

Research Article

On Certain Classes of Convex Functions

Young Jae Sim and Oh Sang Kwon

Department of Mathematics, Kyungsung University, Busan 608-736, Republic of Korea

Correspondence should be addressed to Oh Sang Kwon; oskwon@ks.ac.kr

Received 25 February 2013; Accepted 7 May 2013

Academic Editor: Heinrich Begehr

Copyright © 2013 Y. J. Sim and O. S. Kwon. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

For real numbers α and β such that $0 \leq \alpha < 1 < \beta$, we denote by $\mathcal{K}(\alpha, \beta)$ the class of normalized analytic functions which satisfy the following two sided-inequality: $\alpha < \Re\{1 + (zf''(z)/f'(z))\} < \beta$ ($z \in \mathbb{U}$), where \mathbb{U} denotes the open unit disk. We find some relationships involving functions in the class $\mathcal{K}(\alpha, \beta)$. And we estimate the bounds of coefficients and solve the Fekete-Szegő problem for functions in this class. Furthermore, we investigate the bounds of initial coefficients of inverse functions or biunivalent functions.

1. Introduction

Let \mathcal{A} denote the class of analytic functions in the unit disc

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\} \quad (1)$$

which is normalized by

$$f(0) = 0, \quad f'(0) = 1. \quad (2)$$

Also let \mathcal{S} denote the subclass of \mathcal{A} which is composed of functions which are univalent in \mathbb{U} . And, as usual, we denote by \mathcal{K} the class of functions in \mathcal{A} which are convex in \mathbb{U} .

We say that f is subordinate to F in \mathbb{U} , written as $f < F$ ($z \in \mathbb{U}$), if and only if $f(z) = F(w(z))$ for some Schwarz function $w(z)$ such that

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathbb{U}). \quad (3)$$

If F is univalent in \mathbb{U} , then the subordination $f < F$ is equivalent to

$$f(0) = F(0), \quad f(\mathbb{U}) \subset F(\mathbb{U}). \quad (4)$$

Definition 1. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}(\alpha, \beta)$ if f satisfies the following inequality:

$$\alpha < \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \beta \quad (z \in \mathbb{U}). \quad (5)$$

It is clear that $\mathcal{K}(\alpha, \beta) \subset \mathcal{K}$. And we remark that, for given real numbers α and β ($0 \leq \alpha < 1 < \beta$), $f \in \mathcal{K}(\alpha, \beta)$ if and only if f satisfies each of the following two subordination relationships:

$$1 + \frac{zf''(z)}{f'(z)} < \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (z \in \mathbb{U}), \quad (6)$$

$$1 + \frac{zf''(z)}{f'(z)} < \frac{1 + (1 - 2\beta)z}{1 - z} \quad (z \in \mathbb{U}).$$

Now, we define an analytic function $p : \mathbb{U} \rightarrow \mathbb{C}$ by

$$p(z) = 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))z}}{1 - z} \right). \quad (7)$$

The above function p was introduced by Kuroki and Owa [1], and they proved that p maps \mathbb{U} onto a convex domain

$$\Lambda = \{w : \alpha < \Re\{w\} < \beta\} \quad (8)$$

conformally. Using this fact and the definition of subordination, we can obtain the following lemma, directly.

Lemma 2. Let $f(z) \in \mathcal{A}$ and $0 \leq \alpha < 1 < \beta$. Then $f \in \mathcal{K}(\alpha, \beta)$ if and only if

$$1 + \frac{zf''(z)}{f'(z)} < 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))z}}{1 - z} \right) \quad (z \in \mathbb{U}). \tag{9}$$

And we note that the function p , defined by (7), has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{10}$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} i \left(1 - e^{2n\pi i((1-\alpha)/(\beta-\alpha))} \right) \quad (n \in \mathbb{N}). \tag{11}$$

For given real numbers α and β such that $0 \leq \alpha < 1 < \beta$, we denote by $\mathcal{K}_\sigma(\alpha, \beta)$ the class of biunivalent functions consisting of the functions in \mathcal{A} such that

$$f \in \mathcal{K}(\alpha, \beta), \quad f^{-1} \in \mathcal{K}(\alpha, \beta), \tag{12}$$

where f^{-1} is the inverse function of f .

