On Invariant Subspaces of Essentially Self-Adjoint Operators

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An Extension of Burnside's Theorem:

Proposition. *Let* $\mathcal{A} \subset \mathrm{B}(\mathcal{H})$ *be a convex subset of bounded linear operators acting on a real or complex Hilbert space* $\mathcal{H}$*. Suppose that for every vector* $g \perp f_0$ *and* $\|g\| \leq 1$*, there exists an operator* $A \in \mathcal{A}$*, satisfying the following strict inequality for* $\xi = r/\sqrt{1-r^2}$*:*

$$\mathrm{Re}\left\langle A\left(f_0 + \xi g\right), f_0 - \xi^{-1} g\right\rangle > \|\mathrm{Re}\, A\|_{\mathrm{ess}}\,(1 - \|g\|^2).$$

Then $\mathcal{A}$ *contains an operator* A_0 *with an eigenvalue* λ *satisfying the condition:*

$$|\mathrm{Re}\, \lambda| > \|\mathrm{Re}\, A_0\|_{\mathrm{ess}}\,.$$

Proposition. *Suppose $\mathcal{H}$ is a real or complex Hilbert space, and $\lambda \in \mathbf{C}$ is a point in the spectrum of the operator $A \in \mathbf{B}(\mathcal{H})$, such that*

$$|\mathrm{Re}\,\lambda| > \|\mathrm{Re}\,A\|_{\mathrm{ess}}\,.$$

Then the norm closure of the real algebra generated by A contains a nonzero finite-rank operator.

Note: Applies to real or complex Hilbert space

Applications: Transitive Algebras and Invariant Subspaces

Suppose $A \in \mathbf{B}(\mathcal{H})$ is a set of operators acting on a real or complex Hilbert space $\mathcal{H}$.

Definition. *Let $\mathcal{D} \subset \mathcal{H}$ be the set of all non-zero vectors $x \in \mathcal{H}$ for which there exists a nonzero vector $y \in \mathcal{H}$ satisfying the following inequality for every operator $A \in \mathcal{A}$:*

$$\operatorname{Re} \langle Ax, y \rangle \leq \|\operatorname{Re} A\|_{\text{ess}} \langle x, y \rangle .$$

Alternative: Either the norm closure of the real algebra generated by the operators in $\mathcal{A}$ contains a nonzero finite-rank operator or the set $\mathcal{D}$ is dense in $\mathcal{H}$.

Applications to the invariant subspace problem for *essentially self-adjoint* operators:

Let $\mathcal{H}$ be an infinite-dimensional real or complex Hilbert space. The underlying field of real or complex numbers (respectively) is denoted by $\mathbf{F}$. Suppose $A \in \mathbf{B}(\mathcal{H})$ is a fixed essentially self-adjoint operator without non-trivial closed invariant subspaces and let E denote its essential spectrum. Furthermore, we may assume that $\|A\|_{\text{ess}} < 1$, and consequently: $E \subset (-1, 1)$. Let $\mathcal{A} \subset \mathbf{B}(\mathcal{H})$ be an algebra generated by A, i.e. $\mathcal{A}$ is the algebra of all polynomials $p(A)$, with the coefficients in the underlying field $\mathbf{F}$.

The algebra of all polynomials with the coefficients in $\mathbf{F}$, equipped with the norm

$$\|p\|_\infty = \max_{t \in E} |p(t)|,$$

is denoted by $\mathcal{P}(E)$.

There exist a pair of nonzero vectors x, y such that

$$\text{Re}\,\langle Ax, y\rangle \le \|\text{Re}\,A\|_{\text{ess}}\,\langle x, y\rangle\,.$$

Equivalently, for every $p \in \mathcal{P}(E)$:

$$\text{Re}\,\langle p(A)x, y\rangle \le \|\text{Re}\,p\|_{\infty}\,\langle x, y\rangle\,.$$

Consequently,

$$\tau(p) = \langle p(A)x, y\rangle$$

is a (bounded) positive functional on the space of all polynomials $\mathcal{P}(E)$, equipped with the max norm.

