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Approximations for strongly singular evolution equations ☆

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Abstract

The problem of specification of self-adjoint operators corresponding to singular bilinear forms is very important for applications, such as quantum field theory and theory of partial differential equations with coefficient functions being distributions. In particular, the formal expression $-\Delta + g\delta(\mathbf{x})$ corresponds to a non-trivial self-adjoint operator $\hat{H}$ in the space $L^2(\mathbb{R}^d)$ only if $d \leqslant 3$. For spaces of larger dimensions (this corresponds to the strongly singular case), the construction of $\hat{H}$ is much more complicated: first one should consider the space $L^2(\mathbb{R}^d)$ as a subspace of a wider Pontriagin space, then one implicitly specifies $\hat{H}$. It is shown in this paper that Schrodinger, parabolic and hyperbolic equations containing the operator $\hat{H}$ can be approximated by explicitly defined systems of evolution equations of a larger order. The strong convergence of evolution operators taking the initial condition of the Cauchy problem to the solution of the Cauchy problem is proved.

Keywords: Strong resolvent convergence; Singular bilinear forms; Pontriagin space; Schrodinger equation; Abstract parabolic and hyperbolic equations; Semigroup of operators; Self-adjoint extensions

1. Introduction

1. The main difficulty of the quantum field theory is the problem of divergences [5] which arise since the evolution equations of quantum field theory are ill-defined. It is

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suitable to investigate such problems, making use of the simpler quantum mechanical models which illustrate some of the difficulties of the quantum field theory. One of such models is the Schrodinger equation for the particle moving in the external singular potential

$$i\frac{d\psi(t)}{dt} = \hat{H}\psi(t), \tag{1.1}$$

where $t \in \mathbb{R}$, $\psi(t) \in \mathscr{H} = L^2(\mathbb{R}^d)$, while the Hamiltonian operator $\hat{H}$ is formally written as $\hat{H} = -\Delta + \varphi(\mathbf{x})$, and $\varphi(\mathbf{x})$ is the operator of multiplication by a distribution. Such models were considered in [2,4,6,7,10,13,14,18,20,22–25]. As an example, one can consider the function $\varphi(\mathbf{x}) = a + g\delta(\mathbf{x})$, where $a > 0$, $g \in \mathbb{R}$. Then

$$\hat{H}_g = -\Delta + a + g\delta(\mathbf{x}). \tag{1.2}$$

The more general example of $\hat{H}$ (compared to (1.2)) is the following formal expression:

$$\hat{H}_g\psi = \hat{T}\psi + g\chi(\chi, \psi). \tag{1.3}$$

Here $\hat{T}$ is a positively definite self-adjoint operator in $\mathscr{H}$. Making use of the operator $\hat{T}$, construct the scale of Hilbert spaces $\cdots \subset \mathscr{H}^2 \subset \mathscr{H}^1 \subset \mathscr{H} = \mathscr{H}^0 \subset \mathscr{H}^{-1} \subset \mathscr{H}^{-2} \subset \cdots$. The space $\mathscr{H}^k$ is a completion of the subspace $\bigcap_{n=1}^{\infty} D(\hat{T}^n)$ of the space $\mathscr{H}$ with respect to the norm $||\psi||_k^2 = \langle \psi, \psi \rangle_k = (\psi, \hat{T}^k\psi)$. The function χ entering Eq. (1.3) should belong to the space $\mathscr{H}^{-k}$ for some k. Expression (1.2) is a partial case of (1.3) for $\hat{T} = -\Delta + a$, $\chi(\mathbf{x}) = \delta(\mathbf{x}) \in \mathscr{H}^{-k}$ at $k > d/2$.

To define Eq. (1.1) mathematically, one should specify a self-adjoint operator in $\mathscr{H}$ corresponding to the formal expression (1.3) (in particular, (1.2)). For $\chi \in \mathscr{H}^{-2}$ $\mathscr{H}$ $(d \leqslant 3)$, this problem is solved as follows [4]. One should consider the restriction of the operator $\hat{T}$ to the domain

$$\left\{ \psi \in \bigcap_{n=1}^{\infty} D(\hat{T}^n) | (\hat{T}^{-k}\chi, \hat{T}^k\psi) = 0 \right\} \tag{1.4}$$

(for the partial case (1.2) the domain is $\{\psi \in S(\mathbb{R}^d) | \psi(0) = 0\}$). One justifies that the defect indices of this symmetric operator are (1,1). Making use of the standard procedure (see, for example, [1]), one constructs the one-parametric set $\{\hat{H}_g\}$ of self-adjoint extensions of the operator $\hat{T}$. It is in one-to-one correspondence to the one-parametric set of formal expressions (1.3).

2. For the strongly singular case, i.e. for $\chi \notin \mathscr{H}^{-2}$ $(d > 3)$, the operator $\hat{T}$ considered on domain (1.4) is essentially self-adjoint, so that the considered approach does not allow us to construct a non-trivial self-adjoint operator corresponding to the formal expression (1.3). It was noted in [3,22,23,29] that one should consider an indefinite inner product space instead of the space $\mathscr{H}$ in order to construct a non-trivial self-adjoint operator $\hat{H}$ in the strongly singular cases.

A self-adjoint operator $\hat{H}$ in the Pontriagin space Π_m [8] corresponding to expression (1.3) was specified in [24] (see also [25]), provided that $\chi \in \mathscr{H}^{-k-1} \backslash \mathscr{H}^{-k}$ for some k. Here $m = [k/2]$. One-parametric set of formal expressions (1.3) corresponds to the k-parametric set of operators $\hat{H}$ in $\Pi_{[k/2]}(g_2, \dots, g_k)$; $k-1$ parameters specifies the inner product, while one parameter is an analog of g. Denote the operator constructed in [24] (see Section 2) as $\hat{H}(g_2, \dots, g_k, \alpha)$. Therefore, the equation

$$i \frac{d\psi(t)}{dt} = \hat{H}(g_2, \dots, g_k, \alpha)\psi(t) \tag{1.5}$$

for $\psi(t) \in \Pi_{[k/2]}(g_2, \dots, g_k)$ is defined.

According to the analog of the Stone theorem for the Pontriagin spaces [17,21], the operator $\hat{H}$ is a generator of a one-parametric group of unitary operators $e^{-i\hat{H}t}$ in Π_m. The operator $\hat{U}(t) = e^{-i\hat{H}(g_2, \dots, g_k, \alpha)t}$ restricted to $D(\hat{H}(g_2, \dots, g_k, \alpha))$ is an operator taking the initial condition of the Cauchy problem for Eq. (1.1) to the solution of Eq. (1.1).

3. The problem of constructing approximations of singular equations (1.5) often arises [2]. This problem is also important for quantum field theory [16].

It was shown in [13] that for $m = 0$ the operator transforming the initial condition for the Cauchy problem to the solution of the Cauchy problem for Eq. (1.5) can be approximated in the strong sense as $n \to \infty$ by the evolution operator for the equation

$$i \frac{d\psi_n(t)}{dt} = \hat{T}\psi_n(t) + g_n \chi_n(\chi_n, \psi_n(t)), \quad \psi_n(t) \in \mathscr{H} \tag{1.6}$$

provided that

$$||\hat{T}^{-1}\chi_n - \hat{T}^{-1}\chi|| \to_{n\to\infty} 0, \qquad g_n^{-1} + (\chi_n, \hat{T}^{-1}\chi_n) \to_{n\to\infty} -\alpha^{-1}. \tag{1.7}$$

Note that for all $\chi \in \mathscr{H}^{-2}$ there exist sequences $g_n \in \mathbb{R}$, $\chi_n \in \mathscr{H}$ obeying (1.7), for example, $\chi_n = e^{-\hat{T}/n}\chi$, $g_n = -(\alpha^{-1} + (\chi_n . \hat{T}^{-1}\chi_n))^{-1}$.

This paper deals with the construction of an approximation for Eq. (1.5) for the strongly singular case (for $m \neq 0$ or $k > 1$). Approximation (1.6) cannot be applied then. It happens that the resolving operator for the Cauchy problem for Eq. (1.5) (the t-dependent operator transforming the initial condition of the Cauchy problem to the solution of Eq. (1.5) at fixed t) can be viewed as a limit as $n \to \infty$ of resolving operators for the Cauchy problem of the system of differential equations of a larger order

$$\begin{aligned} & i \frac{d\psi_n(t)}{dt} = \hat{T}\psi_n(t) + c_n(t)\chi_n, \\ & z_{0,n}c_n(t) + iz_{1,n}\frac{dc_n(t)}{dt} + \cdots + i^{k-1}z_{k-1,n}\frac{d^{k-1}c_n(t)}{dt^{k-1}} = (\chi_n, \psi_n(t)). \end{aligned} \tag{1.8}$$

Here $c_n(t) \in \mathbb{C}$ is a complex function, and $\psi_n(t)$ is an element of the space $\mathcal{H}$. The limit should be considered in a generalized strong sense [11,28]. The following conditions are imposed:

$$z_{s,n} + (\chi_n, \hat{T}^{-s-1}\chi_n) \to_{n\to\infty} g_s, \quad s = \overline{0, k-1},$$

$$||\hat{T}^{-\frac{k+1}{2}}(\chi_n - \chi)|| \to_{n\to\infty} 0. \tag{1.9}$$

Here $g_1 = -\alpha^{-1}$.

For the partial case $k = 1$, the left-hand side of the second equation of system (1.8) contains only one term. Therefore, system (1.8) is equivalent to Eq. (1.6). If one increases k, the number of parameters $z_{s,n}$ is also increased, so that the terms with derivatives of higher orders appear. The procedure of adding such terms ("counter-terms") is analogous to quantum field theory procedure of infinite renormalization of the wave function [5].

In particular, the Schrodinger equation with the δ-potential which was constructed in [24] is formally written as

$$i\frac{\partial\psi(x,t)}{\partial t} = [-\Delta + a + g\delta(x)]\psi(x,t), \quad x \in \mathbb{R}^d.$$

It appears to be the limit as $n \to \infty$ of the system of equations on $\psi_n(x,t)$ and $c_n(t)$

$$i\frac{\partial\psi_n(x,t)}{\partial t} = [-\Delta + a]\psi_n(x,t) + c_n(t)\chi_n(x),$$

$$z_{0,n}c_n(t) + \cdots + i^{k-1}z_{k-1,n}\frac{d^{k-1}c_n(t)}{dt^{k-1}} = \int dy\, \chi_n(y)\psi_n(y,t),$$

provided that $k = [d/2]$, $\chi_n \to \delta$ in the $\mathcal{H}^{-k-1}$-norm and sequences $z_{s,n} + (\chi_n, \hat{T}^{-s-1}\chi_n)$, $s = \overline{0, k-1}$, are convergent as $n \to \infty$.

4. Besides Schrodinger equation for the particle moving in the singular potential, other equations appear in the applications. Evolution of relativistic particle in the external scalar field is described by the Klein–Gordon-type equation [5]

$$-\frac{d^2\psi(t)}{dt^2} = \hat{H}(g_2, \ldots, g_k, \alpha)\psi(t). \tag{1.10}$$

The Schrodinger equation in the imaginary time is also considered

$$-\frac{d\psi(t)}{dt} = \hat{H}(g_2, \ldots, g_k, \alpha)\psi(t). \tag{1.11}$$

After specifying the operator $\hat{H}(g_2, \dots, g_k, \alpha)$ Eqs. (1.10) and (1.11) become well defined. It happens that Eq. (1.10) can be approximated by the system

$$-\frac{d^2\psi_n(t)}{dt^2} = \hat{T}\psi_n(t) + c_n(t)\chi_n,$$

$$z_{0,n}c_n(t) - z_{1,n}\frac{d^2c_n(t)}{dt^2} + \dots + (-1)^{k-1}z_{k-1,n}\frac{d^{2k-2}c_n(t)}{dt^{2k-2}} = (\chi_n, \psi_n(t)), \tag{1.12}$$

while the approximation for Eq. (1.11) is

$$-\frac{d\psi_n(t)}{dt} = \hat{T}\psi_n(t) + c_n(t)\chi_n,$$

$$z_{0,n}c_n(t) - z_{1,n}\frac{dc_n(t)}{dt} + \dots + (-1)^{k-1}z_{k-1,n}\frac{d^{k-1}c_n(t)}{dt^{k-1}} = (\chi_n, \psi_n(t)). \tag{1.13}$$

Therefore, the evolution operators for strongly singular evolution equations (1.5), (1.10), (1.11) which was defined in [24,25] with the help of complicated implicit procedure can be approximated in the generalized strong sense [11,28] by evolution operators for explicitly defined systems of equations (1.8), (1.10), (1.11).

2. Formulation of results

2.1. Strongly singular equations

Recall the procedure of constructing the space Π_m and operator $\hat{H}$ entering Eq. (1.5).

First of all, consider the space $\mathscr{P}_m$ containing all linear combinations of the form $\psi = \sum_{l=1}^{2m} c_l T^{-l}\chi + \psi_{\text{reg}}$, where $c_l \in \mathbb{C}$, $\psi_{\text{reg}} \in \mathscr{H}^{2m}$. The inner product in this space is specified by the $k-1$ real parameters $g_2, \dots, g_k$. Set

$$(\chi, \hat{T}^{-s}\chi)_{\text{reg}} = g_s \quad \text{for} \quad s \leqslant k,$$

$$(\chi, \hat{T}^{-s}\chi)_{\text{reg}} = (\chi, \hat{T}^{-s}\chi) \quad \text{for} \quad s \geqslant k+1.$$

The inner product in $\mathscr{P}_m$ is

$$\langle \psi, \psi \rangle = \sum_{l,s=1}^{2m} c_l^* c_s (\chi, \hat{T}^{-l-s}\chi)_{\text{reg}} + (\psi_{\text{reg}}, \psi_{\text{reg}})$$
$$+ \sum_{s=1}^{2m} c_s (\hat{T}^m \psi_{\text{reg}}, \hat{T}^{-m-s}\chi) + \sum_{s=1}^{2m} c_s^* (\hat{T}^{-m-s}\chi, \hat{T}^m \psi_{\text{reg}}).$$

This expression is well defined, since $\hat{T}^{-m-k}\chi \in \mathscr{H}$ for $k \geqslant 1$, while $\hat{T}^m \psi_{\text{reg}} \in \mathscr{H}$.