In our present investigation, we first find some relationships for functions in bounded positive class $\mathcal{K}(\alpha, \beta)$. And we solve several coefficient problems including Fekete-Szegő problems for functions in the class. Furthermore, we estimate the bounds of initial coefficients of inverse functions and biunivalent functions. For the coefficient bounds of functions in special subclasses of \mathcal{S} , the readers may be referred to the works [2–4].

2. Relations Involving Bounds on the Real Parts

In this section, we will find some relations involving the functions in $\mathcal{K}(\alpha, \beta)$. And the following lemma will be needed in finding the relations.

Lemma 3 (see Miller and Mocanu [5, Theorem 2.3.i]). Let Ξ be a set in the complex plane \mathbb{C} and let b be a complex number such that $\Re\{b\} > 0$. Suppose that a function $\psi : \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C}$ satisfies the condition

$$\psi(i\rho, \sigma; z) \notin \Xi \tag{13}$$

for all real $\rho, \sigma \leq -|b - i\rho|^2 / (2\Re\{b\})$ and all $z \in \mathbb{U}$. If the function $p(z)$ defined by $p(z) = b + b_1z + b_2z^2 + \dots$ is analytic in \mathbb{U} and if

$$\psi(p(z), zp'(z); z) \in \Xi, \tag{14}$$

then $\Re\{p(z)\} > 0$ in \mathbb{U} .

Theorem 4. Let $f \in \mathcal{A}$, $0 \leq \alpha < 1$ and

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathbb{U}). \tag{15}$$

Then

$$\Re \left\{ \sqrt{f'(z)} \right\} > \frac{1}{2 - \alpha} \quad (z \in \mathbb{U}). \tag{16}$$

Proof. First of all, we put $\gamma = 1/(2 - \alpha)$ and note that $1/2 \leq \gamma < 1$ for $0 \leq \alpha < 1$. Let

$$p(z) = \frac{1}{1 - \gamma} \left(\sqrt{f'(z)} - \gamma \right). \tag{17}$$

Differentiating (17), we can obtain

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{2(1 - \gamma)zp'(z)}{(1 - \gamma)p(z) + \gamma} = \psi(p(z), zp'(z)), \tag{18}$$

where

$$\psi(r, s) = 1 + \frac{2(1 - \gamma)s}{(1 - \gamma)r + \gamma}. \tag{19}$$

Using (15), we have

$$\left\{ \psi(p(z), zp'(z)) : z \in \mathbb{U} \right\} \subset \{w \in \mathbb{C} : \Re(w) > \alpha\} := \Omega_\alpha. \tag{20}$$

Now for all real $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1 + \rho^2)/2$,

$$\begin{aligned} \Re \{ \psi(i\rho, \sigma) \} &= 1 + \frac{2\gamma(1 - \gamma)\sigma}{(1 - \gamma)^2\rho^2 + \gamma^2} \\ &\leq 1 - \frac{\gamma(1 - \gamma)(1 + \rho^2)}{(1 - \gamma)^2\rho^2 + \gamma^2}. \end{aligned} \tag{21}$$

Define a function $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(\rho) = \frac{1 + \rho^2}{(1 - \gamma)^2\rho^2 + \gamma^2}. \tag{22}$$

Then g is a continuous even function and

$$g'(\rho) = \frac{2(2\gamma - 1)\rho}{((1 - \gamma)^2\rho^2 + \gamma^2)^2}. \tag{23}$$

