\mathcal{H}$, then for every vector $f \in \mathcal{H}$ and every real polynomial p we have:

$$\operatorname{Im} \langle p(A)f, f \rangle = 0. \tag{3.2.1}$$

The condition (3.2.1) in fact characterizes self–adjoint operators on a complex Hilbert space [**?**, p. 103]. Roughly speaking, Proposition 3.2.1 and its corollary establish a similar fact for essentially self–adjoint operators acting on a complex Hilbert space.

3.3 The Space of Vector States

In the previous section we applied our machinery only to the imaginary part of an essentially self–adjoint operator A. An application to the real part yields the existence of "vector states" on the space of all polynomials restricted to the essential spectrum of A. Before proceeding, we make the following conventions that hold through the rest of this chapter:

As usual, let $\mathcal{H}$ be an infinite–dimensional real or complex Hilbert space. The underlying field of real or complex numbers (respectively) is denoted by $\mathbb{F}$. Suppose $A \in \mathbf{B}(\mathcal{H})$ is a fixed essentially self–adjoint operator without non–trivial closed invariant subspaces and let E denote its essential spectrum. Furthermore, we may assume that $\|A\|_{\text{ess}} \le 1$, and consequently, $E \subset [-1, 1]$. Let $\mathcal{A} \subset \mathbf{B}(\mathcal{H})$ be an algebra generated by A, i.e. $\mathcal{A}$ is the algebra of all polynomials $p(A)$ with the coefficients in the underlying field $\mathbb{F}$.

The algebra of all polynomials with the coefficients in $\mathbb{F}$, equipped with the norm

$$\|p\|_\infty = \max_{t \in E} |p(t)| \, ,$$

is denoted by $\mathcal{P}(E)$.

Definition 3.3.1. Let $\mathcal{D} \subset \mathcal{H}$ be the set of all nonzero vectors $x \in \mathcal{H}$ for which there exists a nonzero vector $y \in \mathcal{H}$ satisfying the following inequality for every polynomial $p \in \mathcal{P}(E)$

$$\operatorname{Re} \langle p(A)x, y \rangle \le \|\operatorname{Re} p\|_\infty \langle x, y \rangle \, . \tag{3.3.1}$$

Lemma 3.3.1. *The set $\mathcal{D}$ is dense in $\mathcal{H}$.*

Proof. Since the operator A has no invariant subspaces the condition of Proposition 2.4.1 is never satisfied for the algebra $\mathcal{A}$. More precisely, for every unit vector $f_0 \in \mathcal{H}$ and any positive number $r \in (0,1)$ there exists a vector $g \perp f_0$ such that for every polynomial $p \in \mathcal{P}(E)$

$$\operatorname{Re}\left\langle p(A)\left(f_0 + \tfrac{r}{\sqrt{1-r^2}} g\right), f_0 - \tfrac{\sqrt{1-r^2}}{r} g\right\rangle \le \|\operatorname{Re} p(A)\|_{\mathrm{ess}} (1 - \|g\|^2).$$

Clearly, for every polynomial $p \in \mathcal{P}(E)$ we have

$$\|\operatorname{Re} p(A)\|_{\mathrm{ess}} = \|(\operatorname{Re} p)(A)\|_{\mathrm{ess}} = \|\operatorname{Re} p\|_{\infty}.$$

The vectors

$$x = f_0 + \tfrac{r}{\sqrt{1-r^2}} g \quad \text{and} \quad y = f_0 - \tfrac{\sqrt{1-r^2}}{r} g$$

satisfy the inequality (3.3.1). Letting $r \to 0$, and replacing the vector x by λx, where $\lambda > 0$, implies the required density of $\mathcal{D}$. □

Lemma 3.3.2. *For fixed vectors $x, y \in \mathcal{H}$ define a linear functional $\tau \colon \mathcal{P}(E) \longrightarrow \mathbb{F}$*

$$\tau(p) = \langle p(A)x, y\rangle .$$

Then τ is a bounded positive functional on the space $\mathcal{P}(E)$ if and only if the following inequality is satisfied for every polynomial $p \in \mathcal{P}(E)$:

$$\operatorname{Re}\langle p(A)x, y\rangle \le \|\operatorname{Re} p\|_{\infty} \langle x, y\rangle .$$

Proof. Suppose that τ is a positive functional on $\mathcal{P}(E)$. Then $\operatorname{Re}\langle p(A)x, y\rangle = \langle(\operatorname{Re} p)(A)x, y\rangle$. Since $\|\operatorname{Re} p\|_{\infty} - \operatorname{Re} p$ is a positive polynomial on E, we have

$$\tau(\|\operatorname{Re} p\|_{\infty} - \operatorname{Re} p) = \langle(\|\operatorname{Re} p\|_{\infty} - \operatorname{Re} p)(A)x, y\rangle \ge 0,$$

or equivalently,

$$\operatorname{Re}\langle p(A)x, y\rangle \le \|\operatorname{Re} p\|_\infty \langle x, y\rangle .$$

