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PROPERTY PRESERVING OPERATORS

EVELYN M. SILVIA

Let S denote the class of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ that are analytic and univalent in $|z| < 1$. Given $f \in S$ and a, b, c, real numbers other than $0, -1, -2, \ldots$, let $\Omega(a,b,c;f) = F(a,b,c;z)^{*} f(z)$ where $z^{-1}F(a,b,c;z) = 1 + \sum_{k=1}^{\infty} \big((a)_k (b)_k\big)/\big((c)_k (1)_k\big) z^k$ is a hypergeometric Gauss function with $(a)_0 = 1$ and $(a)_k = a(a+1)\ldots(a+k-1)$ and * denotes the Hadamard product. For $q_n(z) = z + a_2 z^2 + \ldots + a_n z^n$ $(a_n \neq 0,\ n = 5, 6)$ in S, it is shown that $\Omega(\gamma+1, 1, \gamma+2; q_n) = \Phi_\gamma(q_n) = ((\gamma+1)/z^\gamma)\int_0^z t^{\gamma-1} q_n(t)\,dt$, $\gamma > -1$, is univalent in $|z| < 1$. This extends the result previously known for $n = 3$ and $n = 4$. Also, we obtain a necessary and sufficient condition involving a, b, and c such that $\Omega(a,b,c;\cdot)$ preserves the subclass of S consisting of starlike functions of order α, $0 \leqslant \alpha \leqslant 1$, with $a_k \leqslant 0$.

1. INTRODUCTION

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic and univalent in $\Delta = \{z : |z| < 1\}$, with $S^{*}(\alpha)$, $0 \leqslant \alpha \leqslant 1$, designating the subclass of S consisting of functions starlike of order α. We shall denote by T the subclass of S consisting of functions that may be expressed in the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geqslant 0,$$

and will set $T^{*}(\alpha) = T \cap S^{*}(\alpha)$. It is known [10] that $f \in T^{*}(\alpha)$ if and only if its coefficients satisfy the inequality

$$(1) \qquad \sum_{n=2}^{\infty} (n-\alpha) a_n \leqslant (1-\alpha).$$

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The class of polynominals of degree n, $q_n(z) = z + \sum_{k=2}^{n} a_k z^k$, $a_n \neq 0$, that are univalent in Δ will be designated by P_n. In the next section, we will consider the general integral operator

$$\Phi_\gamma(f(z)) = \frac{(\gamma+1)}{z^\gamma} \cdot \int_0^z t^{\gamma-1} f(t)dt \qquad (\gamma > -1).$$

The Hadamard product or convolution of two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } h(z) = \sum_{n=0}^{\infty} c_n z^n$$

is defined as the power series

$$(f \star h)(z) = \sum_{n=0}^{\infty} a_n c_n z^n.$$

For $G(z) = \sum_{n=1}^{\infty} ((\gamma+1)/(\gamma+n))z^n$ we note that $\Phi_\gamma(f(z)) = (f \star G)(z)$. Since $G(z)$ is known [**8**] to be convex for $\gamma > 0$, it follows from the work of Ruscheweyh and Sheil-Small [**9**] that $\Phi_\gamma(f)$, $\gamma > 0$, is close-to-convex or starlike of order α whenever $f(z)$ is such. It was shown in [**12**] that for $f \in T^\star(\alpha)$ we actually have $\Phi_\gamma(f(z)) \in T^\star((2+\alpha\gamma)/(3+\gamma-\alpha))$ which is a little better than we get from closure under convolution with a convex function.

The question of preservation of univalence under Φ_γ is still relatively open for discussion. In [**5**] an example of an $f(z)$ univalent in Δ with $\Phi_0(f)$ not univalent is given. For $\gamma = 1$, the radius of close-to-convexity for S [**4**] assures the univalence of $\Phi_\gamma(f(z))$, $f \in S$, in $|z| < \rho$ where $0.80 < \rho < 0.81$. Whether ρ can be replaced by 1 is still unknown. In [**6**], it is shown that if $f \in P_n$, then $\Phi_0(f)$ is univalent for $|z| < 2\sin(\pi/n)$ and $\Phi_1(f)$ is univalent for $|z| < 2\sin(\pi/(n+1))$. Hence, Φ_0 preserves P_n for $n \leqslant 6$ and Φ_1 preserves P_n for $n \leqslant 5$. Finally, from [**11**], we know that $\Phi_\gamma(P_n) \subset P_n$ for $n = 3$, 4 and for all $\gamma > -1$. In Section 2, we extend the latter result to $n = 5$ and $n = 6$. In Section 3, we will consider a generalisation of the operator Φ_γ.