Hence $g'(0) = 0$ and g is increasing on $(0, \infty)$, since $1/2 \leq \gamma < 1$. Hence g satisfies that

$$\frac{1}{\gamma^2} \leq g(\rho) < \frac{1}{(1 - \gamma)^2}, \tag{24}$$

for all $\rho \in \mathbb{R}$. Therefore, by combining (21) and (24), we can get

$$\Re \{ \psi(i\rho, \sigma) \} \leq 1 - \gamma(1 - \gamma)g(\rho) \leq 2 - \frac{1}{\gamma} = \alpha. \tag{25}$$

And this shows that $\Re\{\psi(i\rho, \sigma)\} \notin \Omega_\alpha$ for all $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1 + \rho^2)/2$. By Lemma 3, we get $\Re\{p(z)\} > 0$ for all $z \in \mathbb{U}$, and this shows that the inequality (16) holds and the proof of Theorem 4 is completed. \square

Theorem 5. Let $f \in \mathcal{A}$, $1 < \beta < 2$ and

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \beta \quad (z \in \mathbb{U}). \tag{26}$$

Then

$$\Re \left\{ \sqrt{f'(z)} \right\} < \frac{1}{2-\beta} \quad (z \in \mathbb{U}). \tag{27}$$

Proof. We put $\delta = 1/(2-\beta)$ and note that $\delta > 1$ for $1 < \beta < 2$. And let

$$p(z) = \frac{1}{1-\delta} \left(\sqrt{f'(z)} - \delta \right),$$

$$\psi(r, s) = 1 + \frac{2(1-\delta)s}{(1-\delta)r + \delta}. \tag{28}$$

As in the proof of Theorem 4, we can get

$$\left\{ \psi(p(z), zp'(z)) : z \in \mathbb{U} \right\} \subset \{w \in \mathbb{C} : \Re(w) < \beta\} := \Omega_\beta, \tag{29}$$

by (26). And for all real ρ, σ with $\sigma \leq -(1 + \rho^2)/2$,

$$\begin{aligned} \Re \{ \psi(i\rho, \sigma) \} &= 1 + \frac{2\delta(1-\delta)\sigma}{(1-\delta)^2\rho^2 + \delta^2} \\ &\geq 1 - \frac{\delta(1-\delta)(1+\rho^2)}{(1-\delta)^2\rho^2 + \delta^2} \\ &= 1 - \delta(1-\delta)g(\rho), \end{aligned} \tag{30}$$

where $g(\rho)$ is given by

$$g(\rho) = \frac{1 + \rho^2}{(1-\delta)^2\rho^2 + \delta^2}. \tag{31}$$

Since $\delta > 1$, g satisfies the inequality

$$\frac{1}{\delta^2} \leq g(\rho) < \frac{1}{(1-\delta)^2}, \tag{32}$$

for all $\rho \in \mathbb{R}$. Therefore,

$$\Re \{ \psi(i\rho, \sigma) \} \geq 1 - \delta(1-\delta)g(\rho) \geq 2 - \frac{1}{\delta} = \beta. \tag{33}$$

And this shows that $\Re \{ \psi(i\rho, \sigma) \} \notin \Omega_\beta$ for all $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1 + \rho^2)/2$. By Lemma 3, we get $\Re \{ p(z) \} > 0$ for all $z \in \mathbb{U}$, and this shows that the inequality (27) holds and the proof of Theorem 5 is completed. \square

By combining Theorems 4 and 5, we can obtain the following result.

Theorem 6. Let $f \in \mathcal{A}$, $0 \leq \alpha < 1 < \beta < 2$ and

$$\alpha < \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \beta \quad (z \in \mathbb{U}). \tag{34}$$

Then

$$\frac{1}{2-\alpha} < \Re \left\{ \sqrt{f'(z)} \right\} < \frac{1}{2-\beta} \quad (z \in \mathbb{U}). \tag{35}$$

3. Coefficient Problems Involving Functions in $\mathcal{K}(\alpha, \beta)$

In the present section, we will solve some coefficient problems involving functions in the class $\mathcal{K}(\alpha, \beta)$. And our first result on the coefficient estimates involves the function class $\mathcal{K}(\alpha, \beta)$ and the following lemma will be needed.