Conversely, suppose τ is not a bounded positive functional on $\mathcal{P}(E)$. Then either there exists a real polynomial p such that $\operatorname{Im}\langle p(A)x, y\rangle \neq 0$, or $\langle p(A)x, y\rangle < 0$ for some positive polynomial $p \in \mathcal{P}(E)$. After replacing p by $\pm ip$ it is easy to see that $\operatorname{Im}\langle p(A)x, y\rangle \neq 0$ contradicts (3.3.1). Similarly, for a positive polynomial p we have

$$\big\| \|p\|_\infty - p\big\|_\infty \le \|p\|_\infty .$$

Therefore $\langle p(A)x, y\rangle < 0$ and $\langle x, y\rangle \ge 0$ imply

$$\big\langle \big(\|p\|_\infty - p(A)\big)x, y\big\rangle > \|p\|_\infty \langle x, y\rangle \ge \big\| \|p\|_\infty - p\big\|_\infty \langle x, y\rangle ,$$

contradicting (3.3.1). Finally, in the case when $\langle x, y\rangle < 0$ the inequality (3.3.1) fails for the polynomial $p \equiv -1$. □

Definition 3.3.2. The set of all bounded positive linear functionals on $\mathcal{P}(E)$ is denoted by $\mathcal{T}$. For each vector $x \in \mathcal{H}$ define the set

$$\mathcal{T}_x = \{y \in \mathcal{H} \mid \tau(p) = \langle p(A)x, y\rangle \in \mathcal{T}\} .$$

Lemma 3.3.3. *For every vector $x \in \mathcal{H}$, $\mathcal{T}_x$ is a closed convex subset of $\mathcal{H}$.*

Proof. Convexity of the set $\mathcal{T}_x$ is obvious. It remains to prove that the complement of $\mathcal{T}_x$ is an open subset of $\mathcal{H}$. If $y \notin \mathcal{T}_x$ then there exists a positive polynomial $p \in \mathcal{P}(E)$ such that $\langle p(A)x, y\rangle \not\geq 0$. In that case there exists a weak neighborhood $\mathcal{W}$ of y such that $\langle p(A)x, z\rangle \not\geq 0$ for every $z \in \mathcal{W}$. Consequently, the complement of the set $\mathcal{T}_x$ is a (weakly) open subset of $\mathcal{H}$. □

Definition 3.3.3. A positive functional $\tau \in \mathcal{T}$ is called a *state* if $\|\tau\| = 1$, or equivalently $\tau(1) = 1$. The space of all states on $\mathcal{P}(E)$ is denoted by $\mathcal{T}'$. Similarly, for every vector $x \in \mathcal{H}$ the set $\mathcal{T}'_x$ is defined by

$$\mathcal{T}'_x = \{y \in \mathcal{H} \mid \tau(p) = \langle p(A)x, y\rangle \in \mathcal{T}'\}.$$

Remark 3.3.1. From Lemma 3.3.1 and Lemma 3.3.2 it follows that the set $\mathcal{D}$ of all vectors $x \in \mathcal{H}$ for which the set $\mathcal{T}_x$ contains a nonzero vector is dense in $\mathcal{H}$. If x and y are nonzero vectors and $y \in \mathcal{T}_x$ then $\langle x, y\rangle \geq 0$. However, since a positive functional always attains its norm on the identity function, the equality $\langle x, y\rangle = 0$ implies that $\tau(p) = \langle p(A)x, y\rangle = 0$ for every polynomial $p \in \mathcal{P}(E)$, contradicting the fact that the operator A has no invariant subspaces. Therefore, the set $\mathcal{T}'_x$ is nonempty for every vector x in a dense set $\mathcal{D} \subset \mathcal{H}$. In fact, for every vector $x \in \mathcal{D}$ the set $\mathcal{T}'_x$ is the intersection of the cone $\mathcal{T}_x$ and the hyperplane $\mathcal{M}_x = \{y \in \mathcal{H} \mid \langle y, x\rangle = 1\}$. Note also, that for nonzero vectors $x \in \mathcal{D}$ and $y \in \mathcal{T}_x$, we have: $\langle x, y\rangle^{-1} y \in \mathcal{T}'_x$.

By Lemma 3.3.3 the set $\mathcal{T}'_x$ is a weakly closed convex subset of $\mathcal{H}$. We show that the set $\mathcal{T}'_x$ has no extreme points.