Consider the completion [8] of the space $\mathscr{P}_m$ which is a Pontriagin space Π_m with $m = [k/2]$. It has the structure $\Pi_m = \mathbb{C}^{2m} \oplus \mathscr{H}$,

$$\Pi_m = \{(\gamma, \rho, \varphi) | \gamma = (\gamma_1, \dots, \gamma_m) \in \mathbb{C}^m, \rho = (\rho_1, \dots, \rho_m) \in \mathbb{C}^m, \varphi \in \mathscr{H}\}.$$

Introduce an indefinite inner product in Π_m as follows:

$$\langle \Phi, \Phi \rangle = \sum_{su=1}^{m} \gamma_s^* \gamma_u (\chi, \hat{T}^{-s-u} \chi)_{\text{reg}} - \sum_{s=1}^{m} (\gamma_s^* \rho_s + \gamma_s \rho_s^*) + (\varphi, \varphi).$$

The one-to-one correspondence $I : \mathscr{P}_m \to \Pi_m$ between $\mathscr{P}_m$ and a dense subset of the space Π_m can be specified as $I\{\sum_{l=1}^{2m} c_l \hat{T}^{-l} \chi + \psi_{\text{reg}}\} = (\gamma, \rho, \varphi)$, where

$$\gamma_1 = -c_1, \dots, \gamma_m = -c_m,$$

$$\rho_1 = \sum_{l=m+1}^{2m} c_l (\chi, \hat{T}^{-l-1} \chi)_{\text{reg}} + (\hat{T}^{-1} \chi, \psi_{\text{reg}}),$$

$$\dots,$$

$$\rho_m = \sum_{l=m+1}^{2m} c_l (\chi, \hat{T}^{-l-m} \chi)_{\text{reg}} + (\hat{T}^{-m} \chi, \psi_{\text{reg}}),$$

$$\varphi = \sum_{l=m+1}^{2m} c_l \hat{T}^{-l} \chi + \psi_{\text{reg}}.$$

The following statement has been proved in [24,25].

Lemma 2.1. *The continuation of the mapping I is a one-to-one correspondence between the completion of the space $\mathscr{P}_m$ and the space Π_m.*

Instead of the unbounded operator $\hat{H}$, it is more convenient to define the bounded operator $\hat{H}^{-1}$. Consider the formal equation $\hat{H}\psi = \phi$, $\hat{T}\psi + g\chi(\chi, \psi) = \phi$ and find (formally) $\psi : \psi = \hat{T}^{-1}\phi + \alpha T^{-1}\chi(\hat{T}^{-1}\chi, \phi)$. Here $\alpha = -\frac{1}{1/g + (\chi, T^{-1}\chi)}$. Therefore, define the operator $\hat{H}^{-1}$ in the space $\mathscr{P}_m$ as follows:

$$\hat{H}^{-1}\phi = \hat{T}^{-1}\phi + \alpha \hat{T}^{-1}\chi \langle \hat{T}^{-1}\chi, \phi \rangle. \tag{2.1}$$

One should also specify a one-to-one correspondence between α and g. For the case $m = 0$, definition (2.1) is in agreement with the approach based on self-adjoint extensions [4].

Operator (2.1) can be continued [24] to the space Π_m. Thus, the operator $\hat{H}^{-1}$ can be viewed as a continuous operator in the Pontriagin space Π_m. It does not have zero eigenvalues for $\alpha \neq 0$. The inverse operator $\hat{H} \equiv (\hat{H}^{-1})^{-1}$ is then [8] a self-adjoint (generally, unbounded) operator in Π_m.

Therefore, space Π_m and operator $\hat{H}$ are constructed.

2.2. Approximation of a strongly singular equation

Formulate now the main results of the paper. The resolving operator for the Cauchy problem for system (1.8) approximates the resolving operator for the Cauchy problem for Eq. (1.5) in the general strong sense. Recall the corresponding definition [11,28].

Let $\mathscr{B}$ and $\mathscr{B}_n$, $n = 1, 2, \ldots$ be Banach spaces, $P_n : \mathscr{B} \to \mathscr{B}_n$, $n = 1, 2, \ldots$ be a sequence of operators with uniformly bounded norms: $||P_n|| \leqslant a < \infty$ for some n-independent quantity a.

Definition 2.1. We say that a sequence of operators $A_n : \mathscr{B}_n \to \mathscr{B}_n$, $n = 1, 2, \ldots$ is $\{P_n\}$-strongly convergent to operator $A : \mathscr{B} \to \mathscr{B}$, if for all $v \in \mathscr{B}$ the property $||P_n A v - A_n P_n v|| \to_{n\to\infty} 0$ is satisfied.

Note that a generalized strong limit of a sequence of operators depends (generally) on the choice of the sequence $\{P_n\}$; this fact is used in the theory of the Maslov canonical operator in abstract spaces [26,27].

Definition 2.2. Let $u_n \in \mathscr{B}_n$, $n = 1, 2, \ldots$, $u \in \mathscr{B}$. We say that a sequence $\{u_n\}$ is of the class $[u]$ (or is $\{P_n\}$-strongly convergent to u), if $||u_n - P_n u|| \to_{n\to\infty} 0$.

Set $\mathscr{B} = \Pi_m$. Denote by $\mathscr{B}_n = \mathbb{C}^{k-1} \oplus \mathscr{H}$ the space of sets $\Phi_n = (c_n^0, \ldots, c_n^{k-2}, \psi_n)$ of numbers $c_n^0, \ldots, c_n^{k-2} \in \mathbb{C}$ and a vector $\psi \in \mathscr{H}$. Define an indefinite inner product in the space $\mathscr{B}_n$ as follows:

$$\langle \Phi_n, \Phi_n \rangle = (\psi_n, \psi_n) + \sum_{js=0}^{k-2} c_n^{j*} c_n^s z_{j+s+1,n}. \tag{2.2}$$

Here $z_{l,n} = 0$ as $l \geqslant k$ by definition.

Lemma 2.2. *Let $z_{k-1,n} \leqslant 0$. Then the inner product* (2.2) *contains m negative squares.*

Note that the condition of Lemma 2.2 is satisfied at sufficiently large n.

System (1.8) can be presented as a differential equation of the first order

$$i \hat{Z}_n \frac{d}{dt} \Phi_n(t) = \hat{H}_n \Phi_n(t) \tag{2.3}$$

on the vector function $\Phi_n(t) \in \mathscr{B}_n$. The operators $\hat{Z}_n$ and $\hat{H}_n$ are defined as

$$\hat{Z}_n(c_n^0, \ldots, c_n^{k-2}, \psi_n) = (c_n^0, \ldots, c_n^{k-3}, z_{k-1,n} c_n^{k-2}, \psi_n),$$

$$\hat{H}_n(c_n^0, \ldots, c_n^{k-2}, \psi_n) = (c_n^1, \ldots, c_n^{k-2}, (\chi_n, \psi_n) - z_{0,n} c_n^0 - \ldots - z_{k-2,n} c_n^{k-2}, \hat{T}\psi_n + c_n^0 \chi_n).$$

Namely, after redefining $i^l \frac{d^l}{dt^l} c_n(t) = c_n^l(t)$ system (1.8) is taken to the form (2.3).

Lemma 2.3. *Let* $\psi_n(0) \in D(\hat{T})$, $c_n^0(0), \dots, c_n^{k-2}(0) \in \mathbb{C}$. *Then there exists a unique solution of the Cauchy problem for Eq.* 2.3. *It continuously depends on the initial conditions for* $t \in [0, T]$.

Define the operator $U_n(t) : \mathbb{C}^{k-1} \oplus D(\hat{T}) \to \mathbb{C}^{k-1} \oplus D(\hat{T})$ taking the initial condition of the Cauchy problem for Eq. (1.9) to the solution of the Cauchy problem. Since the solution of the Cauchy problem continuously depends on the initial condition, the operator $U_n(t)$ can be continued to the space $\mathscr{B}_n$. This continuation $U_n(t) : \mathscr{B}_n \to \mathscr{B}_n$ is unique, provided that it is continuous.

Lemma 2.4. *The operator* $U_n(t)$ *conserves the indefinite inner product* (2.2).

Introduce the operator $P_n : \mathscr{B} \to \mathscr{B}_n$ of the form $P_n : (\gamma, \rho, \varphi) \mapsto (c_n^0, \dots, c_n^{k-2}, \psi_n)$ as follows.

For arbitrary k, set

$$c_n^0 = \gamma_1, \dots, c_n^{m-1} = \gamma_m, \quad \psi_n = -\sum_{j=0}^{m-1} \gamma_{j+1} T^{-j-1} \chi_n + \varphi_n.$$

For $k = 2m + 1$, set $\varphi_n = \varphi$. For $k = 2m$, set

$$\varphi_n = \varphi + \frac{\hat{T}^{-m}\chi_n[\rho_m - (\hat{T}^{-m}\chi_n, \varphi)]}{(\hat{T}^{-m}\chi_n, \hat{T}^{-m}\chi_n)}.$$

Specify the quantities $c_n^m, \dots, c_n^{2m-1}$ from the relations:

$$(\hat{T}^{-1}\chi_n, \psi_n) - z_{m+1,n}c_n^m - \dots - z_{2m,n}c_n^{2m-1} = \rho_1,$$

$$\dots$$

$$(\hat{T}^{-m+1}\chi_n, \psi_n) - z_{2m-1,n}c_n^m - z_{2m,n}c_n^{m+1} = \rho_{m-1},$$

$$(\hat{T}^{-m}\chi_n, \psi_n) - z_{2m,n}c_n^m = \rho_m. \tag{2.4}$$

For sufficiently large n, $c_n^m, \dots, c_n^{2m-1}$ are defined uniquely, since $z_{k-1,n} \neq 0$. The mapping P_n is constructed.

Lemma 2.5. *As* $n \to \infty$, $\langle P_n\Phi, P_n\Phi \rangle \to \langle \Phi, \Phi \rangle$.

Introduce now Hilbert inner products in $\mathscr{B}$ and $\mathscr{B}_n$.

Recall that a Hilbert inner product in a Pontriagin space is introduced as follows [8]. First, an arbitrary m-dimensional subspace $\mathscr{L}_m \subset \Pi_m$ such that the indefinite inner product is negative definite on $\mathscr{L}_m$, is considered. Without loss of generality, one can consider only the case when the subspace $\mathscr{L}_m$ belongs to the domain of H [19]. Otherwise, introduce a basis e_i' in the space $\mathscr{L}_m$, choose some vectors e_i from the domain of the operator H such that the distance between e_i and e_i' is smaller than ε. Consider the span of the set of vectors e_i. At sufficiently small ε the inner product will be negative definite on the span.

By J we denote the operator of the form $J\Phi = \Phi$ at $\Phi \perp \mathscr{L}_m$ and $J\Phi = -\Phi$ at $\Phi \in \mathscr{L}_m$. According to Iokhvidov [8], the bilinear form

$$\langle \Phi, \Phi \rangle_{\mathscr{L}_m} = \langle \Phi, J\Phi \rangle \tag{2.5}$$

specifies a positive definite Hilbert inner product. The topologies corresponding to inner products (2.5) at different $\mathscr{L}_m$ are equivalent.

The inner product (2.5) specified the following norm in $\mathscr{B}$:

$$||\Phi|| = \sqrt{\langle \Phi, \Phi \rangle_{\mathscr{L}_m}}. \tag{2.6}$$

To specify a norm in $\mathscr{B}_n$, let us use the following statement. Let λ be a sufficiently large positive number such that the resolvent of the operator $-\hat{Z}_n^{-1} H_n$ is defined at sufficiently large n. Denote $\mathscr{L}_m^n = (\hat{Z}_n^{-1}\hat{H}_n + \lambda)^{-1} P_n (H + \lambda)\mathscr{L}_m$.

Lemma 2.6. *At sufficiently large n the inner product* (2.2) *is negative definite on the m-dimensional subspace $\mathscr{L}_m^n \subset \mathscr{B}_n$. At sufficiently large n, the inner product $\langle \Phi, \Phi \rangle_{P_n \mathscr{L}_m}$ is positively definite on $\mathscr{B}_n$ and defines a norm $||\Phi_n|| = \sqrt{\langle \Phi, \Phi \rangle_{P_n \mathscr{L}_m}}$.*

Lemma 2.6 implies the following lemma.

Lemma 2.7. *The operators P_n are uniformly bounded, $||P_n|| \leqslant a$ for some n-independent constant a.*

The following lemma gives necessary and sufficient condition for the property $\{(c_n^0, \ldots, c_n^{k-2}, \psi_n) \in \mathscr{B}_n \in [(\gamma, \rho, \varphi)]$. Denote $\phi_n = \psi_n + \sum_{j=0}^{m-1} c_n^j \hat{T}^{-j-1} \chi_n$.

Lemma 2.8. $\{(c_n^0, \ldots, c_n^{k-2}, \psi_n)\} \in [\gamma, \rho, \varphi]$ *if and only if*

$$\begin{aligned}
&\lim_{n\to\infty} c_n^0 = \gamma_1, \ldots, \lim_{n\to\infty} c_n^{m-1} = \gamma_m,\\
&\lim_{n\to\infty} ||\phi_n - \varphi|| = 0,\\
&\lim_{n\to\infty} (\hat{T}^{-1}\chi_n, \phi_n) - z_{m+1,n} c_n^m - \ldots - z_{2m,n} c_n^{2m-1} = \rho_1,\\
&\qquad \ldots\\
&\lim_{n\to\infty} (\hat{T}^{-m+1}\chi_n, \phi_n) - z_{2m-1,n} c_n^m - z_{2m,n} c_n^{m+1} = \rho_{m-1},\\
&\lim_{n\to\infty} (\hat{T}^{-m}\chi_n, \phi_n) - z_{2m,n} c_n^m = \rho_m.
\end{aligned}$$

In particular, Lemma 2.8 shows that the property of $\{P_n\}$-strong convergence does not depend on the choice of the subspace $\mathscr{L}_m$.

The following lemma shows that any initial condition for Eq. (1.5) can be obtained as a $\{P_n\}$-strong limit of the sequence of initial conditions for system (1.8).

Lemma 2.9. *For any* $(\gamma, \rho, \varphi) \in \mathscr{B}$ *there exists a sequence* $\{(c_n^0, \dots, c_n^{k-2}, \psi_n) \in \mathscr{B}_n\}$ *from the class* $[\gamma, \rho, \varphi]$.

To prove the lemma, it is sufficient to choose $(c_n^0, \dots, c_n^{k-2}, \psi_n) = P_n(\gamma, \rho, \varphi)$.

