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Barycentric selectors and a Steiner-type point of a convex body in a Banach space

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Abstract

Let F be a mapping from a metric space $(\mathscr{M}, \rho)$ into the family of all m-dimensional affine subsets of a Banach space X. We present a Helly-type criterion for the existence of a Lipschitz selection f of the set-valued mapping F, i.e., a Lipschitz continuous mapping $f : \mathscr{M} \to X$ satisfying $f(x) \in F(x), x \in \mathscr{M}$. The proof of the main result is based on an inductive geometrical construction which reduces the problem to the existence of a Lipschitz (with respect to the Hausdorff distance) selector $S_X^{(m)}$ defined on the family $\mathscr{K}_m(X)$ of all convex compacts in X of dimension at most m. If X is a Hilbert space, then the classical Steiner point of a convex body provides such a selector, but in the non-Hilbert case there is no known way of constructing such a point. We prove the existence of a Lipschitz continuous selector $S_X^{(m)} : \mathscr{K}_m(X) \to X$ for an *arbitrary* Banach space X. The proof is based on a new result about Lipschitz properties of the center of mass of a convex set.

Keywords: Set-valued mapping; Lipschitz selection; Helly-type theorems; Steiner point; Barycenter

1. Main results

Let $(X, \|\cdot\|)$ be a Banach space and let $\mathscr{K}_m(X)$ be the family of all convex compact subsets of X of dimension at most m. Given a set $\mathscr{M}$ and a mapping $F : \mathscr{M} \to \mathscr{K}_m(X)$, consider the following problem: under what conditions does the family of sets $\{F(x) : x \in \mathscr{M}\}$ have a common point? Recall that the Helly intersection

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theorem [DGK] states that there is such a common point

$$a \in \bigcap_{x \in \mathcal{M}} F(x) \tag{1.1}$$

whenever for every $(m+2)$-point subset $\mathcal{M}' \subset \mathcal{M}$ the family $\{F(x) : x \in \mathcal{M}'\}$ has a common point. Condition (1.1) is equivalent to the existence of a mapping $f : \mathcal{M} \to X$ such that $f(x) \in F(x)$ for every $x \in \mathcal{M}$ and, furthermore, $f(x) = f(y)$ for all $x, y \in \mathcal{M}$. Then a is simply the constant value of the mapping f. This latter condition on f can be considered as a special case of a *Lipschitz condition*, i.e., $\|f(x) - f(y)\| \leqslant \rho(x, y)$ for all $x, y \in \mathcal{M}$, where, in this case, $\rho \equiv 0$ is the trivial distance function on $\mathcal{M}$.

We recall that a (single-valued) mapping $f : \mathcal{M} \to X$ is called a *selection* of a set-valued mapping $F : \mathcal{M} \to 2^X$ if $f(x) \in F(x)$ for all $x \in \mathcal{M}$. Let $(\mathcal{M}, \rho)$ be a *pseudometric* space, i.e., $\rho : \mathcal{M} \times \mathcal{M} \to \mathbf{R}_+$ is symmetric and satisfies the triangle inequality, but may admit the value 0 for $x \neq y$. Then a selection f is said to be Lipschitz if it satisfies the Lipschitz condition

$$\|f(x) - f(y)\| \leqslant \lambda \rho(x, y), \quad x, y \in \mathcal{M}, \tag{1.2}$$

for some constant λ. We let $\mathrm{Lip}(\mathcal{M}; X)$ denote the space of all Lipschitz continuous mappings from $\mathcal{M}$ into X equipped with the seminorm $|f|_{\mathrm{Lip}(\mathcal{M};X)} := \inf \lambda$ where the infimum is taken over all constants λ which satisfy (1.2).

Thus Helly's theorem can be reformulated in terms of these particular Lipschitz selections (i.e., for $\rho \equiv 0$) as follows: *if the restriction of F to every $(m+2)$-point subset $\mathcal{M}' \subset \mathcal{M}$ has a Lipschitz selection, then F itself has a Lipschitz selection.*

This observation leads us to the following general selection problem: Given a set-valued mapping F from a pseudometric space $(\mathcal{M}, \rho)$ into a family of convex finite-dimensional subsets of X, find a Helly-type criterion for the existence of a Lipschitz selection of F.

We have recently considered this problem for the family of sets $\mathcal{K}_m(X)$ in [S4]. In this paper we shall study the same problem for the family $\mathcal{A}_m(X)$ of all *affine* subsets of X of dimension at most m. Our main result is the following

Theorem 1.1. *Assume we have a set-valued mapping $F : \mathcal{M} \to \mathcal{A}_m(X)$ such that, for every subset $\mathcal{M}' \subset \mathcal{M}$ consisting of at most 2^{m+1} points, the restriction $F|_{\mathcal{M}'}$ of F to $\mathcal{M}'$ has a Lipschitz selection $f_{\mathcal{M}'}$ whose seminorm satisfies $|f_{\mathcal{M}'}|_{\mathrm{Lip}(\mathcal{M}';X)} \leqslant 1$. Then F has a Lipschitz selection f with $|f|_{\mathrm{Lip}(\mathcal{M};X)}$ bounded by a constant depending only on m.*

The number 2^{m+1} is in general sharp, see [S2] ($X = \mathbf{R}^n$) and [S4, Remark 5.4].

For the case of a Hilbert space Theorem 1.1 has been proved in [S3]. The proof in [S3] uses certain properties of orthogonal projectors in a Hilbert space, and therefore cannot be adapted to the non-Hilbert case. In this paper we present another approach to the problem which works when X is an arbitrary Banach space. It is based on a result about Lipschitz properties of the barycenter of a variable set (Theorem 1.3). Other important ingredients of the proof are Lemmas 3.8 and 3.9, which are due to Przesławski and Rybinski [PR], Aubin and Frankowska [AF] and Artstein [Ar] and give information about neighborhoods of intersections of balls and convex sets.