Lemma 3.3.4. *For every vector $x \in \mathcal{H}$ the set $\mathcal{T}'_x$ has no extreme points.*

Proof. Suppose y_0 is an extreme point in $\mathcal{T}'_x$. By definition of the set $\mathcal{T}'_x$, the functional $\tau'(p) = \langle p(A)x, y_0\rangle$ is a state on $\mathcal{P}(E)$. Hence,

$$\omega(p) = \tau((1-t)p(t)) = \langle p(A)x, (1-A^*)y_0\rangle$$

is a positive functional on $\mathcal{P}(E)$. Consequently,

$$y_1 = \langle (1-A)x, y_0\rangle^{-1}(1-A^*)y_0 \in \mathcal{T}'_x.$$

Similarly,

$$y_2 = \langle (1+A)x, y_0 \rangle^{-1} (1+A^*) y_0 \in \mathcal{T}'_x.$$

From

$$y_0 = \frac{\langle (1-A)x, y_0 \rangle}{2} y_1 + \frac{\langle (1+A)x, y_0 \rangle}{2} y_2,$$

we conclude that $y_0 = y_1 = y_2$. Therefore, $(1 - A^*)y_0 = \langle (1-A)x, y \rangle\, y_0$ implies that y_0 is an eigenvector for A^*, contradicting the nonexistence of invariant subspaces for the operator A. □

Corollary 3.3.5. *For every vector $x \in \mathcal{H}$ the set $\mathcal{T}'_x$ is either empty or unbounded.*

Proof. By the Krein–Milman Theorem the set $\mathcal{T}'_x$ cannot be weakly compact due to the lack of extreme points. □

Although the set $\mathcal{T}'_x$ is unbounded for every vector $x \in \mathcal{D}$, the following lemma shows that it contains no line segments of infinite length. In particular, $\mathcal{T}'_x$ is a *proper* subset of the hyperplane

$$\mathcal{M}_x = \{ y \in \mathcal{H} \mid \ \langle y, x \rangle = 1 \}.$$

Lemma 3.3.6. *Every line segment in $\mathcal{T}'_x$ has a finite length.*

Proof. Suppose the set $\mathcal{T}'_x$ contains a line segment of infinite length. Then there exists a vector $y \in \mathcal{T}'_x$, and a unit vector $u \perp x$ such that $y + \lambda u \in \mathcal{T}'_x$ for every $\lambda \geq 0$. For every power $k = 0, 1, \ldots$, and every vector $z \in \mathcal{T}'_x$, we have: $\left| \left\langle A^k x, z \right\rangle \right| \leq 1$. Applying this inequality to a vector $y + \lambda u$ and letting $\lambda \to \infty$ implies that $\left\langle A^k x, u \right\rangle = 0$, contradicting the fact that x is a cyclic vector for A. □

3.4 Invariant Subspaces on a Real Hilbert Space

In this section we use vector states in order to establish the existence of invariant subspaces for essentially self–adjoint operators acting on an infinite–dimensional real Hilbert space. The invariant subspace problem for essentially self–adjoint operator will be translated into an extreme problem and the solution will be obtained upon differentiating certain functions at their extreme. Once again we will employ the differentiability of the Hilbert norm. We start with the following lemma.

Lemma 3.4.1. *Suppose x and y are any vectors in $\mathcal{H}$ such that* $\operatorname{Re}\langle x, y\rangle = 1$. *Fix a nonzero operator $T \in \mathbf{B}(\mathcal{H})$ and let $a = (\|T\|\,\|x\|\,\|y\|)^{-1}$. Then for every vector $z \in \mathcal{H}$ the function $\psi(\lambda)\colon (-a, a) \longrightarrow [0, \infty)$, defined by*

$$\psi(\lambda) = \left\| \big(\operatorname{Re}\langle (1 + \lambda T)y, x\rangle \big)^{-1} (1 + \lambda T)y - z \right\|^2$$

is differentiable on $(-a, a)$. Furthermore, if ψ' denotes the derivative of ψ then

$$\psi'(0) = 2 \operatorname{Re}\left\langle Ty, y - z - (\|y\|^2 - \operatorname{Re}\langle y, z\rangle)x \right\rangle .$$

Proof. Since for $\lambda \in (-a, a)$ we have $\operatorname{Re}\langle (1 + \lambda T)y, x\rangle > 0$, it follows that the function ψ is well defined on $(-a, a)$. In order to compute its derivative $\psi'(0)$ first apply the polar identity to ψ and then use the product and chain rules for differentiation. A straightforward calculation yields the required formula. □

Definition 3.4.1. For every vector $x \in \mathcal{D}$, define $P_x\colon \mathcal{H} \longrightarrow \mathcal{T}'_x$ to be the projection to the set $\mathcal{T}'_x$, i.e. for every $z \in \mathcal{H}$

$$\|P_x z - z\| = \inf_{y \in \mathcal{T}'_x} \|y - z\| .$$

Remark 3.4.1. Since for $x \in \mathcal{D}$ the set $\mathcal{T}'_x$ is nonempty, closed, and convex it follows that the projection P_x is well defined on the whole space $\mathcal{H}$.