For $f \in S$, and a, b, c real numbers other than $0, -1, -2, \ldots$, let

$$\Omega(a,b,c;f) = F(a,b,c;z) \star f(z)$$

where

$$z^{-1}F(a,b,c;z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k$$

is a hypergeometric Gauss function and $(a)_0 = 1$, $(a)_k = a(a+1)\cdots(a+k-1)$. Note that $\Omega(\gamma+1, 1, \gamma+2; f) = \Phi_\gamma(f)$.

In [**11**], it was shown that for $q \in P_3$ and $c \geqslant |a| > 0$, $\Omega(a, 1, c; q) \in P_3$. Let $\Sigma_k(a, b, c; z)$ denote the kth partial sum of $F(a, b, c; z)$. We know that $\Sigma_2(a, b, c; z)$ is convex in Δ if and only if

$$4|a|\,|b| \leqslant |c|. \tag{2}$$

In [**2**], it is shown that for $f(z) = z + \beta z^2 + \delta z^3$, β, δ real, and $0 \leqslant \delta \leqslant 1/15$, the condition $(1+9\delta)/4 \geqslant \beta \geqslant 8\delta/(1+5\delta)$ implies that f is convex. Thus, $\Sigma_3(a, b, c; z)$ is convex for $0 \leqslant (a)_2(b)_2/(c)_2 \leqslant 2/15$ and

$$\frac{2(c)_2 + 9(a)_2(b)_2}{8(c)_2} \geqslant \frac{a \cdot b}{c} \geqslant \frac{8(a)_2(b)_2}{2(c)_2 + 5(a)_2(b)_2}. \tag{3}$$

It follows that $\Omega_2(a, b, c; \cdot)$ and $\Omega_3(a, b, c; \cdot)$ preserve the subsets of P_2 and P_3 consisting of functions that are convex, starlike of order α and close-to-convex as long as (2) and (3) are satisfied, respectively. In the last section, we obtain a necessary and sufficient condition involving a, b and c such that $\Omega(a, b, c; \cdot)$ preserves the class $T^\star(\alpha)$.

2. The operator Φ_γ

In order to show that P_n is preserved under Φ_γ for $n = 5$ and $n = 6$, we will use two lemmas.

LEMMA A. [**11**] *For $q_k(z) = z + a_2 z^2 + \ldots + a_k z^k \in P_k$, a sufficient condition for $\Phi_\gamma(q_k)$ to be in P_k is that the polynomial*

$$G_{k-1,\gamma}(z) = \sum_{j=0}^{k-1} \binom{k-1}{j} \cdot \left[\frac{\gamma+1}{\gamma+k-j}\right] z^j$$

have all of its zeros in $|z| \leqslant 1$.

LEMMA B. [**7**] *(Cohn's Rule) For $f(z) = a_0 + a_1 z + \ldots + a_n z^n$, let $f^\star(z) = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \ldots + \bar{a}_n$. Then, if $|a_0| < |a_n|$, the polynomial f_1 given by $z f_1(z) = \bar{a}_n f(z) - a_0 f^\star(z)$ has one zero less than f has in Δ.*

Given a polynomial of degree n, as long as it is applicable, we can use Lemma B successively $n-1$ times to obtain a first degree polynomial. It follows that if the zero of the first degree polynomial is in Δ, then all n zeros of the original polynomial lie in Δ. Using this method we have:

Theorem 1. *For $2 \leqslant k \leqslant 6$, if $q_k \in P_k$, then $\Phi_\gamma(q_k) \in P_k$ for all $\gamma > -1$.*