The main result of the paper is formulated as follows.

Theorem 1. *The sequence of operators* $U_n(t)$ *is* $\{P_n\}$*-strongly convergent to* $U(t)$.

Corollary. *Let* $\{(c_n^0(0), \dots, c_n^{k-2}(0), \psi_n(0))\} \in [(\gamma, \rho, \varphi)]$. *Then* $\{(c_n^0(t), \dots, c_n^{k-2}(t), \psi_n(t))\} \in [U^t(\gamma, \rho, \varphi)]$.

For non-strongly singular case ($k = 1$ or $m = 0$) Theorem 1 gives the result of [13].

Formulate analogs of Theorem 1 for approximations of Eqs. (1.10) and (1.11).

Lemma 2.10. *Let* $\psi_n \in D(\hat{T})$, $c_n^0(0), \dots, c_n^{k-2}(0) \in \mathbb{C}$. *Then there exists a unique solution of the Cauchy problem for system* 1.13. *It continuously depends on the initial condition. For* $\psi(0) \in D(H)$, *there exists a unique solution of the Cauchy problem for Eq.* (1.11). *It also continuously depends on the initial condition.*

By $\tilde{U}_n(t)$, $\tilde{U}(t)$ we denote the operators transforming the initial conditions for the Cauchy problems for Eqs. (1.13) and (1.11) to the solution of the Cauchy problems for Eqs. (1.13), (1.11) correspondingly.

Theorem 2. *The sequence of operators* $\tilde{U}_n(t)$ *is* $\{P_n\}$*-strongly convergent to* $\tilde{U}(t)$.

Lemma 2.11. *Let* $\psi_n(0) \in D(\hat{T})$, $c_n(0), \dots, c_n^{(2k-3)} \in \mathbb{C}$. *Then there exists a unique solution of the Cauchy problem for Eq.* (1.12). *It continuously depends on the initial condition.*

Note that system (1.12) can be presented as

$$-\frac{d^2}{dt^2}\Phi_n(t) = \hat{Z}_n^{-1}\hat{H}_n\Phi_n. \tag{2.7}$$

Introduce operators $V_n(t)$ and $W_n(t)$ on $D(\hat{T})$ from the relation

$$\Phi_n(t) = V_n(t)\Phi_n(0) + W_n(t)\frac{d\Phi_n}{dt}(0).$$

The operator taking $\Phi_n(0)$ to the solution of the Cauchy problem for Eq. (2.7) at $d\Phi_n(0)/dt = 0$ is denoted as $V_n(t)$. The operator taking $d\Phi_n(0)/dt$ to $\Phi_n(t)$ at $\Phi_n(0) = 0$ is denoted as $W_n(t)$. Since the solution continuously depends on the initial conditions, the operators $V_n(t)$ and $W_n(t)$ are bounded. They are uniquely continued to the whole space $\mathcal{B}_n$.

Analogously, define the operators $V(t)$ and $W(t)$ from the relations

$$\psi(t) = V(t)\psi(0) + W(t)\frac{d\psi}{dt}(0), \tag{2.8}$$

where $\psi(t) \in D(H)$ is a solution of Eq. (1.10), $\psi(0) \in D(H)$, $\dot{\psi}(0) \in D(H)$ are initial conditions.

Theorem 3. *The sequence of operators $V_n(t)$ is $\{P_n\}$-strongly convergent to $V(t)$. The sequence of operators $W_n(t)$ is $\{P_n\}$-strongly convergent to $W(t)$.*

3. Approximation of the space and resolvent convergence

This section deals with the proof of Lemmas 2.2 and 2.5–2.8. We also justify that the sequence of resolvents of the operators $\hat{Z}_n^{-1}\hat{H}_n$ converges in a general strong sense to the resolvent of the operator $\hat{H}$.

1. Lemma 2.2 is a corollary of the following statement. Consider the real matrices A and B of the dimensions $m \times m$, which consist of elements A_{ij} and B_{ij}, $i,j = \overline{1,m}$.

Lemma 3.1. *Let the matrix B be invertible, while the matrix A be Hermitian. Then the quadratic form*

$$\sum_{ij=1}^{m} [x_i^* A_{ij} x_j + y_i^* B_{ij} x_j + x_i^* B_{ji}^* y_i] \tag{3.1}$$

contains m negative and m positive squares.

Proof. Since the matrix A is Hermitian, it can be taken to the diagonal form $U^T A U = diag[\alpha_1, \ldots, \alpha_m]$ with the help of a unitary transformation. After substitution $x_i = \sum_{s=1}^m U_{is}\xi_s$ and transformation $\eta_s = \sum_{ij=1}^m B_{ji}^* U_{is}^* y_j$ the quadratic form (3.1) is taken to the form

$$\sum_{s=1}^{m} [\alpha_s \xi_s^* \xi_s + \xi_s^* \eta_s + \eta_s^* \xi_s]. \tag{3.2}$$

One has

$$\alpha_s \xi_s^* \xi_s + \xi_s^* \eta_s + \eta_s^* \xi_s = \alpha_s (\xi_s^* + \alpha_s^{-1}\eta_s^*)(\xi_s + \alpha_s^{-1}\eta_s) - \alpha_s^{-1}\eta_s^*\eta_s, \quad \alpha_s \neq 0,$$

$$\xi_s^* \eta_s + \eta_s^* \xi_s = \tfrac{1}{2}[(\xi_s^* + \eta_s^*)(\xi_s + \eta_s) - (\xi_s^* - \eta_s^*)(\xi_s - \eta_s)], \quad \alpha_s = 0.$$

For both cases, the quadratic form $\alpha_s\xi_s^*\xi_s + \xi_s^*\eta_s + \eta_s^*\xi_s$ contains one negative and one positive square. Therefore, the form (3.2) contains m positive and m negative squares. Lemma 3.1 is proved. □

Proof of Lemma 2.2. It is sufficient to justify that the quadratic form

$$\sum_{js=0}^{k-2} c_n^{j*} c_n^s z_{j+s+1,n} \tag{3.3}$$

contains m negative squares (we set $z_{l,n} = 0$ for $l \geqslant k$). Take it to the form (3.1). Consider 2 cases.

1. Let $k = 2m+1$. Denote $x_j = c_n^{j-1}$, $y_j = c_n^{m+j-1}$, $j = \overline{1,m}$, $A_{ij} = z_{i+j-1,n}$, $B_{ij} = z_{m+i+j-1,n}$, $i,j = \overline{1,m}$. Since matrix elements B_{ij} vanish as $i+j > m+1$, while $B_{ij} = z_{2m,n} \neq 0$ as $i+j = m+1$, $\det B \neq 0$, and the matrix B is invertible. Therefore, the quadratic form (3.3) is taken to the form (3.1) and contains m negative squares.

2. Let $k = 2m$. Denote $x_j = c_n^{j-1}$, $y_j = c_n^{m+j-1}$, $j = \overline{1, m-1}$, $\sigma = c_n^{m-1}$. The quadratic form (3.3) is taken to the form

$$\sum_{ij=1}^{m-1} [x_i^* \tilde{A}_{ij} x_j + y_j^* \tilde{B}_{ij} x_j + x_i^* \tilde{B}_{ji} y_j] + z_{2m-1,n}\sigma^*\sigma + \sum_{s=1}^{m-1} [\sigma^* z_{m+s-1,n} x_s + \sigma z_{m+s-1,n} x_s^*], \tag{3.4}$$

where $\tilde{A}_{ij} = z_{i+j-1,n}$, $\tilde{B}_{ij} = z_{m+i+j-1,n}$, $i,j = \overline{1, m-1}$. The matrix elements $\tilde{B}_{ij}$ vanish at $i+j > m$ and are non-zero at $i+j = m$. Therefore, the matrix $\tilde{B}$ is invertible. Formula (3.4) is taken to the form

$$\sum_{ij=1}^{m-1} \left[x_i^* \left(\tilde{A}_{ij} - \frac{z_{i+m-1,n} z_{m+j-1,n}}{z_{2m-1,n}} \right) x_j + y_j^* \tilde{B}_{ij} x_j + x_i^* \tilde{B}_{ji} y_j \right] + z_{2m-1,n} \left(\sigma^* + \sum_{s=1}^{m-1} \frac{z_{m+s-1,n}}{z_{2m-1,n}} x_s^* \right) \left(\sigma + \sum_{s=1}^{m-1} \frac{z_{m+s-1,n}}{z_{2m-1,n}} x_s \right) \tag{3.5}$$

Since $z_{2m-1,n} < 0$, the quadratic form (3.5) contains m negative squares. Lemma 2.2 is proved. □

2. The following statement will be used later.

Lemma 3.2. *The sequence $\Phi^{(n)} = (\gamma^{(n)}, \rho^{(n)}, \varphi^{(n)}) \in \mathscr{B}$ strongly converges to zero if and only if*

$$||\Phi^{(n)}||_1 = \max_s \, [||\varphi^{(n)}||, |\gamma_s^{(n)}|, |\rho_s^{(n)}|] \tag{3.6}$$

tends to zero.

Proof. First of all, prove the statement for the special choice of the subspace $\mathscr{L}_m$ entering the definition of the norm (2.6). Denote by $\mathscr{L}^{(0)}$ the subspace of the space $\mathscr{B}$, which consists of all vectors of the form $(\gamma, \rho, 0)$. The quadratic form $\langle \Phi, \Phi \rangle$, considered on $\mathscr{L}^{(0)}$, contains m negative squares, so that for some subspace $\mathscr{L}_m \subset \mathscr{L}^{(0)}$ it is negatively definite. Consider the Hilbert inner product (2.6) corresponding to $\mathscr{L}_m$. It has the structure

$$\sum_{sl=1}^{2m} x_s^* M_{sl} x_l + (\varphi, \varphi), \tag{3.7}$$

where $x_1 = \gamma_1, \ldots, x_m = \gamma_m$, $x_{m+1} = \rho_1, \ldots, x_{2m} = \rho_m$, M_{sl} is a some matrix. Since the inner product (3.7) is positively definite, the matrix M_{sl} is also positively definite.

Thus, $\Phi^{(n)}$ strongly converges to zero if and only if $||\varphi^{(n)}||$ tends to zero and $||x^{(n)}||_M = \sqrt{\sum_{sl=1}^{2m} x_s^{(n)*} M_{sl} x_l^{(n)}} \to 0$. Since all norms in finite-dimensional space are equivalent, the latter property is equivalent to $\max|x_s^{(n)}| \to 0$. Since all norms of type (2.5) in the Pontriagin space are equivalent, we obtain the statement of the lemma for arbitrary choice of $\mathscr{L}_m$. The lemma is proved. □

Corollary. *For some A_1 the following property is satisfied*: $A_1^{-1}||\Phi||_1 \geqslant ||\Phi|| \geqslant A_1 ||\Phi||_1$.

Proof. Suppose that statement of corollary is not satisfied. Then it is possible to choose a sequence $\Phi^{(n)}$ which obeys one of the following properties:

$$\frac{||\Phi^{(n)}||_1}{||\Phi^{(n)}||} \to_{n\to\infty} 0, \quad \frac{||\Phi^{(n)}||}{||\Phi^{(n)}||_1} \to_{n\to\infty} 0.$$

For definiteness, consider the first case. Consider the sequence $\Psi^{(n)} = \frac{\Phi^{(n)}}{\sqrt{||\Phi^{(n)}|| ||\Phi^{(n)}||_1}}$, tending to zero in the $||\cdot||_1$-norm and to infinity in the $||\cdot||$-norm. This contradicts to Lemma 3.2. The corollary is proved. □

Consider the operator $Q_n : \mathscr{B}_n \to \mathscr{B}$ of the form $Q_n : (c_n^0, \ldots, c_n^{k-2}, \psi_n) \mapsto (\gamma_n, \rho_n, \varphi_n)$. Here

$$\begin{aligned} &\varphi_n = \psi_n + \sum_{j=0}^{m-1} c_n^j \hat{T}^{-j-1} \chi_n, \\ &\gamma_n^j = c_n^{j-1}, \quad j = \overline{1,m}, \\ &\rho_n^j = (\hat{T}^{-j} \chi_n, \varphi_n) - z_{m+j.n} c_n^m - \ldots - z_{2m,n} c_n^{2m-j}. \end{aligned} \tag{3.8}$$

Introduce in $\mathscr{B}$ an additional indefinite inner product:

$$\langle \Phi, \Phi \rangle_n = \sum_{su=1}^{m} \gamma_s^* \gamma_u g_{s+u}^{(n)} - \sum_{s=1}^{m} (\gamma_s^* \rho_s - \gamma_s \rho_s^*) + (\varphi, \varphi), \tag{3.9}$$

where

$$g_l^{(n)} = (\chi_n, \hat{T}^{-l}\chi_n) + z_{l-1,n}. \tag{3.10}$$

Lemma 3.3. *The following property* $\langle \Phi_n, \Psi_n \rangle = \langle Q_n\Phi_n, Q_n\Psi_n \rangle_n$, *is satisfied for* $\Phi_n, \Psi_n \in \mathscr{B}_n$.

To prove Lemma 3.3, it is sufficient to substitute formulas (3.8) in the inner product (2.2).

Corollary. *Let* $\Phi_n, \Psi_n \in \mathscr{B}_n$ *be such sequences that* $||Q_n\Phi_n|| \leqslant C$, $||Q_n\Psi_n|| \leqslant C$ *for some* C. *Then* $\langle Q_n\Phi_n, Q_n\Psi_n \rangle - \langle \Phi_n, \Psi_n \rangle \to_{n\to\infty} 0$.