Lemma 7 (see Rogosinski [6, Theorem 10]). Let $p(z) = \sum_{n=1}^\infty C_n z^n$ be analytic and univalent in \mathbb{U} and suppose that $p(z)$ maps \mathbb{U} onto a convex domain. If $q(z) = \sum_{n=1}^\infty A_n z^n$ is analytic in \mathbb{U} and satisfies the following subordination:

$$q(z) < p(z) \quad (z \in \mathbb{U}), \tag{36}$$

then

$$|A_n| \leq |C_n| \quad (n = 1, 2, \dots). \tag{37}$$

Theorem 8. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. If the functions $f(z) = z + \sum_{n=2}^\infty a_n z^n \in \mathcal{K}(\alpha, \beta)$, then

$$|a_n| \leq \begin{cases} \frac{1}{2} |B_1|, & n = 2, \\ \frac{|B_1|}{n(n-1)} \prod_{k=1}^{n-2} \left(1 + \frac{|B_1|}{k} \right), & n = 3, 4, 5, \dots, \end{cases} \tag{38}$$

where $|B_1|$ is given by

$$|B_1| = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}. \tag{39}$$

Proof. Let us define

$$q(z) = 1 + \frac{zf''(z)}{f'(z)}, \tag{40}$$

$$p(z) = 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))z}}{1 - z} \right). \tag{41}$$

Then, the subordination (9) can be written as follows:

$$q(z) < p(z) \quad (z \in \mathbb{U}). \tag{42}$$

Note that the function $p(z)$ defined by (41) is convex in \mathbb{U} and has the form

$$p(z) = 1 + \sum_{n=1}^\infty B_n z^n, \tag{43}$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} i \left(1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))} \right). \tag{44}$$

If we let

$$q(z) = 1 + \sum_{n=1}^\infty A_n z^n, \tag{45}$$

then by Lemma 7, we see that the subordination (42) implies that

$$|A_n| \leq |B_1| \quad (n = 1, 2, \dots), \tag{46}$$

where

$$|B_1| = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}. \tag{47}$$

Now, equality (40) implies that

$$f'(z) + zf''(z) = q(z)f'(z) \quad (z \in \mathbb{U}). \tag{48}$$

Then, the coefficients of z^{n-1} in both sides lead to

$$a_n = \frac{1}{n(n-1)} (A_{n-1} + 2a_2A_{n-2} + 3a_3A_{n-3} + \dots + (n-1)a_{n-1}A_1). \tag{49}$$

A simple calculation combined with the inequality (46) yields that

$$\begin{aligned} |a_n| &= \frac{1}{n(n-1)} \\ &\times |A_{n-1} + 2a_2A_{n-2} + 3a_3A_{n-3} \\ &\quad + \dots + (n-1)a_{n-1}A_1| \\ &\leq \frac{1}{n(n-1)} \\ &\times (|A_{n-1}| + 2|a_2||A_{n-2}| + 3|a_3||A_{n-3}| \\ &\quad + \dots + (n-1)|a_{n-1}||A_1|) \\ &\leq \frac{|B_1|}{n(n-1)} \sum_{k=2}^n (k-1)|a_{k-1}|, \end{aligned} \tag{50}$$

where $|B_1|$ is given by (47) and $|a_1| = 1$. Hence, we have $|a_2| \leq |B_1|/2$. To prove the assertion of the theorem, we need to show that

$$\begin{aligned} |a_n| &\leq \frac{|B_1|}{n(n-1)} \sum_{k=2}^n (k-1)|a_{k-1}| \\ &\leq \frac{|B_1|}{n(n-1)} \prod_{k=1}^{n-2} \left(1 + \frac{|B_1|}{k}\right) \quad (n = 3, 4, 5, \dots). \end{aligned} \tag{51}$$