Proof: For $k = 2$, the result is trivial. The cases $k = 3$ and $k = 4$ were obtained earlier [**11**]. For $k = 5$, from Lemma A, it suffices to show that all the zeros of

$$G(z) = \frac{\gamma+1}{\gamma+5} + 4\left(\frac{\gamma+1}{\gamma+4}\right)z + 6\left(\frac{\gamma+1}{\gamma+3}\right)z^2 + 4\left(\frac{\gamma+1}{\gamma+2}\right)z^3 + z^4$$

lie in $|z| \leqslant 1$. Since $(\gamma+1)/(\gamma+5) < 1$, Lemma B applies and we can form

$$z\tilde{G}_1(z) = 1 \cdot G(z) - \frac{\gamma+1}{\gamma+5} \cdot G^\star(z)$$

from which we obtain

$$\begin{aligned} G_1(z) &= \frac{(\gamma+5)^2}{8(\gamma+3)} \cdot \tilde{G}_1(z) \\ &= \frac{(\gamma+1)(\gamma+5)}{(\gamma+2)(\gamma+4)} + 3\frac{(\gamma+1)(\gamma+5)}{(\gamma+3)^2}z + 3\frac{(\gamma+1)(\gamma+5)}{(\gamma+2)(\gamma+4)}z^2 + z^3. \end{aligned}$$

For $\gamma > -1$, we have $(\gamma+1)(\gamma+5)/((\gamma+2)(\gamma+4)) < 1$. To apply Lemma B we form

$$z\tilde{G}_2(z) = 1 \cdot G_1(z) - \frac{(\gamma+1)(\gamma+5)}{(\gamma+2)(\gamma+4)} \cdot G_1^\star(z).$$

This leads to

$$\begin{aligned} G_2(z) &= \frac{(\gamma+2)^2(\gamma+4)^2}{3(2\gamma^2+12\gamma+13)} \cdot \tilde{G}_2(z) \\ &= \frac{(\gamma+1)(\gamma+5)(2\gamma^2+12\gamma+19)}{(\gamma+3)^2(2\gamma^2+12\gamma+13)} + 4\frac{(\gamma+1)(\gamma+2)(\gamma+4)(\gamma+5)}{(\gamma+3)^2(2\gamma^2+12\gamma+13)}z + z^2 \\ &\equiv \mu + 4\lambda z + z^2. \end{aligned}$$

Once again we have $\mu < 1$ for $\gamma > -1$, so we let

$$z\tilde{G}_3(z) = 1 \cdot G_2(z) - \mu \cdot G_2^\star(z)$$

and obtain

$$G_3(z) = \frac{1}{1-\mu^2} \cdot \tilde{G}_3(z) = \frac{4\lambda}{1+\mu} + z.$$

Now, since $0 < \mu < 1$, $4\lambda/(1+\mu) < 1$ if and only if

$$\begin{aligned} 4(\gamma+1)(\gamma+2)(\gamma+4)(\gamma+5) < (\gamma+3)^2&(2\gamma^2+12\gamma+13) \\ &+ (\gamma+1)(\gamma+5)(2\gamma^2+12\gamma+19) \end{aligned}$$

which is equivalent to $4(2\gamma^2+12\gamma+13) > 0$. This last inequality is satisfied for $\gamma > -1$. Therefore, G_3 has one root in Δ. Applying Lemma B sequentially, it follows that G_2 has 2 roots in Δ, G_1 has 3 roots there and, finally, G has all 4 roots in Δ. Thus, by Lemma A, $\Phi_\gamma(P_5) \subset P_5$.

The process detailed for $k=5$ goes just as smoothly for $k=6$. To apply Lemma A, we consider

$$H(z) = \sum_{j=0}^{5} \binom{5}{j} [\frac{\gamma+1}{\gamma+6-j}] z^j.$$

We obtain the following finite sequence of auxiliary polynomials

$$\begin{aligned} H_1(z) &= \frac{(\gamma+1)(\gamma+6)}{(\gamma+2)(\gamma+5)} + 4\frac{(\gamma+1)(\gamma+6)}{(\gamma+3)(\gamma+4)}z + 6\frac{(\gamma+1)(\gamma+6)}{(\gamma+3)(\gamma+4)}z^2 \\ &\quad + 4\frac{(\gamma+1)(\gamma+6)}{(\gamma+2)(\gamma+5)}z^3 + z^4 \\ H_2(z) &= \frac{(\gamma+1)(\gamma+6)(\gamma^2+7\gamma+14)}{(\gamma+3)(\gamma+4)(\gamma^2+7\gamma+8)} + 3\frac{(\gamma+1)(\gamma+2)(\gamma+5)(\gamma+6)}{(\gamma+3)(\gamma+4)(\gamma^2+7\gamma+8)}z \\ &\quad + 3\frac{(\gamma+1)(\gamma+2)(\gamma+5)(\gamma+6)}{(\gamma+3)(\gamma+4)(\gamma^2+7\gamma+8)}z^2 + z^3, \\ H_3(z) &= \xi + \xi z + z^2 \qquad \text{for } \xi = \frac{(\gamma+1)(\gamma+2)(\gamma+5)(\gamma+6)}{\gamma^4+14\gamma^3+69\gamma^2+140\gamma+90}, \end{aligned}$$