Proof. Define $Q_n\Phi_n = X_n = (\gamma_n, \rho_n, \varphi_n)$, $Q_n\Psi_n = \tilde{X}_n = (\tilde{\gamma}_n, \tilde{\rho}_n, \tilde{\varphi}_n)$. Statement of the corollary means that

$$\sum_{su=1}^{m} \gamma_{n,s}^* \tilde{\gamma}_{n,u}(g_{s+u}^{(n)} - g_{s+u}) \to_{n\to\infty} 0.$$

This property is a corollary of Lemma 3.2. The corollary is proved. □

Lemma 3.4. *For some quantity* C *that does not depend on* n, Φ *and* Ψ, *the estimation* $|\langle \Phi, \Psi \rangle_n| \leqslant C||\Phi||||\Psi||$ *is satisfied.*

Proof. Let $\Phi = (\gamma, \rho, \varphi)$, $\Psi = (\tilde{\gamma}, \tilde{\rho}, \tilde{\varphi})$. It follows from (3.9) that

$$|\langle \Phi, \Psi \rangle_n| \leqslant \sum_{su=1}^{m} (|\gamma_s||\tilde{\gamma}_u||g_{s+u}^{(n)}| + |\gamma_s||\tilde{\rho}_s| + |\tilde{\gamma}_s||\rho_s|) + ||\varphi||||\tilde{\varphi}||$$

$$\leqslant \sum_{su=1}^{m} ||\Phi||_1||\Psi||_1(g_{s+u}^{(n)} + 2) + ||\Phi||_1||\Psi||_1$$

$$\leqslant A_1^2||\Phi||||\Psi||\left(\sum_{su=1}^{m} |g_{s+u}^{(n)}| + 2m^2 + 1\right).$$

Since the sequences $g_{s+u}^{(n)}$ are convergent, they are bounded. We obtain statement of the lemma. □

Corollary. *Let* $\Phi_n, \Psi_n \in \mathscr{B}$. *Then the following estimation is satisfied*: $|\langle \Phi_n, \Psi_n \rangle| \leqslant C||Q_n\Phi_n||||Q_n\Psi_n||$.

Let us check that the sequence of the operators $Q_nP_n : \mathscr{B} \to \mathscr{B}$ strongly converges to 1. First, let us justify the following statement.

Lemma 3.5. *Let $\xi \in \mathscr{H}$ and $||\hat{T}^{1/2}\theta(b-\hat{T})\xi|| \leqslant C$ for some b-independent quantity C. Then $\xi \in \mathscr{H}^1 \subset \mathscr{H}$.*

Proof. Consider the sequence $\xi_n = \theta(n-\hat{T})\xi$. Suppose it to be not fundamental in $\mathscr{H}^1$. Then for some $\varepsilon > 0$ there exists an increasing sequence $n_1, n_2, n_3, \ldots$, such that $||\xi_{n_{2s}} - \xi_{n_{2s-1}}||_{\mathscr{H}^1} = ||I_{[n_{2s-1},n_{2s}]}(\hat{T})\xi||_{\mathscr{H}^1} > \varepsilon$ (here $I_{[m,n]}(\lambda) = 1$ at $\lambda \in [m,n]$ and $I_{[m,n]}(\lambda) = 0$ at $\lambda \notin [m,n]$). Therefore,

$$(\xi, \hat{T}\theta(n_{2l}-\hat{T})\xi) \geqslant \sum_{s=1}^{l} (\xi, \hat{T}I_{[n_{2s-1},n_{2s}]}(\hat{T})\xi) \geqslant \varepsilon l.$$

For $l > C/\varepsilon$, we obtain a contradiction with the conditions of lemma. Therefore, $\xi = \lim_{n\to\infty} \xi_n \in \mathscr{H}^1$. Lemma 3.5 is proved. □

Corollary. *Let $\chi \in \mathscr{H}^{-k-1}$ and $||\hat{T}^{-k/2}\theta(b-\hat{T})\chi|| \leqslant C$. Then $\chi \in \mathscr{H}^{-k}$ for some b-independent quantity C.*

Lemma 3.5 implies the following statement.

Lemma 3.6. (1) *The sequence $||\hat{T}^{-k/2}\chi_n||$ tends to infinity as $n \to \infty$.*

(2) *The sequence of elements of $\mathscr{H}$ of the form $\frac{\hat{T}^{-k/2}\chi_n}{||\hat{T}^{-k/2}\chi_n||}$ weakly converges to zero as $n \to \infty$.*

Proof. (1) Suppose that the sequence $||\hat{T}^{-k/2}\chi_n||$ does not tend to infinity. Choose from it the bounded subsequence $||\hat{T}^{-k/2}\chi_{n_j}|| \leqslant C$. One has

$$||\hat{T}^{-k/2}\theta(b-\hat{T})\chi_{n_j}|| \leqslant ||\hat{T}^{-k/2}\chi_{n_j}|| \leqslant C.$$

Consider the limit of the left-hand side as $j \to \infty$. Use the fact that the operator $\hat{T}^{1/2}\theta(b-\hat{T})$ is bounded. We obtain $||\hat{T}^{-k/2}\theta(b-\hat{T})\chi|| \leqslant C$. It follows form Lemma 3.5 and property $\hat{T}^{-\frac{k+1}{2}}\chi \in \mathscr{H}$ that $\hat{T}^{-\frac{k}{2}}\chi \in \mathscr{H}$, so that $\chi \in \mathscr{H}^{-k}$. This contradicts the condition $\chi \in \mathscr{H}^{-k-1}\ \mathscr{H}^{-k}$.

(2) Denote $\eta_n = \frac{\hat{T}^{-k/2}\chi_n}{||\hat{T}^{-k/2}\chi_n||}$. If $\xi \in D(\hat{T}^{1/2})$, one has

$$(\eta_n, \xi) = \frac{(\hat{T}^{-\frac{k+1}{2}}\chi_n, \hat{T}^{1/2}\xi)}{||\hat{T}^{-k/2}\chi_n||} \to_{n\to\infty} 0,$$

since $(\hat{T}^{-\frac{k+1}{2}}\chi_n, \hat{T}^{1/2}\xi) \to_{n\to\infty} (\hat{T}^{-\frac{k+1}{2}}\chi, \hat{T}^{1/2}\xi) \neq \infty$, $||\hat{T}^{-k/2}\chi_n|| \to_{n\to\infty} \infty$. Thus, the sequence η_n, $n = 1, 2, \ldots$ of the elements of the unit sphere in $\mathscr{H}$ weakly converges to zero on dense subset of $\mathscr{H}$. Therefore [11], the sequence η_n weakly converges to zero. Lemma 3.6 is proved. □

Lemma 3.6 implies that $z_{s,n} < 0$ for sufficiently large n.

Corollary 1. *Let $\Phi \in \mathscr{B}$. The following property is satisfied*: $Q_n P_n \Phi \to_{n\to\infty} \Phi$.

Proof. It follows from the definitions of the operators Q_n and P_n (3.8) and (2.4) that $Q_n P_n(\gamma, \rho, \varphi) = (\gamma, \rho, \varphi_n)$, where $\varphi_n = \varphi$ for odd values of k and

$$\varphi_n = \varphi + \frac{\hat{T}^{-m}\chi_n[\rho_m - (\hat{T}^{-m}\chi_n, \varphi)]}{(\chi_n, \hat{T}^{-2m}\chi_n)}$$

for $k = 2m$. It follows from Lemma 3.5 that φ_n strongly converges to φ as $n \to \infty$. Lemma 3.6 is proved. □

Corollary 2. *The sequence $Q_n P_n$ is uniformly bounded.*

Namely, any strongly convergent sequence is uniformly bounded [9].

Proof of Lemma 2.5. Let $\Phi \in \mathscr{B}$. It follows from Lemma 3.6 that the sequence $||Q_n P_n \Phi||$ is bounded. Corollary of Lemma 3.4 tells us that

$$\langle P_n\Phi, P_n\Phi \rangle - \langle Q_n P_n \Phi, Q_n P_n \Phi \rangle \to_{n\to\infty} 0.$$

It follows from Lemma 3.6 that $\langle Q_n P_n \Phi, Q_n P_n \Phi \rangle \to_{n\to\infty} \langle \Phi, \Phi \rangle$. We obtain statement of Lemma 2.5.

3. Let us obtain the commutation rule between operator Q_n and resolvent of the operator $\hat{Z}_n^{-1}\hat{H}_n$.

Denote by $\tilde{R}_n(\lambda)$ the operator in $\mathscr{B}$ that takes the set $(\gamma_{n,1}, \dots, \gamma_{n,m}, \rho_{n,1}, \dots, \rho_{n,m}, \varphi_n)$, $\gamma_{n,s}, \rho_{n,s} \in \mathbb{C}$, $\varphi_n \in \mathscr{H}$, to the set $(\tilde{\gamma}_{n,1}, \dots, \tilde{\gamma}_{n,m}, \tilde{\rho}_{n,1}, \dots, \tilde{\rho}_{n,m}, \tilde{\varphi}_n)$, which is specified from the relations

$$\begin{aligned}
&\gamma_{n,s} = \lambda\tilde{\gamma}_{n,s} + \tilde{\gamma}_{n,s+1}, \quad s = \overline{1, m-1},\\
&\gamma_{n,m} = \lambda\tilde{\gamma}_{n,m} + \tilde{c}_n^m,\\
&\varphi_n = (\hat{T} + \lambda)\tilde{\varphi}_n + \tilde{c}_n^m \hat{T}^{-m}\chi_n,\\
&\rho_{n,j} = \tilde{\rho}_{n,j-1} + \lambda\tilde{\rho}_{n,j} + g_{j+m}^{(n)}\tilde{c}_n^m, \quad j = \overline{2, m},\\
&\tilde{\rho}_{n,m} = (\hat{T}^{-m}\chi_n, \tilde{\varphi}_n) - z_{2m,n}\tilde{c}_n^m,\\
&g_1^{(n)}\tilde{\gamma}_{n,1} + \dots + g_m^{(n)}\tilde{\gamma}_{n,m} + g_{m+1}^{(n)}\tilde{c}_n^m = \rho_{n,1} - \lambda\tilde{\rho}_{n,1},
\end{aligned} \tag{3.11}$$

where $g_n^{(s)}$ has the form (3.10).

Lemma 3.7. *The following property is satisfied*: $Q_n(\hat{Z}_n^{-1}\hat{H}_n + \lambda)^{-1} = \tilde{R}_n(\lambda)Q_n$.

Proof. Let $\Phi_n = (c_n^0, \dots, c_n^{k-2}, \psi_n) \in \mathscr{B}_n$, $\tilde{\Phi}_n = (\hat{Z}_n^{-1}\hat{H}_n + \lambda)^{-1}\Phi_n = (\tilde{c}_n^0, \dots, \tilde{c}_n^{k-2}, \tilde{\psi}_n) \in \mathscr{B}_n$. Define

$$Q_n\tilde{\Phi}_n = (\tilde{\gamma}_{n,1}, \dots, \tilde{\gamma}_{n,m}, \tilde{\rho}_{n,1}, \dots, \tilde{\rho}_{n,m}, \tilde{\varphi}_n),$$

$$Q_n\Phi_n = (\gamma_{n,1}, \dots, \gamma_{n,m}, \rho_{n,1}, \dots, \rho_{n,m}, \varphi_n).$$

Check that $Q_n\tilde{\Phi}_n = \tilde{R}_n(\lambda)Q_n\Phi_n$. It follows from definitions of operators $\hat{Z}_n$ and $\hat{H}_n$ that

$$\begin{aligned}
&c_n^0 = \lambda\tilde{c}_n^0 + \tilde{c}_n^1,\\
&\qquad \dots,\\
&c_n^{k-3} = \lambda\tilde{c}_n^{k-3} + \tilde{c}_n^{k-2},\\
&z_{k-1,n}c_n^{k-2} = \lambda z_{k-1,n}\tilde{c}_n^{k-2} + (\chi_n, \tilde{\psi}_n) - z_{0,n}\tilde{c}_n^0 - \dots - z_{k-2,n}\tilde{c}_n^{k-2},\\
&\psi_n = \lambda\tilde{\psi}_n + \hat{T}\tilde{\psi}_n + \tilde{c}_n^0\chi_n.
\end{aligned} \tag{3.12}$$

Formulas (3.8) imply 3 first equations of system (3.11). We obtain the fourth and the fifth equation from formulas for ρ and $\tilde{\rho}$. The last equation is a corollary of Eqs. (3.12). Lemma 3.7 is proved. □

Denote

$$a_n(\lambda) = \sum_{s=1}^{2m+1} g_s^{(n)}(-\lambda)^{s-1-2m} - \lambda(\chi_n, \hat{T}^{-2m-1}(\hat{T} + \lambda)^{-1}\chi_n),$$

$$a(\lambda) = \lim_{n\to\infty} a_n(\lambda) = \sum_{s=1}^{2m+1} g_s(-\lambda)^{s-1-2m} - \lambda(\chi, \hat{T}^{-2m-1}(\hat{T} + \lambda)^{-1}\chi). \tag{3.13}$$

Lemma 3.8. *Under condition $a_n(\lambda) \neq 0$, the quantities $\tilde{\gamma}$, $\tilde{\rho}$, $\tilde{\varphi}$ are defined uniquely from system* (3.11). *Under condition $a(\lambda) \neq 0$ the sequence of operators $\tilde{R}_n(\lambda)$ being defined for $n \geqslant n_0$ is strongly convergent as $n \to \infty$.*

Proof. Let $(\gamma, \rho, \varphi) \in \mathscr{B}$. Set $\gamma_n = \gamma$, $\rho_n = \rho$, $\varphi_n = \varphi$, $\tilde{R}_n(\lambda)(\gamma, \rho, \varphi) = (\tilde{\gamma}_n, \tilde{\rho}_n, \tilde{\varphi}_n)$. It follows from (3.11) that $\tilde{\gamma}_{n,1}$ has the form

$$\tilde{\gamma}_{n,1} = (a_n(\lambda)(-\lambda)^{2m})^{-1}B_n(\lambda), \tag{3.14}$$

where

$$B_n(\lambda) = -\sum_{s=1}^{m} g_s^{(n)} \sum_{j=0}^{s-2} (-\lambda)^j\gamma_{n,s-j-1} + \sum_{j=0}^{m-1}(-\lambda)^j\rho_{n,j+1}$$

$$+ (-\lambda)^m((\hat{T}+\lambda)^{-1}\hat{T}^{-m}\chi_n, \varphi_n) - \left(\sum_{j=0}^{m-1}(-\lambda)^j g^{(n)}_{m+j+1} \right.$$

$$\left. + (-\lambda)^m(z_{2m,n} + (\chi_n, \hat{T}^{-2m}(\hat{T}+\lambda)^{-1}\chi_n)) \right) \sum_{j=0}^{m-1} (-\lambda)^j \gamma_{n,m-j}.$$

For $a_n(\lambda)\neq 0$, $\tilde{\gamma}_{n,1}$ is not defined. For this case, other components of the vector $\tilde{\gamma}_n$, vectors $\tilde{\rho}_n$ and $\tilde{\varphi}_n$ are defined uniquely from system (3.11).