We now use the mathematical induction for the proof of the theorem. For the case $n = 3$, it is clear. We assume that the

inequality (51) holds for $n = m$. Then, some calculation gives us that

$$\begin{aligned} |a_{m+1}| &\leq \frac{|B_1|}{(m+1)m} \sum_{k=2}^{m+1} (k-1)|a_{k-1}| \\ &\leq \frac{|B_1|}{(m+1)m} \prod_{k=1}^{m-2} \left(1 + \frac{|B_1|}{k}\right) \\ &\quad + \frac{|B_1|^2}{m(m+1)(m-1)} \prod_{k=1}^{m-2} \left(1 + \frac{|B_1|}{k}\right) \\ &= \frac{|B_1|}{(m+1)m} \prod_{k=1}^{m-1} \left(1 + \frac{|B_1|}{k}\right), \end{aligned} \tag{52}$$

which implies that the inequality (51) is true for $n = m + 1$. Hence, by the mathematical induction, we prove that

$$|a_n| \leq \frac{|B_1|}{n(n-1)} \prod_{k=1}^{n-2} \left(1 + \frac{|B_1|}{k}\right) \quad (n = 3, 4, 5, \dots), \tag{53}$$

where $|B_1|$ is given by (47). This completes the proof of Theorem 8. \square

And now, we will solve the Fekete-Szegő problem for $f \in \mathcal{K}(\alpha, \beta)$, and we will need the following lemma.

Lemma 9 (see Keogh and Merkes [7]). *Let $p(z) = 1 + c_1z + c_2z^2 + \dots$ be a function with positive real part in \mathbb{U} . Then, for any complex number ν ,*

$$|c_2 - \nu c_1^2| \leq 2 \max\{1, |1 - 2\nu|\}. \tag{54}$$

Now, the following result holds for the coefficient of $f \in \mathcal{K}(\alpha, \beta)$.

Theorem 10. *Let $0 \leq \alpha < 1 < \beta$ and let the function $f(z)$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{K}(\alpha, \beta)$. Then, for a complex number μ ,*

$$\begin{aligned} &|a_3 - \mu a_2^2| \\ &\leq \frac{\beta - \alpha}{3\pi} \sin\left(\frac{1 - \alpha}{\beta - \alpha}\pi\right) \\ &\quad \cdot \max\left\{1; \left|\frac{1}{2} + \left(1 - \frac{3}{2}\mu\right)\frac{\beta - \alpha}{\pi}i\right| \right. \\ &\quad \left. + \left(\frac{1}{2} - \left(1 - \frac{3}{2}\mu\right)\frac{\beta - \alpha}{\pi}i\right) \right. \\ &\quad \left. \times e^{2\pi i((1-\alpha)/(\beta-\alpha))} \right\}. \end{aligned} \tag{55}$$

The result is sharp.

Proof. Let us consider a function $q(z)$ given by $q(z) = 1 + zf''(z)/f'(z)$. Then, since $f \in \mathcal{K}(\alpha, \beta)$, we have $q(z) < p(z)$ ($z \in \mathbb{U}$), where

$$p(z) = 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))} z}{1 - z} \right) \tag{56}$$

$$= 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where B_n is given by (11). Let

$$h(z) = \frac{1 + p^{-1}(q(z))}{1 - p^{-1}(q(z))} = 1 + h_1 z + h_2 z^2 + \dots \quad (z \in \mathbb{U}). \tag{57}$$

Then h is analytic and has positive real part in the open unit disk \mathbb{U} . We also have

$$q(z) = p \left(\frac{h(z) - 1}{h(z) + 1} \right) \quad (z \in \mathbb{U}). \tag{58}$$

We find from (58) that

$$a_2 = \frac{1}{4} B_1 h_1, \tag{59}$$

$$a_3 = \frac{1}{12} B_1 h_2 - \frac{1}{24} B_1 h_1^2 + \frac{1}{24} B_2 h_1^2 + \frac{1}{24} B_1^2 h_1^2,$$

which imply that

$$a_3 - \mu a_2^2 = \frac{1}{12} B_1 (h_2 - \nu h_1^2), \tag{60}$$

where

$$\nu = \frac{1}{2} \left(1 - \frac{B_2}{B_1} - B_1 + \frac{3}{2} \mu B_1 \right). \tag{61}$$