and

$$H_4(z) = z + \frac{\xi}{1+\xi}.$$

Since $\gamma^4+14\gamma^3+69\gamma^2+140\gamma+90 = \frac{1}{2}(A+B)$ where

$$A = (\gamma+3)(\gamma+4)(\gamma^2+7\gamma+8)$$

and

$$B = (\gamma+1)(\gamma+6)(\gamma^2+7\gamma+14),$$

we know that $1/2(A+B) > 0$ for $\gamma > -1$ and $\xi > 0$. Also, $\xi < 1$ if and only if

$$(\gamma+1)(\gamma+2)(\gamma+5)(\gamma+6) < (\gamma^4+14\gamma^3+69\gamma^2+140\gamma+90)$$

which is equivalent to

$$2(\gamma^2+14\gamma+15) > 0$$

and is satisfied for $\gamma > -1$. Therefore, $\xi/(\xi+1) < 1$. We conclude that H_4 has one root in Δ and H has 5 roots there. ∎

Remarks 1. To see that the sufficient condition is not met for $k = 7$, $\gamma > -1$, Lemma B proves to be a bit unwieldy. Instead we can appeal to the Schur-Cohn Criteria [7]: If for the polynomial $f(z) = a_0 + a_1 z + \ldots + a_n z^n$, all the determinants

$$\Delta_k = \begin{vmatrix} a_0 & 0 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_{n-k+1} \\ a_1 & a_0 & 0 & \cdots & 0 & 0 & a_n & \cdots & a_{n-k+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_0 & 0 & 0 & \cdots & a_n \\ \bar{a}_n & 0 & 0 & \cdots & 0 & \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_{k-1} \\ \bar{a}_{n-1} & \bar{a}_n & 0 & \cdots & 0 & 0 & \bar{a}_0 & \cdots & \bar{a}_{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{n-k+1} & \bar{a}_{n-k+2} & \bar{a}_{n-k+3} & \cdots & \bar{a}_n & 0 & 0 & \cdots & \bar{a}_0 \end{vmatrix}$$

for $k = 1, 2, 3, \ldots, n$ are different from 0, then f has no zeros on the circle $|z| = 1$ and p zeros in this circle, where p is the number of variations of sign in the sequence $1, \Delta_1, \Delta_2, \ldots, \Delta_n$. Thus in order for f to have n zeros in Δ the sequence must have alternating signs. For the case $k = 7$ we consider

$$F(z) = \sum_{j=0}^{6} \binom{6}{j} \left(\frac{\gamma + 1}{\gamma + 7 - j} \right) z^j.$$

Then, for $\gamma > -1$, $\Delta_0 = 1$, $\Delta_1 = (-12(\gamma + 4))/\left((\gamma + 7)^2\right) < 0$, and

$$\Delta_2 = \frac{720(\gamma + 4)^2 (2\gamma^2 + 16\gamma + 19)}{(\gamma + 2)^2 (\gamma + 6)^2 (\gamma + 7)^4} > 0.$$

However,

$$\Delta_3 = \frac{345,600(\gamma + 4)^3 (\gamma^2 + 8\gamma - 3)(2\gamma^4 + 32\gamma^3 + 179\gamma^2 + 408\gamma + 279)}{(\gamma + 2)^4 (\gamma + 3)^2 (\gamma + 5)^2 (\gamma + 6)^4 (\gamma + 7)^6}$$

is positive for $\gamma > -4 + \sqrt{19}$. Thus, at least for $\gamma > -4 + \sqrt{19}$, the sufficient condition given in Theorem 1 is not met.