For $a(\lambda)\neq 0$, the sequence $\tilde{\gamma}_{n,1}$ is convergent. We prove by induction that the sequences

$$\tilde{\gamma}_{n,s} = \sum_{j=0}^{s-2}(-\lambda)^j\gamma_{s-j-1} + (-\lambda)^{s-1}\tilde{\gamma}_{n,1},$$

$$\tilde{c}^m_n = \sum_{j=0}^{m-1}(-\lambda)^j\gamma_{n,m-j} + (-\lambda)^m\tilde{\gamma}_{n,1} \tag{3.15}$$

are also convergent as $n\to\infty$. Therefore, the sequence for elements $\mathscr{H}$ of the form

$$\tilde{\varphi}_n = (\hat{T}+\lambda)^{-1}\varphi - \tilde{c}^m_n\hat{T}^{-m}(\hat{T}+\lambda)^{-1}\chi_n \tag{3.16}$$

is also strongly convergent as $n\to\infty$. The sequence $\tilde{\rho}_{n,m}$ is taken to the form

$$\tilde{\rho}_{n,m} = ((\hat{T}+\lambda)^{-1}\hat{T}^{-m}\chi_n, \varphi) - \tilde{c}^m_n[z_{2m,n} + (\chi_n, \hat{T}^{-2m}(\hat{T}+\lambda)^{-1}\chi_n)] \tag{3.17}$$

and has a limit as $n\to\infty$. Therefore, sequences

$$\tilde{\rho}_{n,m-s} = \sum_{j=0}^{s-1}\rho_{n,m-s-j-1}(-\lambda)^j + (-\lambda)^s\tilde{\rho}_{n,m} - \sum_{j=0}^{s-1} g^{(n)}_{2m-s+j+1}(-\lambda)^j\tilde{c}^m_n \tag{3.18}$$

are convergent. Therefore, the sequence $(\tilde{\gamma}_n, \tilde{\rho}_n, \tilde{\varphi}_n)$ is convergent in the $||\cdot||_1$-norm. Because of corollary of Lemma 3.2, it is convergent in the norm $||\cdot||$. The lemma is proved. □

Denote $R(\lambda) = \lim_{n\to\infty}\tilde{R}_n(\lambda)$. It follows from proof of Lemma 3.8 that $R(\lambda)$ is a bounded operator.

We will use further

Lemma 3.9. *Let $A_n : \mathscr{B}\to\mathscr{B}$, $n = 1, 2, \ldots$ be a strongly convergent as $n\to\infty$ sequence of operators, $A_n \to_{n\to\infty} A$ and $A_nQ_n = 0$. Then $A = 0$.*

Proof. It follows from the condition of lemma that $A_n Q_n P_n = 0$. Lemma 3.6 implies that the sequence of operators $Q_n P_n : \mathscr{B} \to \mathscr{B}$ is strongly convergent to 1, so that $A_n Q_n P_n \to_{n\to\infty} A$ in a strong sense. Therefore, $A = 0$. □

Lemma 3.10. *Let* $a(\lambda) \neq 0$, $a(\mu) \neq 0$. *Then*

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu). \tag{3.19}$$

Proof. Consider the following sequence of operators A_n: $A_n = \tilde{R}_n(\lambda) - \tilde{R}_n(\mu) + (\lambda - \mu)\tilde{R}_n(\lambda)\tilde{R}_n(\mu)$. It satisfies the property $A_n Q_n = 0$ and strongly converges as $n \to \infty$ to $R(\lambda) - R(\mu) + (\lambda - \mu) R(\lambda) R(\mu)$. We obtain statement of the lemma. □

Lemma 3.11. *Under condition* $a(\lambda) \neq 0$ *the following property is satisfied*:

$$R(\lambda) = (\lambda + \hat{H})^{-1}. \tag{3.20}$$

Proof. Justify that for $\lambda = 0$ the operator $R(\lambda)$ coincides with the operator $\hat{H}^{-1}$ defined in Section 2. Find an explicit form of $\hat{H}^{-1}$. It follows from (2.1) that

$$\hat{H}^{-1}\left[\sum_{l=1}^{2m} c_l \hat{T}^{-l}\chi + \psi_{\text{reg}}\right] = \alpha a \hat{T}^{-1}\chi + \sum_{l=1}^{2m} c_l \hat{T}^{-l-1}\chi + \hat{T}^{-1}\psi_{\text{reg}},$$

where $a = \langle \hat{T}^{-1}\chi, \sum_{l=1}^{2m} c_l \hat{T}^{-l}\chi + \psi_{\text{reg}} \rangle$.

For the vectors $I[\sum_{l=1}^{2m} c_l \hat{T}^{-l}\chi + \psi_{\text{reg}}] = (\gamma, \rho, \varphi)$, $I[\sum_{l=1}^{2m} \alpha a \hat{T}^{-1}\chi + c_l \hat{T}^{-l-1}\chi + \hat{T}^{-1}\psi_{\text{reg}}] = (\tilde{\gamma}, \tilde{\rho}, \tilde{\varphi})$, one has

$$\begin{aligned}
&\tilde{\varphi} = -\gamma_m \hat{T}^{-m-1}\chi + \hat{T}^{-1}\varphi,\\
&\tilde{\gamma}_1 = -\alpha a, \qquad \tilde{\gamma}_2 = \gamma_1, \quad , \dots, \tilde{\gamma}_m = \gamma_{m-1},\\
&\tilde{\rho}_s = -(\chi, \hat{T}^{-m-s-1}\chi)_{\text{reg}}\gamma_m + \rho_{s+1}, \quad s = \overline{1, m-1},\\
&\tilde{\rho}_m = -(\chi, \hat{T}^{-2m-1}\chi)_{\text{reg}}\gamma_m + (\hat{T}^{-m-1}\chi, \varphi).
\end{aligned} \tag{3.21}$$

Formula (3.14) can be presented in the following form as $n \to \infty$: $\tilde{\gamma}_1 = g_1^{-1}(\rho_1 - \sum_{s=0}^{m} g_{s+1}\gamma_s)$, For the case $\alpha = -g_1^{-1}$ it coincides with $\tilde{\gamma}_1 = -\alpha a$. Formulas (3.15)–(3.18) also coincide with (3.21). Therefore, property (3.20) is satisfied as $\lambda = 0$. It follows from (3.19) that $R(\lambda)$ is a pseudoresolvent [11]. Therefore, property (3.20) is satisfied for all λ obeying the condition $a(\lambda) \neq 0$.

Lemma 3.12. *Under condition* $a(\lambda)\neq 0$ *the following property is satisfied*:

$$\langle (\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_n\Phi, (\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_n\Psi\rangle \to_{n\to\infty},$$

$$\langle (\hat{H}+\lambda)^{-1}\Phi, (\hat{H}+\lambda)^{-1}\Psi\rangle, \quad \Phi,\Psi\in\mathscr{B}.$$

Proof. Check that the conditions of the corollary of Lemma 3.3 are satisfied. Namely, for $\Phi\in\mathscr{B}$ one has

$$||Q_n\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_n\Phi|| = ||(\hat{H}+\lambda)^{-1}Q_nP_n\Phi||$$

$$\leqslant ||(\hat{H}+\lambda)^{-1}||\max_n ||Q_nP_n\Phi||\leqslant C.$$

An analogous property is correct for Ψ also. Therefore,

$$\langle (\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_n\Phi, (\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_n\Psi\rangle$$

$$-\ \langle (\hat{H}+\lambda)^{-1}Q_nP_n\Phi, (\hat{H}+\lambda)^{-1}Q_nP_n\Psi\rangle \to_{n\to\infty} 0.$$

The properties $Q_nP_n\Phi\to\Phi$, $Q_nP_n\Psi\to\Psi$ imply the statement of the lemma.

Proof of Lemma 2.6. Choose such a basis $e_1,\dots,e_m$ in $\mathscr{L}_m$ that obeys the condition $\langle e_i,e_j\rangle = -\delta_{ij}$. To prove negative definiteness of the inner product on $\mathscr{L}_m^n = (\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_n(\hat{H}+\lambda)\mathscr{L}_m$, it is sufficient to check the positive definiteness of the matrix

$$A_{ij}^{(n)} = -\langle (\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_n(\hat{H}+\lambda)e_i, (\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_n(\hat{H}+\lambda)e_j\rangle. \tag{3.22}$$

Its components tend to the components of the unit matrix according to Lemma 3.12. At sufficiently large n $||A^{(n)}-1||<1/2$, so that

$$(\xi, A^{(n)}\xi)-\frac{1}{2}(\xi,\xi) = \frac{1}{2}(\xi,\xi)+(\xi,(A^{(n)}-1)\xi)$$

$$\geqslant \frac{1}{2}||\xi||^2 - ||A^{(n)}-1||||\xi||^2\geqslant 0.$$

Positive definiteness of the inner product $\langle\Phi,\Phi\rangle_{\mathscr{L}_m^n}$ is a corollary of general results of [8]. Lemma 2.6 is proved. □

Lemma 3.13. *Let* $|\langle P_n\Phi,P_n\Phi\rangle|\leqslant B_1||\Phi||^2$ *for some constant* B_1, $\langle (\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_ne_i, P_n\Phi\rangle|\leqslant C_i||\Phi||$, $i=\overline{1,m}$ *for some* $C_1,\dots,C_m$. *Then* $||P_n||\leqslant a$.

Proof. It follows from formula (2.5) that

$$||P_n\Phi||^2 = \langle P_n\Phi, P_n\Phi \rangle_{\mathscr{L}^n_m} = \langle P_n\Phi, P_n\Phi \rangle + 2\sum_{ij=1}^{m} \langle P_n\Phi, (\hat{Z}_n^{-1}\hat{H}_n + \lambda)^{-1} P_n e_i \rangle M_{ij}^{(n)} \times \langle (\hat{Z}_n^{-1}\hat{H}_n + \lambda)^{-1} P_n e_j, P_n\Phi \rangle,$$

where $M^{(n)}$ is a matrix being inverse to (3.22). It follows from the conditions of lemma that

$$||P_n\Phi||^2 \leqslant \left(B_1 + 2\sum_{ij=1}^{m} C_i M_{ij}^{(n)} C_j\right) ||\Phi||^2 \leqslant \left(B_1 + 2 \sup_n ||M^{(n)}|| ||C||^2\right) ||\Phi||^2.$$

We obtain statement of Lemma 3.13. □

Proof of Lemma 2.7. Check that conditions of Lemma 3.13 are satisfied. Use the corollary of Lemma 3.4.

$$\begin{aligned} |\langle P_n\Phi, P_n\Phi \rangle| &\leqslant C\left(\sup_n ||Q_n P_n \Phi||^2 \leqslant C_1, \right.\\ &\leqslant |(\hat{Z}_n^{-1}\hat{H}_n + \lambda)^{-1} P_n e_i, P_n\Phi \rangle| \\ &\leqslant C \sup_n ||Q_n(\hat{Z}_n^{-1}\hat{H}_n + \lambda)^{-1} P_n e_i|| \sup_n ||Q_n P_n \Phi|| \\ &\leqslant C||(H+\lambda)^{-1}|| \left(\sup_n ||Q_n P_n||\right)^2 ||(H+\lambda)e_i|| ||\Phi||. \end{aligned}$$

Lemma 2.7 is proved. □

Lemma 3.14. $||\Phi_n|| \leqslant A_3 ||Q_n\Phi_n||$ *for some constant* A_3.

Proof. One has

$$||\Phi_n||^2 = \langle \Phi_n, \Phi_n \rangle + 2\sum_{ij=1}^{m} \langle \Phi_n, (\hat{Z}_n^{-1}\hat{H}_n + \lambda)^{-1} P_n e_i \rangle \times M_{ij}^{(n)} \langle (\hat{Z}_n^{-1}\hat{H}_n + \lambda)^{-1} P_n e_j, \Phi_n \rangle.$$

It follows from Lemma 3.4 that

$$||\Phi_n||^2 \leqslant C||Q_n\Phi_n||^2 + 2\sum_{ij=1}^{m} |M_{ij}^{(n)}| C^2 ||Q_n\Phi_n||^2 ||\tilde{R}_n(\lambda)||^2 \times ||Q_n P_n||^2 ||(\hat{H}+\lambda)e_i|| ||(\hat{H}+\lambda)e_j||.$$

We obtain statement of the lemma. □

Lemma 3.15. *Let the condition $a(\lambda)\neq 0$ be satisfied. Then the sequence of operators $(\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}:\mathscr{B}_n\to\mathscr{B}_n$ is $\{P_n\}$-strongly convergent to the operator $(\hat{H}+\lambda)^{-1}:\mathscr{B}\to\mathscr{B}$.*

Proof. It is sufficient to check that for any $\Phi\in\mathscr{B}$

$$||(\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_n\Phi-P_n(\hat{H}+\lambda)^{-1}\Phi||\to_{n\to\infty}0.$$

It follows from Lemma 3.14 that $||\Phi_n||\to 0$, provided that $||Q_n\Phi_n||\to 0$. It is sufficient to prove then that

$$||Q_n(\hat{Z}_n^{-1}\hat{H}+\lambda)^{-1}P_n\Phi-Q_nP_n(\hat{H}+\lambda)^{-1}\Phi||\to_{n\to\infty}0.$$

This property is a corollary of the relation

$$Q_n(\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}P_n\Phi=\tilde{R}_n(\lambda)Q_nP_n\Phi\to_{n\to\infty}(H+\lambda)^{-1}\Phi,$$

$$Q_nP_n(\hat{H}+\lambda)^{-1}\Phi\to_{n\to\infty}(\hat{H}+\lambda)^{-1}\Phi.$$

Lemma 3.15 is proved. □

Lemma 3.16. *For some constant A_2 the estimation $||Q_n\Phi_n||\leqslant A_2||\Phi_n||$ is satisfied.*

Proof. Since the norm of the operator J entering Eq. (2.5) is equal to 1, the following estimation is satisfied for the indefinite inner product:

$$|\langle\Phi_n,\Psi_n\rangle|\leqslant||\Phi_n||||\Psi_n||,\quad \Psi_n,\Phi_n\in\mathscr{B}_n.$$