Lemma 3.4.2. *If $x \in \mathcal{D}$ then for every vector $z \in \mathcal{H}$ and every power $k = 0, 1, \ldots,$ the following condition is satisfied:*

$$\operatorname{Re}\left\langle A^k\big((\|P_x z\|^2 - \operatorname{Re}\langle P_x z, z\rangle)x + (I - P_x)z\big), P_x z\right\rangle = 0.$$

Proof. Let $T = A^{*k}$, and fix a vector $y \in \mathcal{T}'_x$. The function $\Phi(\lambda)\colon (-1,1) \longrightarrow \mathcal{H}$ is defined by

$$\Phi(\lambda) = \langle (1+\lambda T)y, x\rangle^{-1}(1+\lambda T)y.$$

The same argument as in the proof of Lemma 3.3.4 shows that Φ is well defined and $\Phi(\lambda) \in \mathcal{T}'_x$ for every $\lambda \in (-1,1)$.

Choose any vector $z \in \mathcal{H}$ and consider the function $\psi(\lambda)\colon (-1,1) \longrightarrow [0,\infty)$, defined by

$$\psi(\lambda) = \|\Phi(\lambda) - z\|^2.$$

By Lemma 3.4.1 the function ψ is differentiable, and

$$\psi'(0) = 2\operatorname{Re}\left\langle Ty, y - z - (\|y\|^2 - \operatorname{Re}\langle y, z\rangle)x\right\rangle.$$

By definition of the projection P_x the function ψ attains its global minimum at the point $\lambda = 0$ whenever $y = P_x z$. Consequently, $\psi'(0) = 0$ for $y = P_x z$, which completes the proof. □

A remarkable fact is that Lemma 3.4.2 holds on a real or complex infinite–dimensional Hilbert space. It is now easy to establish the existence of proper invariant subspaces for essentially self–adjoint operators acting on a real Hilbert space.

Theorem 3.4.3. *Every essentially self–adjoint operator acting on a real infinite–dimensional Hilbert space $\mathcal{H}$ has a nontrivial closed invariant subspace.*

Proof. Suppose A is an essentially self–adjoint operator acting on a real infinite–dimensional Hilbert space $\mathcal{H}$. We may assume that $\|A\|_{\text{ess}} \leq 1$. If the operator A has no nontrivial invariant subspaces then we can apply Lemma 3.4.2 and Lemma 3.3.6. We will show that this contradicts the non–existence of invariant subspaces for A.

On a real Hilbert space Lemma 3.4.2 implies that for every $k = 0, 1, \ldots$:

$$\operatorname{Re}\big\langle A^k\big((\|P_x z\|^2 - \operatorname{Re}\langle P_x z, z\rangle)x + (I - P_x)z\big), P_x z\big\rangle =$$

$$\big\langle A^k\big((\|P_x z\|^2 - \operatorname{Re}\langle P_x z, z\rangle)x + (I - P_x)z\big), P_x z\big\rangle = 0.$$

Since $P_x z \neq 0$ it follows that

$$y_z = \big(\, \|P_x z\|^2 - \operatorname{Re}\langle P_x z, z\rangle\,\big)x + (I - P_x)z$$

is a non–cyclic vector for $\mathcal{A}$ whenever $x \in \mathcal{D}$. The proof is therefore completed if we show that $y_z \neq 0$ for a suitable choice of the vector $z \in \mathcal{H}$.

Recall that the set $\mathcal{T}_x'$ lies in the hyperplane $\mathcal{M}_x = \{y \in \mathcal{H} \mid \ \langle y, x\rangle = 1\}$. By definition of the projection P_x, the vector $y_z = 0$ for $z \in \mathcal{M}_x$ if and only if $z \in \mathcal{T}_x'$. Lemma 3.3.6 implies that $\mathcal{T}_x'$ is a proper subset of the hyperplane $\mathcal{M}_x$ and thus completes the proof. □

Remark 3.4.2. Theorem 3.4.3 yields the existence of invariant subspaces for an essentially self–adjoint operator A acting on a complex Hilbert space, whenever the operator A has a matrix representation with real coefficients. Although considerable efforts have been made to reduce the general complex case to the real one, so far all such attempts have been unsuccessful.

We suggest that further research in this direction is likely going to reveal additional properties of essentially self–adjoint operators and thus contribute to our understanding of how such operators act on the underlying Hilbert space in terms of invariant subspaces.