2. As noted earlier, Φ_0 does not preserve the class S [5]. Thus, we know that there exists a univalent polynomial p, such that $\Phi_0(p) \notin S$. We've also noted that it is an open problem as to whether $\Phi_\gamma(P_n) \subset S$ for $\gamma > 0$ and all $n = 1, 2, \ldots$. The sufficient condition of univalence of $\Phi_\gamma(P_n)$ not being met for $n = 7$ suggests that we try to show that $\Phi_0(P_7) \not\subseteq P_7$. It is natural to consider the polynominals

$$p(z; n; j) = z + \sum_{k=2}^{n} \left(\frac{n - k + 1}{n} \cdot \frac{\sin(kj\pi/(n + 1))}{\sin(j\pi/(n + 1))} \right) z^k$$

which were shown to be univalent in Δ by Suffridge [**13**]. On the other hand, there are reasons for doubting that $\Phi_\gamma(p(z;7;j)) \notin S$ for $j = 1,2,\ldots,7$. In particular, it can be shown directly that for $\Phi_\gamma(p(z;7;j)) = z + \sum_{k=2}^{7} b_{j,k}z^k$,

$$|b_{j,k} + b_{j,8-k}| \leqslant (1 + b_{j,7}) \cdot \frac{\sin(k\pi/8)}{\sin(\pi/8)}, \qquad (k = 2,3,\ldots,7)$$

for each $j = 1,2,\ldots,7$ and for all $\gamma > -1$. This set of coefficient conditions was shown in [**13**] to be necessary for univalence. In addition, we have used a symbolic manipulation program and the Schur-Cohn Criteria to verify that, for $j = 1,2,\ldots,7$, the derivative of each $\Phi_\gamma(p(z;7;j))$ is nonzero in Δ for $\gamma = 0,1,2,\ldots,15$. Since neither of the conditions is sufficient for univalence, this leaves us with the following

Open Problem 1. Find a univalent polynomial of degree 7, p, such that $\Phi_\gamma(p)$ is not univalent for some $\gamma > -1$.

3. Since for $q_n \in P_n$, $\lim_{\gamma\to\infty} \Phi_\gamma(q_n) = q_n \in P_n$, and $(\gamma+1)/(\gamma+n) < 1$, it is also natural to pose

Open Problem 2. For γ large enough, show that $\Phi_\gamma(P_n) \subset P_n$ for all n.

3. THE OPERATOR $\Omega(a,b,c;\cdot)$.

Using a method due to Khokhlov [**3**], we obtain:

THEOREM 2. *A necessary and sufficient condition such that* $\Omega(a,b,c;T^*(\alpha)) \subset T^*(\alpha)$ *is that* $a > 0$, $b > 0$, $c > a+b$ *and* $\Gamma(c-a-b)\Gamma(c) \leqslant 2\Gamma(c-a)\Gamma(c-b)$.

PROOF: For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T^*(\alpha)$, let

$$g(z) = \Omega(a,b,c;f) = z - \sum_{n=2}^{\infty} d_n z^n$$

where $d_n = \big((a)_{n-1}(b)_{n-1}\big)/\big((c)_{n-1}(1)_{n-1}\big) \cdot a_n \geqslant 0$. From (1), $g \in T^*(\alpha)$ if and only if $\sum_{n=2}^{\infty} ((n-\alpha)/(1-\alpha))|d_n| \leqslant 1$. We also know [**10**] that $|a_n| \leqslant \dfrac{1-\alpha}{n-\alpha}$. Thus,

$$\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}\right)|d_n| \leqslant \sum_{n=2}^{\infty}\left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right) = \left\{1 + \sum_{n=1}^{\infty}\left(\frac{(a)_n(b)_n}{(c)_n(1)_n}\right)\right\} - 1.$$

It is well-known [**14**] that $F(a,b,c;z)$ is convergent in Δ for $c > a+b$ and $F(a,b,c;1) = \Gamma(c-a-b)\Gamma(c)/(\Gamma(c-a)\Gamma(c-b))$. Therefore, for $c > a+b$, we have $\sum_{n=2}^{\infty} ((n-\alpha)/(1-\alpha))|d_n| \leqslant 1$ if and only if $(\Gamma(c-a-b)\Gamma(c)/\Gamma(c-a)\Gamma(c-b)) - 1 \leqslant 1$. ∎

Remark. For $c \geqslant 3$, we note that $\Omega(1,1,c;T^\star(\alpha)) \subset T^\star(\alpha)$. Therefore, for $n \geqslant 2$, the generalised Biernacki operators

$$n!z^{1-n}\int_0^z\int_0^{\tau_n}\dots\int_0^{\tau_2}\frac{f(\tau_1)}{\tau_1}d\tau_1\dots d\tau_n$$

preserve the class $T^\star(\alpha)$.

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Department of Mathematics,
University of California, Davis,
Davis, CA 95616
United States of America.