Therefore,

$$|\langle\Phi_n,\Phi_n\rangle|\leqslant||\Phi_n||^2,\quad |\langle\Phi_n,P_n\Phi\rangle|\leqslant a||\Phi_n||||\Phi|| \tag{3.23}$$

for all $\Phi\in\mathscr{B}$, $\Phi_n\in\mathscr{B}_n$. Lemma 3.3 implies that property (3.23) can be presented as

$$|\langle Q_n\Phi_n,Q_n\Phi_n\rangle_n|\leqslant||\Phi_n||^2,$$

$$|\langle Q_n\Phi_n,Q_nP_n\Phi\rangle_n|\leqslant a||\Phi_n||||\Phi||. \tag{3.24}$$

Choose $\Phi=(\tilde{\gamma},\tilde{\rho},0)$. Then $Q_nP_n\Phi=\Phi$. Denote $Q_n\Phi_n=(\gamma_n,\rho_n,\varphi_n)$. It follows from the second property (3.24) that

$$\left|\sum_{su=1}^{m}\gamma_{n,s}^*\tilde{\gamma}_u g_{s+u}^{(n)}-\sum_{s=1}^{m}(\gamma_{n,s}^*\tilde{\rho}_s+\tilde{\gamma}_s\rho_{n,s}^*)\right|\leqslant a||(\tilde{\gamma},\tilde{\rho},0)||||\Phi_n||.$$

Choose $\tilde{\rho}_s^{(l)} = \delta_{sl}$, $\tilde{\gamma}_s = 0$. For different l we obtain

$$|\gamma_{n,s}| \leqslant a \max_l ||(0, \tilde{\rho}^{(l)}, 0)|| \, ||\Phi||_n \leqslant C_1 ||\Phi_n||$$

for some constant C_1. Analogously, $|\rho_{n,s} - \sum_{u=1}^m \gamma_{n,s} g_{s+u}^{(n)}| \leqslant C_2 ||\Phi_n||$ for some constant C_2. Therefore

$$|\rho_{n,s}| \leqslant C_3 ||\Phi_n||.$$

It follows from the first inequality (3.24) that

$$|(\varphi_n, \varphi_n)| \leqslant \left| \sum_{su=1}^m \gamma_{n,s}^* \gamma_{n,m} g_{s+u}^{(n)} \right| + \left| \sum_{s=1}^m (\gamma_{n,s}^* \rho_{n,s} + \rho_{n,s}^* \gamma_{n,s}) \right| + ||\Phi_n||^2 \leqslant C_4 ||\Phi_n||^2.$$

Therefore, $||\varphi_n|| \leqslant C_4^{1/2} ||\Phi_n||$. For norm (3.6) of the vector $Q_n \Phi_n$, the following estimation is satisfied: $||Q_n \Phi_n||_1 \leqslant C ||\Phi_n||$. Making use of the corollary of Lemma 3.2, we obtain statement of Lemma 3.16. $\square$

Lemma 3.17. *The sequence* $\{\Phi_n\}$ *is of the class* $[\Phi]$ *if and only if* $Q_n \Phi_n \to \Phi$.

Proof. The condition $\{\Phi_n\} \in [\Phi]$ means that $||\Phi_n - P_n \Phi|| \to 0$. It follows from Lemmas 3.15 and 3.17 that it is equivalent to

$$||Q_n \Phi_n - Q_n P_n \Phi|| \to 0. \tag{3.25}$$

Since $||Q_n P_n \Phi - \Phi|| \to 0$ according to Lemma 3.6, condition (3.25) is equivalent to $Q_n \Phi \to \Phi$. Lemma 3.17 is proved. $\square$

Lemma 3.17 implies Lemma 2.8.

4. Some properties of solutions of evolution equations

This section deals with investigations of properties of evolution operators for Eqs. (1.5), (1.8), (1.10), (1.11), (1.12) and (1.13). Lemmas 2.3, 2.4 and first parts of Theorems 2,3 are proved.

1. Investigate properties of the operators entering the right-hand side of evolution equations. As usual, we call operators which are self-adjoint with respect to the indefinite inner product in $\mathscr{B}$ or $\mathscr{B}_n$ as J-self-adjoint operators, while operators being self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathscr{L}_m}$ or $\langle \cdot, \cdot \rangle_{\mathscr{L}_m^n}$ will be called H-self-adjoint.

Lemma 4.1. *The operators* $\hat{Z}_n^{-1}\hat{H}_n$ *and* H *are* J*-self-adjoint.*

Proof. It follows from [8] that it is sufficient to check that the bounded operator $(\hat{Z}_n^{-1}\hat{H}_n + \lambda)^{-1}$ is J-self-adjoint for some real λ. Lemmas 3.3 and 3.7 imply that this property is equivalent to self-adjointness of the operator $\tilde{R}_n(\lambda) : (\gamma, \rho, \varphi) \mapsto (\tilde{\gamma}, \tilde{\rho}, \tilde{\varphi})$ with respect to the inner product $\langle \cdot, \cdot \rangle_n$. To justify the latter property, it is sufficient to check that for all $\Phi = (\gamma, \rho, \varphi)$ the inner product

$$\langle \Phi, \tilde{R}_n(\lambda)\Phi \rangle_n = \sum_{su=1}^{m} \gamma_s^* \tilde{\gamma}_u g_{s+u}^{(n)} - \sum_{s=1}^{m} (\gamma_s^* \tilde{\rho}_s + \tilde{\gamma}_s \rho_s^*) + (\varphi, \tilde{\varphi}) \tag{4.1}$$

is real. It follows from (3.11) that

$$\sum_{su=1}^{m} \gamma_s^* \tilde{\gamma}_u g_{s+u}^{(n)} = \lambda \sum_{su=1}^{m} \tilde{\gamma}_s^* \tilde{\gamma}_u g_{s+u}^{(n)} + \sum_{su=1}^{m} g_{s+u-1}^{(n)} \tilde{\gamma}_s^* \tilde{\gamma}_u$$

$$+ \tilde{c}_m^* \sum_{u=1}^{m} g_{m+u}^{(n)} \tilde{\gamma}_u - \tilde{\gamma}_1^* \sum_{u=1}^{m} g_u^{(n)} \tilde{\gamma}_u,$$

$$\sum_{s=1}^{m} \gamma_s^* \tilde{\rho}_s = \sum_{s=1}^{m} \lambda \tilde{\gamma}_s^* \tilde{\rho}_s + \sum_{s=1}^{m-1} \tilde{\gamma}_{s+1}^* \tilde{\rho}_s + \tilde{c}_m^* \tilde{\rho}_m,$$

$$\sum_{s=1}^{m} \tilde{\gamma}_s \rho_s^* = \sum_{s=2}^{m} \tilde{\gamma}_s \tilde{\rho}_{s-1}^* + \lambda \sum_{s=1}^{m} \tilde{\gamma}_s \tilde{\rho}_s^*$$

$$+ \sum_{s=1}^{m} g_{m+s}^{(n)} \tilde{\gamma}_s \tilde{c}_m^* + \tilde{\gamma}_1 \sum_{u=1}^{m} g_u^{(n)} \tilde{\gamma}_u^*,$$

$$(\varphi, \tilde{\varphi}) = (\tilde{\varphi}, (\hat{T} + \lambda)\tilde{\varphi}) + \tilde{c}_m^* (\tilde{\rho}_m + z_{2m,n} \tilde{c}_m^*).$$

Therefore, expression (4.1) is real.

Self-adjointness of the operator H is checked analogously [24,25]. Lemma 4.1 is proved. □

Lemma 4.1 and analog of the theorem for the Pontriagin spaces [17] imply statements of Lemmas 2.3 and 2.4.

Lemma 4.2. *The operator* $\hat{Z}_n^{-1}\hat{H}_n$ *is presented as a sum*

$$\hat{Z}_n^{-1}\hat{H}_n = H_n^1 + H_n^2 \tag{4.2}$$

of a H-self-adjoint operator $\hat{H}_n^1$ and a bounded operator $\hat{H}_n^2$; for some n-independent quantities B_1 and B_2

$$\hat{H}_n^1 \geqslant B_1, ||\hat{H}_n^2|| \leqslant B_2. \tag{4.3}$$

The operator $\hat{H}$ is a sum $\hat{H}_1 + \hat{H}_2$ of a H-self-adjoint operator $\hat{H}_1$ being semi-bounded below and a bounded operator $\hat{H}_2$.

To prove this lemma, let us prove Lemmas 4.3–4.7.

Lemma 4.3. *The function $f(\lambda) = \lambda(\chi, \hat{T}^{-k}(\hat{T}+\lambda)^{-1}\chi)$ increases and tends to infinity as $\lambda \to \infty$.*

Proof. Since the operator $\hat{T}$ is positive and self-adjoint, the difference

$$f(\lambda_1) - f(\lambda_2) = (\lambda_1 - \lambda_2)(\chi, \hat{T}^{-k+1}(\hat{T}+\lambda_1)^{-1}(\hat{T}+\lambda_2)^{-1})\chi)$$

is positive as $\lambda_1 > \lambda_2$. Thus, f increases.

Check that $f(\lambda)$ tends to infinity as $\lambda \to \infty$. Suppose, that $f(\lambda) < C$ for some C. Then the property of positive definiteness of the operator $\hat{T}$ implies that for all b

$$\lambda(\chi, \hat{T}^{-k}\theta(b - \hat{T})(\hat{T}+\lambda)^{-1}\chi) \leqslant C.$$

Consider the limit $\lambda \to \infty$. We find $(\chi, \hat{T}^{-k}\theta(b-\hat{T})\chi) \leqslant C$. According to corollary of Lemma 3.5, we obtain a contradiction with the condition $\chi \notin \mathscr{H}^{-k}$. Lemma 4.3 is proved. □

Lemma 4.4. *For all $C > 0$ there exist some λ_0 and n_0 such that for all $\lambda > \lambda_0$* and *$n > n_0$ $f_n(\lambda) = \lambda(\chi_n, \hat{T}^{-k}(\hat{T}+\lambda)^{-1}\chi_n) > C$.*

Proof. Suppose that for some C for all λ_0 and n_0 there exist $\lambda > \lambda_0$ and $n > n_0$ such that $f_n(\lambda) \leq C$. Analogous to the previous subsection, we justify that the function $f_n(\lambda)$ is increasing. This implies that $f_n(\lambda_0) \leqslant C$. Therefore, for some sequence $n_p \to \infty$ $f_{n_p}(\lambda_0) \leqslant C$. Consider a limit $p \to \infty$. We find $f(\lambda_0) \leqslant C$ for all λ_0. Lemma 4.4 is proved. □

Let $\Phi = (\gamma, \rho, \varphi) \in \mathscr{B}$. Denote $\tilde{\Phi}_n = \tilde{R}_n(\lambda)\Phi = (\tilde{\gamma}_n(\lambda), \tilde{\rho}_n(\lambda), \tilde{\varphi}_n(\lambda)) \in \mathscr{B}$. $\tilde{\Phi}_n$ is determined from system (3.11).

Lemma 4.5. *For some constants λ_0, n_0 and A_4 for $\lambda \geqslant \lambda_0$ and $n \geqslant n_0$ the operator $\tilde{R}_n(\lambda)$ is well defined and obeys properties*

$$|\tilde{c}_n^m| \leqslant A_4 ||\Phi||_1,$$

$$|\tilde{c}_n^m| ||\lambda \hat{T}^{-m}(\hat{T}+\lambda)^{-1}\chi_n|| \leqslant A_4 ||\Phi||_1. \tag{4.4}$$

Proof. It follows from system (3.11) that $\tilde{c}_n^m = (a_n(\lambda))^{-1} b_n(\lambda)$, where $a_n(\lambda)$ has the form (3.13), while

$$b_n(\lambda) = (\hat{T}^{-m}(\hat{T}+\lambda)^{-1}\chi_n, \varphi) + \sum_{s=1}^{m} (-\lambda)^{-s}\rho_{m-s+1} + \sum_{s=1}^{m}\sum_{l=1}^{m+1-s} g_s^{(n)}(-\lambda)^{-l-m}\gamma_{s+l-1}.$$

For some A_5, the following property is satisfied:

$$|b_n(\lambda)| \leqslant (||\hat{T}^{-m}(\hat{T}+\lambda)^{-1}\chi_n|| + A_5\lambda^{-1})||\Phi||_1.$$

Obtain an estimation for $a_n(\lambda)$.

1. At $k = 2m$ $z_{2m,n} = 0$. Lemma 4.4 implies

$$a_n(\lambda) \geqslant \tfrac{1}{2}(\chi_n, \hat{T}^{-2m}(\hat{T}+\lambda)^{-1}\chi_n) + \tfrac{1}{2},$$

for sufficiently large λ_0 and n_0. Therefore,

$$|b_n(\lambda)/a_n(\lambda)| \leqslant 2A_1\lambda^{-1}||\Phi||_1 - \frac{2||\hat{T}^{-m}(\hat{T}+\lambda)^{-1}\chi_n||}{(\chi_n, \hat{T}^{-2m}(\hat{T}+\lambda)^{-1}\chi_n)}||\Phi||_1.$$

The inequalities $||\hat{T}^{-m}(\hat{T}+\lambda)^{-1}\chi_n|| \leqslant ||\hat{T}^{-m-1}\chi_n|| \leqslant C_1$, $\lambda||\hat{T}^{-m}(\hat{T}+\lambda)^{-1}\chi_n||^2 - (\chi_n, \hat{T}^{-2m}(\hat{T}+\lambda)^{-1}\chi_n) = -(\chi_n, \hat{T}^{-2m}\hat{T}(\hat{T}+\lambda)^{-2}\chi_n) \leqslant 0$ imply Eq. (4.4).

2. Let $k = 2m+1$. Then

$$(-\lambda)^{-1}a_n(\lambda) = \sum_{s=1}^{2m+1} g_s^{(n)}(-\lambda)^{s-1-2m} + (\chi_n, \hat{T}^{-2m-1}(\hat{T}+\lambda)^{-1}\chi_n) \geqslant \frac{1}{2}(\chi_n, \hat{T}^{-2m-1}(\hat{T}+\lambda)^{-1}\chi_n) + \frac{1}{2}.$$

We obtain the following inequality: $|b_n(\lambda)/a_n(\lambda)| \leqslant \lambda^{-1}C_2||\Phi||_1$ and Eq. (4.4).

Existence of the operator $\tilde{R}_n(\lambda)$ for $\lambda \geqslant \lambda_0$ and $n \geqslant n_0$ is a corollary of the proved property $a_n(\lambda) \neq 0$. Lemma 4.5 is proved. □

Lemma 4.6. *For some constant B_3 the following property is satisfied*: $\lambda||\tilde{R}_n(\lambda)||_1 = \sup_{\Phi\in\mathscr{B}} \frac{\lambda||\tilde{R}_n(\lambda)\Phi||_1}{||\Phi||_1} \leqslant B_3$.