Applying Lemma 9, we can obtain

$$|a_3 - \mu a_2^2| = \frac{1}{12} |B_1| |h_2 - \nu h_1^2| \tag{62}$$

$$\leq \frac{1}{6} |B_1| \cdot \max \{1; |1 - 2\nu|\}.$$

And substituting

$$B_1 = \frac{\beta - \alpha}{\pi} i \left(1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))} \right), \tag{63}$$

$$B_2 = \frac{\beta - \alpha}{2\pi} i \left(1 - e^{4\pi i((1-\alpha)/(\beta-\alpha))} \right) \tag{64}$$

in (62), we can obtain the result as asserted. The estimate is sharp for the function $f : \mathbb{U} \rightarrow \mathbb{C}$ defined by

$$f(z) = \int_0^z \left\{ \exp \left(\int_0^\xi \frac{p(\xi) - 1}{\xi} d\xi \right) \right\} d\xi, \tag{65}$$

where the function p is given by (7). Hence the proof of Theorem 10 is completed. \square

Using Theorem 10, we can get the following result.

Corollary 11. Let $0 \leq \alpha < 1 < \beta$ and let the function f , given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, be in the class $\mathcal{K}(\alpha, \beta)$. Also let the function f^{-1} , defined by

$$f^{-1}(f(z)) = z = f(f^{-1}(z)), \tag{66}$$

be the inverse of f . If

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n \quad \left(|w| < r_0; r_0 > \frac{1}{4} \right), \tag{67}$$

then

$$|b_2| \leq \frac{\beta - \alpha}{\pi} \sin \left(\frac{1 - \alpha}{\beta - \alpha} \pi \right),$$

$$|b_3| \leq \frac{\beta - \alpha}{3\pi} \sin \left(\frac{1 - \alpha}{\beta - \alpha} \pi \right)$$

$$\cdot \max \left\{ 1; \left| \frac{1}{2} - \frac{2(\beta - \alpha)}{\pi} i + \left(\frac{1}{2} + \frac{2(\beta - \alpha)}{\pi} i \right) e^{2\pi i((1-\alpha)/(\beta-\alpha))} \right| \right\}.$$
(68)

Proof. The relations (66) and (67) give

$$b_2 = -a_2, \quad b_3 = 2a_2^2 - a_3. \tag{69}$$

Thus, we can get the estimate for $|b_2|$ by

$$|b_2| = |a_2| \leq \frac{1}{2} |B_1| = \frac{\beta - \alpha}{\pi} \sin \left(\frac{1 - \alpha}{\beta - \alpha} \pi \right), \tag{70}$$

immediately. Furthermore, an application of Theorem 10 (with $\mu = 2$) gives the estimates for $|b_3|$; hence, the proof of Corollary 11 is completed. \square

Finally, we will estimate some initial coefficients for the bi-univalent functions $f \in \mathcal{K}_\sigma(\alpha, \beta)$.

Theorem 12. For given α and β such that $0 \leq \alpha < 1 < \beta$, let f given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, be in the class $\mathcal{K}_\sigma(\alpha, \beta)$. Then

$$|a_2| \leq \frac{|B_1| \sqrt{|B_1|}}{\sqrt{2|B_1^2 - 2B_1 + 2B_2|}}, \tag{71}$$

$$|a_3| \leq \frac{1}{2} (|B_1| + |B_2 - B_1|), \tag{72}$$

where B_1 and B_2 are given by (63) and (64).