Recall that such a functional is called a *vector state* if $\|\tau\| = 1$, or equivalently $\langle x, y\rangle = 1$.

Note: If A is self-adjoint than

$$\tau(p) = \langle p(A)x, x\rangle$$

is a vector state for every vector $x \in \mathcal{H}$.

Let $\mathcal{T}$ be the set of all vectors $y \in \mathcal{H}$ for which the functional $\tau(p) = \langle p(A)x, y \rangle$ is a vector state on $\mathcal{P}(E)$.

Then $\mathcal{T}$ is a proper closed and convex subset of the hyperplane $\{y \in \mathcal{H} | \langle x, y \rangle = 1\}$.

For $y \in \mathcal{T}$:

$$\widehat{\tau}(p) = \tau((1-t)p(t)) = \langle p(A)x, (1-A^*)y \rangle$$

is a positive functional, and consequently,

$$\langle x, (1-\lambda A^*)y \rangle^{-1} (1-\lambda A^*)y \in \mathcal{T},$$

for any $\lambda \in (-1, 1)$.

This observation immediately implies that an extreme point in $\mathcal{T}$ is an eigenvector for A^*.

Hence: $\mathcal{T}$ has no extreme points.

Fix a vector $y \in \mathcal{T}$ and let $T = A^{*k}$ $(k \geq 0)$. Define $\Phi(\lambda)\colon (-1,1) \longrightarrow \mathcal{T}$ by

$$\Phi(\lambda) = \langle x, (1-\lambda T)y \rangle^{-1} (1-\lambda T)y.$$

Choose any vector $z \in \mathcal{H}$ and consider the function $\psi\colon (-1,1) \longrightarrow \mathbf{R}^+$, defined by

$$\psi(\lambda) = \|\Phi(\lambda) - z\|^2 .$$

The function ψ is differentiable:

$$\psi'(0) = 2\,\mathrm{Re}\left\langle Ty, y - z - (\|y\|^2 - \mathrm{Re}\,\langle y, z\rangle)x \right\rangle .$$

Let $P \colon \mathcal{H} \longrightarrow \mathcal{T}$ be the projection on a convex set $\mathcal{T}$, i.e. for every vector $z \in \mathcal{H}$

$$\|Pz - z\| = \inf_{y \in \mathcal{T}} \|y - z\| \, .$$

Recall: for $z \notin \mathcal{T}$ the point Pz is called a support point of $\mathcal{T}$ and $z - Pz$ is called a support functional at Pz.

Although the set $\mathcal{T}$ has no extreme points it has "plenty" of support points. We will show that these points are non-cyclic vectors for the real algebra generated by A^*.

Choose a vector $z \in \mathcal{H}$ and let $y = Pz$. Then the function $\psi : (-1, 1) \longrightarrow \mathbf{R}^+$, (as before) defined by

$$\psi(\lambda) = \left\| \langle x, (1 - \lambda T) y \rangle^{-1} (1 - \lambda T) y - z \right\|^2$$

attains minimum at $\lambda = 0$:

$$\psi(0) = \| Pz - z \|^2 = \inf_{y \in \mathcal{T}} \| y - z \| .$$

Hence: $\psi'(0) = 0$.

Equivalently:

$$\mathrm{Re} \left\langle Ty, y - z - (\|y\|^2 - \mathrm{Re} \langle y, z \rangle) x \right\rangle = 0.$$

Conclusion: y is a non-cyclic vector for the real algebra generated by A^*.

Theorem. *Every essentially self-adjoint operator acting on a real infinite-dimensional Hilbert space has a nontrivial invariant subspace.*

Complex Case: Only Real Subspaces.

Our technique applies if A admits an essentially self-adjoint matrix representation with real entries.

Conjecture. *Every essentially self-adjoint operator with real spectrum admits an essentially self-adjoint matrix representation with real entries.*

True in finite dimensions.

Seems hard to prove.

Invariant Subspaces may exist even if the conjecture is false.

The structure of the space of vector states: subject to further research...