Proof. It follows from the second equation of system (3.11) that $\lambda|\tilde{\gamma}_{n,m}| \leqslant C_1||\Phi||_1$. We obtain from the first equation by induction that $\lambda|\tilde{\gamma}_{n,s}| \leqslant C_1||\Phi||_1$ for $s = \overline{1, m-1}$. It follows from the positive definiteness of the operator $\hat{T}$ and from the third equation that

$$|\lambda\tilde{\varphi}_n(\lambda)|| \leqslant ||\lambda(\hat{T}+\lambda)^{-1}\varphi|| + |\tilde{c}_n^m(\lambda)|||\hat{T}^{-m}\lambda(\hat{T}+\lambda)^{-1}\chi_n||$$
$$\leqslant ||\varphi|| + A_4||\Phi||_1 \leqslant (A_4+1)||\Phi||_1.$$

The latter equation of system (3.11) implies $\lambda|\tilde{\rho}_{n,1}| \leqslant C_3||\Phi||_1$. The fourth equation implies $\lambda|\tilde{\rho}_{n,s}| \leqslant C_4||\Phi||_1$, $s=\overline{2,m}$. We obtain statement of Lemma 4.6. □

Corollary of Lemma 3.2 implies

Corollary. *For some constant B_4 the following property is satisfied*: $\lambda||\tilde{R}_n(\lambda)|| \leqslant B_4$.

Lemma 4.7. *There exist constants B_5, λ_0 and n_0 such that for $\lambda \geqslant \lambda_0$ and $n \geqslant n_0$*

$$\lambda||(\lambda+\hat{Z}_n^{-1}\hat{H}_n)^{-1}|| \leqslant B_5,$$
$$\lambda||(\lambda+\hat{H})^{-1}|| \leqslant B_5. \tag{4.5}$$

Proof. Lemma 3.11 implies the second property of (4.5). It follows from Lemmas 3.14, 3.16 and 3.7 that

$$\lambda||(\lambda+\hat{Z}_n^{-1}\hat{H}_n)^{-1}|| = \sup_{\Phi_n\in\mathscr{B}_n} \frac{||\lambda(\lambda+\hat{Z}_n^{-1}\hat{H}_n)^{-1}\Phi_n||}{||\Phi_n||}$$
$$\leqslant \sup_{\Phi_n\in\mathscr{B}_n} \frac{A_3||Q_n\lambda(\lambda+\hat{Z}_n^{-1}\hat{H}_n)^{-1}\Phi_n||}{A_2||Q_n\Phi_n||}$$
$$= \sup_{\Phi_n\in\mathscr{B}_n} \frac{A_3||\lambda\tilde{R}_n(\lambda)Q_n\Phi_n||}{A_2||Q_n\Phi_n||} \leqslant B_4A_3/A_2.$$

The lemma is proved. □

Proof of Lemma 4.2. By $R_n^{\parallel}$ we denote the orthogonal with respect to the inner product (2.2) projector on the subspace $\mathscr{L}_m^n$, by $R_n^{\perp}$ denote the orthogonal projector on $(\mathscr{L}_m^n)^{\perp}$. Set

$$H_n^1 = R_n^{\perp}\hat{Z}_n^{-1}\hat{H}_nR_n^{\perp},$$
$$H_n^2 = \hat{Z}_n^{-1}\hat{H}_n - H_n^1 = R_n^{\parallel}\hat{Z}_n^{-1}\hat{H}_n + \hat{Z}_n^{-1}\hat{H}_nR_n^{\parallel} + R_n^{\parallel}\hat{Z}_n^{-1}\hat{H}_nR_n^{\parallel}.$$

Check that the operators H_1 and H_2 obey properties (4.3). Since the inner products $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathscr{L}_m^n}$ coincide on $(\mathscr{L}_m^n)^{\perp}$, H_1^n is a H-self-adjoint operator. Find an estimation on the norm of the operator H_n^2. The operator $R_n^{\parallel}$ is rewritten as $R_n^{\parallel} = -\sum_{ij=1}^m e_i^{(n)}\langle e_j^{(n)},\Phi_n\rangle M_{ij}^{(n)}$, where $M_{ij}^{(n)}$ is a matrix being inverse to (3.22),

$e_i^{(n)} = (\hat{Z}_n\hat{H}_n + \lambda)^{-1}P_n(\hat{H}+\lambda)e_i$. For the norm of the operator $\hat{Z}_n^{-1}\hat{H}_nR_n^{\parallel}$, we obtain the following estimation:

$$||\hat{Z}_n^{-1}\hat{H}_nR_n^{\parallel}|| \leqslant m^2 \max_{ij}|M_{ij}^{(n)}| \max_i ||\hat{Z}_n^{-1}\hat{H}_ne_i^{(n)}|| \max_i ||e_i^{(n)}||. \tag{4.6}$$

Since $M_{ij}^{(n)} \to_{n\to\infty} \delta_{ij}$,

$$\hat{Z}_n^{-1}\hat{H}_ne_i^{(n)} = P_n(H+\lambda)e_i - \lambda e_i^{(n)},$$

$$||e_i^{(n)}|| \leqslant ||(\hat{Z}_n^{-1}\hat{H}_n+\lambda)^{-1}||||P_n||||(H+\lambda)e||,$$

quantity (4.6) is bounded uniformly with respect to n. An analogous estimation can be obtained for norms of the operators $\hat{Z}_n^{-1}\hat{H}_nR_n^{\parallel}$–$R_n^{\parallel}\hat{Z}_n^{-1}\hat{H}_nR_n^{\parallel}$. Therefore, $||H_n^2|| \leqslant B$.

To check that the operator $\hat{H}_n^1$ is semibounded below, present it as a sum of an absolutely convergent in the norm-topology series: $(\lambda+\hat{H}_n^1)^{-1} = \sum_{k=0}^{\infty}(\lambda+\hat{Z}_n^{-1}\hat{H}_n)^{-1}(H_n^2(\lambda+\hat{Z}_n^{-1}\hat{H}_n)^{-1})^k$ provided that $\lambda \geqslant BB_5$ and $\lambda \geqslant \lambda_0$. Namely, for this case the norm of the kth term of the series is not larger than $\frac{B^kB_5^{k+1}}{\lambda^{k+1}}$. Therefore, for sufficiently large λ and $n \geqslant n_0$ the resolvent of the H-self-adjoint operator $\hat{H}_n^1$ is bounded. Therefore, the spectrum of the operator $\hat{H}_n^1$ is semibounded below by an n-independent quantity. Analogously, we prove statement of Lemma 4.2 for the operator $\hat{H}$. Lemma 4.2 is proved. □

Without loss of generality, suppose that the quantity C entering Lemma 4.2 obeys the property $C>0$. Otherwise, one can redefine the operators $\hat{H}_n^1$ and $\hat{H}_n^2$.

Representation (4.2) and results of [11] imply the following properties of evolution operators for Eqs. (1.8)–(1.13) on $[0,t]$.

Lemma 4.8. *The following properties are satisfied*:

$$||e^{-it\hat{Z}_n^{-1}\hat{H}_n}|| \leqslant e^{Bt}, \quad ||e^{-itH}|| \leqslant e^{Bt}.$$

$$||e^{-t\hat{Z}_n^{-1}H_n}|| \leqslant e^{(B-C)t}, \quad ||e^{-tH}|| \leqslant e^{(B-C)t}.$$

Proof. It was shown in [11] that if T is a generator for a one-parametric semigroup e^{-Tt} such that

$$||e^{-Tt}|| \leqslant Me^{\beta t}, \tag{4.7}$$

while A is a bounded operator, then $T+A$ is also a generator of a semigroup. Moreover, $||e^{-(T+A)t}|| \leqslant Me^{(\beta+M||A||)t}$. The operator $i\hat{H}_n^1$ for the case of a H-self-adjoint $\hat{H}_n^1$ is a generator of a one-parametric semigroup of H-unitary operators.

This means that property (4.7) is satisfied for $M = 1$, $\beta = 0$. Therefore, $||e^{-it\hat{Z}_n^{-1}H_n}|| \leqslant e^{||H_n^2||t} \leqslant e^{Bt}$. The second inequality is checked analogously.

Since the operator $\hat{H}_1$ satisfies the property $\hat{H}_1 \geqslant C$, it is a generator of a one-parametric semigroup, while $||e^{-\hat{H}_n^1 t}|| \leqslant e^{-Ct}$. We proved Lemma 4.8. $\square$

Note also that since $\hat{H}_n^1 + \hat{H}_n^2$ is a generator of a one-parametric semigroup, there exists a unique solution of the Cauchy problems for Eqs. (1.13) and (1.11) for $\Phi_n(0) \in D(\hat{H}_n^1 + \hat{H}_n^2)$. This solution continuously depends on the initial conditions. Lemma 2.10 is proved. $\square$

To prove Lemma 2.11, let us justify some auxiliary statements analogously to Ref. [15].

Consider the following differential equation in the Banach space $\mathscr{B}$:

$$-\frac{d^2\Phi}{dt^2} = \hat{A}\Phi, \qquad \Phi(t) \in D(\hat{A}) \subset \mathscr{B}, \quad t \in [0, T] \tag{4.8}$$

with closed operator $\hat{A}$.

Definition 4.1. We say that the Cauchy problem for Eq. (4.8) is formulated uniformly correct if for all $\Phi(0)$ and $\dot{\Phi}(0)$ from $D(\hat{A})$ there exists a unique two-times continuously differentiable function $\Phi(t) \in D(\hat{A})$ satisfying Eq. (4.8) and initial conditions. The dependence of $\Phi(t)$ on initial conditions is uniformly continuous.

Define on $D(A)$ the operators $V(t)$ and $W(t)$ from property (2.8), $\Phi(t) = V(t)\Phi(0) + W(t)\dot{\Phi}(0)$. Denote by $\dot{V}(t)$ and $\dot{W}(t)$ the operators from $D(A)$ to $\mathscr{B}$ which are defined from the relation $\dot{\Phi}(t) = \dot{V}(t)\Phi(0) + \dot{W}(t)\dot{\Phi}(0)$.

Let $\mathscr{B}$ be a Hilbert space.

Lemma 4.9. *Let A be a H-self-adjoint semibounded below operator in $\mathscr{B}$: $A \geqslant C_1 > 0$. Then the Cauchy problem for Eq.* (4.8) *is uniformly correct and*

$$||V(t)|| \leqslant 1, \quad ||W(t)|| \leqslant 1/\sqrt{C_1}. \tag{4.9}$$

Proof. The function of the form

$$\Phi(t) = \cos(\sqrt{\hat{A}}t)\Phi(0) + \frac{\sin(\sqrt{\hat{A}}t)}{\sqrt{\hat{A}}}\dot{\Phi}(0) \tag{4.10}$$

is a solution of the Cauchy problem for Eq. (4.8) [15]. Prove the property of uniqueness. Let $\Phi(0) = 0$, $\dot{\Phi}(0) = 0$. Consider the function $f(t) = \frac{1}{2}(\dot{\Phi}(t), \dot{\Phi}(t)) + \frac{1}{2}(\Phi(t), \hat{A}\Phi(t))$. It satisfies the conditions $f(0) = 0$, $df/dt = 0$. Therefore, $f(t) = 0$. Since the operator A is semibounded below, one has $(\dot{\Phi}, \dot{\Phi}) = 0$, $(\Phi, \hat{A}\Phi) = 0$.

Therefore, $\Phi = 0$. The property of uniqueness is proved. It follows from the explicit form of solution of Eq. (4.10) the property of uniform correctness of the Cauchy problem and relations (4.9). Lemma 4.10 is proved. □

Suppose that there exists such ζ that the operator $(A+\zeta)^{-1}$ is well defined.

Lemma 4.10. *Let the Cauchy problem for Eq.* (4.8) *be uniformly correct. Consider the equation*

$$-\frac{d^2\Phi(t)}{dt^2} = \hat{A}\Phi(t) + \xi(t), \qquad \Phi(t) \in D(\hat{A}) \subset \mathscr{B}, \quad t \in [0,T], \tag{4.11}$$

where $\xi(t) \in D(\hat{A}^2)$. $(\hat{A}+\zeta)^2\xi(t)$ *is a continuous function on* $[0,T]$. *Then the Cauchy problem for Eq.* (4.11) *has a unique solution of the form*

$$\Phi(t) = V(t)\Phi(0) + W(t)\dot{\Phi}(0) - \int_0^t d\tau\, W(t-\tau)\xi(\tau). \tag{4.12}$$

Proof. The uniqueness is obvious. Let Φ_1 and Φ_2 be two solutions of the Cauchy problem. Then their difference satisfies Eq. (4.8) and zero initial condition. It follows from uniform correctness of the Cauchy problem for Eq. (4.8) that $\Phi_1 - \Phi_2 = 0$.