Proof. If $f \in \mathcal{K}_\sigma(\alpha, \beta)$, then $f \in \mathcal{K}(\alpha, \beta)$ and $g \in \mathcal{K}(\alpha, \beta)$, where $g = f^{-1}$. Hence

$$Q(z) := 1 + \frac{zf''(z)}{f'(z)} < p(z) \quad (z \in \mathbb{U}), \tag{73}$$

$$L(z) := 1 + \frac{zg''(z)}{g'(z)} < p(z) \quad (z \in \mathbb{U}),$$

where $p(z)$ is given by (7). Let

$$h(z) = \frac{1 + p^{-1}(Q(z))}{1 - p^{-1}(Q(z))} = 1 + h_1z + h_2z^2 + \dots \quad (z \in \mathbb{U}), \tag{74}$$

$$k(z) = \frac{1 + p^{-1}(L(z))}{1 - p^{-1}(L(z))} = 1 + k_1z + k_2z^2 + \dots \quad (z \in \mathbb{U}).$$

Then h and k are analytic and have positive real part in \mathbb{U} . Also, we have

$$Q(z) = p\left(\frac{h(z) - 1}{h(z) + 1}\right) \quad (z \in \mathbb{U}), \tag{75}$$

$$L(z) = p\left(\frac{k(z) - 1}{k(z) + 1}\right) \quad (z \in \mathbb{U}).$$

By suitably comparing coefficients, we get

$$a_2 = \frac{1}{4}B_1h_1, \tag{76}$$

$$3a_3 - 2a_2^2 = \frac{1}{4}B_1h_2 - \frac{1}{8}B_1h_1^2 + \frac{1}{8}B_2h_1^2, \tag{77}$$

$$b_2 = \frac{1}{4}B_1k_1, \tag{78}$$

$$3b_3 - 2b_2^2 = \frac{1}{4}B_1k_2 - \frac{1}{8}B_1k_1^2 + \frac{1}{8}B_2k_1^2, \tag{79}$$

where B_1 and B_2 are given by (63) and (64), respectively. Now, considering (76) and (78), we get

$$h_1 = -k_1. \tag{80}$$

Also, from (77), (78), (79), and (80), we find that

$$a_2^2 = \frac{B_1^3(h_2 + k_2)}{8(B_1^2 - 2B_1 + 2B_2)}. \tag{81}$$

Therefore, we have

$$|a_2^2| \leq \frac{|B_1|^3}{8|B_1^2 - 2B_1 + 2B_2|} (|h_2| + |k_2|) \leq \frac{|B_1|^3}{2|B_1^2 - 2B_1 + 2B_2|}. \tag{82}$$

This gives the bound on $|a_2|$ as asserted in (71). Now, further computations from (77), (79), (80), and (81) lead to

$$a_3 = \frac{1}{12}B_1(2h_2 + k_2) + \frac{1}{8}h_1^2(B_2 - B_1). \tag{83}$$

This equation, together with the well-known estimates

$$|h_1| \leq 2, \quad |h_2| \leq 2, \quad |k_2| \leq 2, \tag{84}$$

leads us to the inequality (72). Therefore, the proof of Theorem 12 is completed. \square

Acknowledgment

The research was supported by Kyung Sung University Research Grants in 2013.

References

- [1] K. Kuroki and S. Owa, "Notes on new class for certain analytic functions," *RIMS Kokyuroku*, vol. 1772, pp. 21–25, 2011.
- [2] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, "Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 344–351, 2012.
- [3] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1188–1192, 2010.
- [4] Q.-H. Xu, Y.-C. Gui, and H. M. Srivastava, "Coefficient estimates for a certain subclass of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 25, no. 6, pp. 990–994, 2012.
- [5] S. S. Miller and P. T. Mocanu, *Differential Subordinations, Theory and Applications*, Marcel Dekker, New York, NY, USA, 2000.
- [6] W. Rogosinski, "On the coefficients of subordinate functions," *Proceedings of the London Mathematical Society*, vol. 48, pp. 48–82, 1943.
- [7] F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proceedings of the American Mathematical Society*, vol. 20, pp. 8–12, 1969.