To prove the lemma, it is sufficient to justify that the function

$$\Phi(t) = -\int_0^t d\tau\, W(t-\tau)\xi(\tau)$$

obeys Eq. (4.11) and zero initial condition. Check that

$$\frac{d\Phi(t)}{dt} = -\int_0^t d\tau\, \dot{W}(t-\tau)\xi(\tau). \tag{4.13}$$

Consider the difference

$$\begin{aligned}
&-\frac{\Phi(t+\delta t)-\Phi(t)}{\delta t} + \dot{\Phi}(t)\\
&= \int_t^{t+\delta t} \frac{d\tau}{\delta t} W(t+\delta t-\tau)\xi(\tau) + \int_0^t d\tau \left(\frac{W(t+\delta t-\tau)-W(t-\tau)}{\delta t} - \dot{W}(t-\tau)\right)\xi(\tau)\\
&= \int_0^1 ds\, W(\delta t(1-s))\xi(t+\delta ts) + \int_0^t d\tau \int_0^1 ds(\dot{W}(t+\delta ts-\tau) - \dot{W}(t-\tau))\xi(\tau)\\
&= \int_0^1 ds\, W(\delta t(1-s))\xi(t) + \int_0^1 ds\, W(\delta t(1-s))(\xi(t+s\delta t)-\xi(t))\\
&\quad + \delta t \int_0^t d\tau \int_0^1 ds \int_0^s ds' A W(t+s'\delta t-\tau)\xi(\tau).
\end{aligned}$$

The norm of this expression is not larger than

$$\int_0^1 ds||W(\delta t(1-s))||(||\xi(t)|| + ||\xi(t+s\delta t) - \xi(t)||$$
$$+ \delta t \int_0^t d\tau \int_0^1 ds \int_0^s ds'||\hat{A}W(t+s'\delta t - \tau)\xi(\tau)||.$$

According to the Lesbegue theorem (see, for example, [12]) this expression tends to zero as $\delta t \to 0$. Therefore, property (4.13) is checked. Initial conditions are obviously satisfied. Check Eq. (4.11). One has

$$-\frac{\dot{\Phi}(t+\delta t) - \dot{\Phi}(t)}{\delta t} - A\Phi(t) - \xi(t)$$
$$= \int_0^1 ds(\dot{W}(\delta t(1-s))\xi(t+s\delta t) - \xi(t))$$
$$+ \int_0^1 d\tau \int_0^1 ds(-W(t-\tau+s\delta t) + W(t-\tau))\hat{A}\xi(\tau).$$

According to the Lesbegue theorem, this expression tends to zero. The lemma is proved. □

Corollary. *Let the function* $\xi(t) \in \mathscr{B}$ *is continuous on* $[0, T]$, *while the function* $\Phi(t)$ *is a solution of Eq.* 4.11. *Then formula* 4.12 *is satisfied.*

Proof. It is sufficient to consider the case if initial conditions vanish; the general case can be reduced to it by the substitution of $\Phi(t)$ by $\Phi(t) - V(t)\Phi(0) - W(t)\dot{\Phi}(0)$. Consider the function $v(t) = (\hat{A} + \zeta)^{-2}\Phi(t)$ satisfying the following equation:

$$-\frac{d^2v(t)}{dt^2} = \hat{A}v(t) + (\hat{A} + \zeta)^{-2}\xi(t),$$

and zero initial condition. Therefore,

$$v(t) = -(\hat{A} + \zeta)^{-2} \int_0^t d\tau\, W(t-\tau)\xi(\tau).$$

We obtain statement of the corollary.

It happens that the condition that $(\hat{A} + \zeta)^{-2}\xi$ is continuous can be substituted by the condition that ξ is two times continuously differentiable.

Lemma 4.11. *Let all the conditions of Lemma* 4.10 *be satisfied, except for continuity of* $(\hat{A} + \zeta)^{-2}\xi$. *Let also the function* $\xi(t)$ *be two times continuously differentiable and* $\xi(0) \in D(\hat{A})$. *Then statement of Lemma* 4.10 *is satisfied.*

Proof. The property of uniqueness of the solution of the Cauchy problem is checked analogously to Lemma 4.10. Corollary of Lemma 4.10 tells us that the solution of the Cauchy problem is given by formula (4.12), provided it exists. It is sufficient then to check that expression (4.12) satisfies Eq. (4.11) and initial condition. It is sufficient to consider the case $\Phi(0)=0$, $\dot{\Phi}(0)=0$. Define $W_1(t)=\int_0^t d\tau\, W(\tau)$, $W_2(\tau)=\int_0^\tau W_1(\tau)$. Substituting $\xi(t)=\xi(0)+\int_0^t d\tau \xi(\tau)$, we find

$$\int_0^t d\tau\, W(t-\tau)\xi(\tau) = W_1(\tau)\xi(0) + \int_0^t ds\, W_1(t-s)\dot{\xi}(s).$$

Applying this formula again, we obtain

$$\int_0^t W(t-\tau)\xi(\tau)d\tau = W_1(t)\xi(0) + W_2(t)\dot{\xi}(0) + \int_0^t ds\, W_2(t-s)\ddot{\xi}(s). \tag{4.14}$$

It follows from the definition of the operator W that it satisfies the following equation:

$$\ddot{W}(t)\Phi = -\hat{A}W(t)\Phi, \quad \Phi \in D(A) \tag{4.15}$$

and commutes on $D(A)$ with the operator A. Integrating twice Eq. (4.15), we find

$$-\hat{A}W_2(t) = W(t) - W(0) - \dot{W}(0)t = W(t) - t \tag{4.16}$$

on $D(A)$. Operator (4.16) is bounded and can therefore be continued on $\mathscr{B}$. It follows from Eqs. (4.16) and (4.14) that

$$-\hat{A}\int_0^t W(t-\tau)\xi(\tau)d\tau = \int_0^t ds[W(t-s)-(t-s)]\ddot{\xi}(s) + \dot{W}(t)\xi(0) + (W(t)-t)\dot{\xi}(0). \tag{4.17}$$

Furthermore,

$$\frac{d^2}{dt^2}\left[\int_0^t W(\tau)\xi(t-\tau)\,d\tau\right] = W(t)\dot{\xi}(0) + \dot{W}(t)\xi(0) + \int_0^t W(\tau)\ddot{\xi}(t-\tau)\,d\tau. \tag{4.18}$$

Comparing Eqs. (4.17) and (4.18), we obtain statement of the lemma. □

Lemma 4.12. *Let the operator $\hat{A}$ be a sum of a H-self-adjoint semibounded below operator $\hat{T}_1 \geqslant C_1 > 0$ and a bounded operator $\hat{T}_2$, $||\hat{T}_2|| \leqslant C_2$. Then the Cauchy problem for Eq.* (4.8) *is uniformly correct and*

$$||V(t)|| \leqslant e^{C_2 t/\sqrt{C_1}}, \qquad ||W(t)|| \leqslant e^{C_2 t/\sqrt{C_1}}/\sqrt{C_1}.$$

Proof. According to corollary of Lemma 4.10, the function $\Phi(t)$ is a solution of the Cauchy problem for Eq. (4.8) if and only if

$$\Phi(t) = V_1(t)\Phi(0) + W_1(t)\dot{\Phi}(0) - \int_0^t d\tau\, W_1(t-\tau)\hat{T}_2\Phi(\tau), \tag{4.19}$$

where $V_1(t) = \cos(\sqrt{\hat{T}_1}t)$, $W_1(t) = \frac{\sin(\sqrt{\hat{T}_1}t)}{\sqrt{\hat{T}_1}}$. The abstract Volterra equation (4.19) has a unique solution (see, for example, proof of [12]), which can be presented as a sum of an absolutely convergent in the norm-topology series:

$$\Phi(t) = \sum_{n=0}^{\infty} (-1)^n \int_{t.\tau_1 > ,,, > \tau_n > 0} d\tau_1 \dots d\tau_n \, W_1(t - \tau_1)\hat{T}_2 \dots W_1(\tau_{n-1} - \tau_n)\hat{T}_2 \times (V_1(\tau_n)\Phi(0) + W_1(\tau_n)\dot{\Phi}(0)).$$

Therefore,

$$||V(t)|| \leqslant \sum_{n=0}^{\infty} \frac{(C_2 t/\sqrt{C_1})^n}{n!} = e^{C_2 t/\sqrt{C_1}},$$

$$||V(t)|| \leqslant \sum_{n=0}^{\infty} \frac{(C_2 t/\sqrt{C_1})^n}{n!\sqrt{C_1}} = e^{C_2 t/\sqrt{C_1}}/\sqrt{C_1}.$$

Lemma 4.12 is proved. □

Lemmas 4.12 and 4.2 imply

Corollary. *The statement of Lemma* 2.11 *is satisfied. For* $t \in [0, T]$ *there exists an n-independent quantity* M *such that* $||V_n(t)|| \leqslant M$, $||V(t)|| \leqslant M$, $||W_n(t)|| \leqslant M$, $||W(t)|| \leqslant M$.

5. Convergence in generalized strong sense

Let us justify the property of generalized strong convergence of the operators U_n, V_n and W_n entering Theorems 1–3. Let us first investigate some properties of generalized strong convergence. Formulate an analog of the Banach–Steinhaus theorem.

Lemma 5.1. *Let* $A_n : \mathcal{B} \to \mathcal{B}_n$, $n = 1, 2, \dots$, *be a sequence of operators satisfying the property* $||A_n|| \leqslant M < \infty$ *for some n-independent constant* M; $\mathcal{D} \subset \mathcal{B}$*—is a dense subset of* $\mathcal{B}$, $||A_n v|| \to_{n\to\infty} 0$ *for* $v \in \mathcal{D}$. *Then* $||A_n v|| \to_{n\to\infty} 0$ *for* $v \in \mathcal{B}$.

Proof. Let $v \in \mathcal{B}$, $\varepsilon > 0$. Choose such $v' \in \mathcal{D}$ that $||v - v'|| \leqslant \frac{\varepsilon}{2M}$. Choose n_0 such that for $n \geqslant n_0 ||A_n v'|| \leqslant \varepsilon/2$. Then $||A_n v|| \leqslant ||A_n v'|| + ||A_n|| ||v - v'|| \leqslant \varepsilon$. We obtain statement of lemma. □

Remark. The proof of Ref. [9] of the Banach–Steinhaus theorem cannot be generalized to the case of $\{P_n\}$-strong convergence. Proof of [28] uses also the condition $||P_n v|| \to_{n\to\infty} ||v||$.

Lemma 5.2. *Let $A_n : \mathscr{B} \to \mathscr{B}_n$, $n = 1, 2, \ldots$, be a sequence of operators satisfying the following property: for each $v \in \mathscr{B}$ the sequence $||A_n v||$ is bounded. Then $||A_n|| \leqslant M$ for some n-independent quantity M.*

Proof. Analogous to [9].

Lemma 5.3. *Let $B_n : \mathscr{B}_n \to \mathscr{B}_n$, $n = 1, 2, \ldots$, be a sequence of operators which $\{P_n\}$-strongly converges to the operator $B : \mathscr{B} \to \mathscr{B}$. Then the sequence $||B_n P_n||$ is bounded.*

Proof. Denote $A_n = B_n P_n$. For all $v \in \mathscr{B}$ $||A_n v - P_n B v|| \to_{n\to\infty} 0$, so that the sequence $||A_n v - P_n B v||$ is bounded, $||A_n v - P_n B v|| \leqslant M$. Therefore, $||A_n v|| \leqslant ||A_n v - P_n B v|| + ||P_n|| ||Bv|| \leqslant M + a||Av||$. Lemma 5.2 implies statement of the lemma. □

Lemma 5.4. *Let $u_n \in \mathscr{B}_n$, $n = 1, 2, \ldots$, is a sequence of vectors from the class $[u]$, $u \in \mathscr{B}$, $A_n : \mathscr{B}_n \to \mathscr{B}_n$, $n = 1, 2, \ldots$, is a uniformly bounded ($||A_n|| \leqslant M$) sequence of operators which $\{P_n\}$-strongly converges to the operator $A : \mathscr{B} \to \mathscr{B}$. Then the sequence $\{A_n u_n\}$ is of the class $[Au]$.*

Proof. One has

$$||A_n u_n - P_n A u|| \leqslant ||A_n|| ||u_n - P_n u|| + ||A_n P_n u - P_n A u|| \to_{n\to\infty} 0.$$

Proofs of Theorems 1 and 2 are identical to Ref. [11].

Proof of Theorem 3. Let $v \in \mathscr{B}$, ζ satisfy the condition $a(\lambda) \neq 0$. Consider the function $w_n(t)$ of the form

$$v_n(t) = V_n(t)(\hat{Z}_n^{-1}\hat{H}_n + \zeta)^{-1} P_n (\hat{H} + \zeta)^{-1} v - (\hat{Z}_n^{-1}\hat{H}_n + \zeta)^{-1} P_n V(t)(\hat{H} + \zeta)^{-1} v.$$

It obeys the following condition:

$$-\frac{d^2 v_n(t)}{dt^2} = \hat{Z}_n^{-1}\hat{H}_n w_n(t) + \xi_n(t), \tag{5.1}$$

where

$$\xi_n(t) = (P_n(\hat{H} + \zeta)^{-1} - (\hat{Z}_n^{-1}\hat{H}_n + \zeta)^{-1} P_n) V(t) v.$$

The initial condition for Eq. (5.1) has the form $v_n(0) = 0$, $\dot{v}_n(0) = 0$. Corollary of Lemma 4.10 implies that

$$v_n(t) = \int_0^t d\tau \, W_n(t - \tau) \xi_n(\tau).$$

Therefore,

$$||w_n(t)|| \leqslant M \int_0^t d\tau ||\xi_n(\tau)||. \tag{5.2}$$

or each $\tau ||\xi_n(t)||$ tends to zero because of Lemma 3.15. Furthermore,

$$||\xi_n(\tau)|| \leqslant (||(\hat{Z}_n^{-1}\hat{H}_n + \zeta)^{-1} P_n|| + ||P_n|| ||(\hat{H} + \zeta)^{-1}||) M ||v||,$$

so that the sequence $||\xi_n(\tau)||$ is uniformly bounded according to Lemma 5.3. The Lesbegue theorem (see, for example, [12]) implies that the integral in the right-hand side of formula (5.2) tends to zero. Therefore, $||v_n(t)|| \to_{n\to\infty} 0$, so that

$$||V_n(t)(\hat{Z}_n^{-1}\hat{H}_n + \zeta)^{-1} P_n \Phi - (\hat{Z}_n^{-1}\hat{H}_n + \zeta)^{-1} P_n V(t)\Phi|| \to_{n\to\infty} 0. \tag{5.3}$$

for $\Phi = (\hat{H} + \zeta)^{-1} v$. Property (5.3) is satisfied for all $\Phi \in D(H)$, on the dense subset of $\mathscr{B}$. Therefore, property (5.3) is satisfied for all $\Phi \in \mathscr{B}$. Furthermore,

$$||V_n(t)((H + \zeta)^{-1} P_n - P_n(H + \zeta)^{-1})\Phi|| \to_{n\to\infty} 0,$$

$$||(\hat{Z}_n^{-1}\hat{H}_n + \zeta)^{-1} P_n - P_n(H + \zeta)^{-1}) V(t)\Phi|| \to_{n\to\infty} 0. \tag{5.4}$$

Eqs. (5.3) and (5.4) imply that

$$||(V_n(t)P_n - P_n V(t))\tilde{\Phi}|| \to_{n\to\infty} 0 \tag{5.5}$$

under condition $\tilde{\Phi} = (\hat{H} + \zeta)^{-1}\Phi$. Relation (5.5) is satisfied on the dense subset $D(\hat{H})$ of $\mathscr{B}$. Therefore, it is satisfied on $\mathscr{B}$. First statement of Theorem 3 is proved. Second statement is proved analogously. □

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