After we obtain a number of preliminary results (which seem interesting in their own right) in Sections 2 and 3, we present the crucial step of the proof of Theorem 1.1 in Section 4. This is an inductive geometrical construction which establishes the existence of a mapping $G : \mathcal{M} \to \mathcal{K}_m(X)$ such that $G(x) \subset F(x), x \in \mathcal{M}$ and

$$d_H(G(x), G(y)) \leqslant \gamma(m)\rho(x, y), \quad x, y \in \mathcal{M}.$$

Here d_H stands for the Hausdorff distance, i.e.,

$$d_H(C_1, C_2) := \inf\{\varepsilon > 0 : C_2 \subset C_1 + B(0, \varepsilon), C_1 \subset C_2 + B(0, \varepsilon)\},$$

where $B(x, r) := \{y \in X : \|y - x\| \leqslant r\}$ is the closed ball in X of radius r centered at x. Having constructed G we complete the proof by letting $f : \mathcal{M} \to X$ be the selection

$$f(x) := S^{(m)}(G(x)), \tag{1.3}$$

where $S^{(m)}$ is a Lipschitz mapping from $(\mathcal{K}_m(X), d_H)$ into X such that $S^{(m)}(C) \in C$ for all $C \in \mathcal{K}_m(X)$.

We construct the mapping $S^{(m)}$ in Section 3. Let us give a few more details about the content of Section 3: We obtain the mapping $S^{(m)}$ via the solution of a slightly more general problem related to set-valued mappings into the family $\mathcal{K}(X) = \bigcup_{m=1}^{\infty} \mathcal{K}_m(X)$ of all *convex finite-dimensional* compact subsets of X. Throughout this paper we will refer to a selection map from $(\mathcal{K}(X), d_H)$ to X as a *selector*.

We recall that for an infinite dimensional Banach space X there does not exist a Lipschitz continuous selector which is defined on *all* of the family $\mathcal{K}(X)$. (See e.g., [PY1] and references therein.) Moreover, Proposition 4.7 in [PY1] states that such a selector does not exist even on a subfamily of $\mathcal{K}(X)$ consisting of all finite-dimensional polytopes in X (i.e., the convex hulls of all finite subsets of X). Our main result in Section 3 (Theorem 1.2) implies that, in contrast to these negative results, *there exists a selector* $S_X : \mathcal{K}(X) \to X$ *which is Lipschitz continuous on every family* $\mathcal{K}_m(X),\ m \in \mathbf{N}$.

For the case of a Hilbert space H the classical *Steiner point* [St] $s(C)$ of a convex body $C \subset H$ provides such a selector. Recall that if $C \in \mathcal{K}(H)$ is a subset of an n-dimensional subspace $L \subset H$, then its Steiner point $s(C)$ is defined by the formula

$$s(C) := n \int_{S_H \cap L} u h_C(u)\, d\sigma(u).$$

Here S_H is the unit sphere in H, $h_C(u) := \sup\{\langle u, x \rangle : x \in C\}$ is the support function of C, and σ denotes the normalized Lebesgue measure on $S_H \cap L$ which is calculated with respect to an arbitrary predetermined Euclidean basis for L.

The Steiner point map has been widely studied from various points of view (see, e.g., [Gr,Sh,Sal,Sc1,S-P,Pos,BL, p. 48], and references therein). In particular, this map is a continuous selector which is *additive* with respect to Minkowski addition and *commutes with the affine isometries of* H. These properties uniquely define the Steiner point and shows that $s(C)$ is well defined, i.e., its definition does not depend on the choice of the finite-dimensional subspace L containing C, nor on the choice of the Euclidean basis of L (see, e.g. [PY1,Vit,S3]). Moreover, the Steiner point map is Lipschitz continuous on every n-dimensional subspace of H and its Lipschitz constant equals $c_n = 2\pi^{-\frac{1}{2}}\Gamma(\frac{n}{2}+1)/\Gamma(\frac{n+1}{2})$. (This value is sharp and is the smallest possible for selectors from $\mathcal{K}(H)$ to H, see, e.g. [Pos,Vit,BL, p. 52]. Note also that the Steiner point does not have a *continuous* extension to *all* convex bodies in H, see [Vit].) Since the linear hull of any two convex compact subsets C_1 and C_2 has dimension not exceeding $n = \dim C_1 + \dim C_2 + 2$, we see that

$$\|s(C_1) - s(C_2)\| \leqslant c_n \ \mathrm{d_H}(C_1, C_2)$$

for every $C_1, C_2 \in \mathcal{K}(H)$. Consequently the restriction $s|_{\mathcal{K}_m(H)}$ is Lipschitz continuous for every $m \in \mathbf{N}$.

We can now give the exact statement of our main result in Section 3, which gives an analogue of the Steiner point for the non-Hilbert case. To formulate this result we let $\mathrm{Conv}_F(X)$ denote the family of all *convex finite-dimensional* subsets of X. Given a centrally symmetric subset A and a positive constant γ we let $\gamma \circ A$ denote the dilation of A with respect to its center by a factor of γ.

Theorem 1.2. *For each Banach space* X, *there is a mapping* $S_X : (\mathrm{Conv}_F(X), \mathrm{d_H}) \to X$ *such that* $S_X(C) \in C$ *for each* $C \in \mathrm{Conv}_F(X)$ *and, for every* $C_1, C_2 \in \mathrm{Conv}_F(X)$,

$$\|S_X(C_1) - S_X(C_2)\| \leqslant \gamma_1 \mathrm{d_H}(C_1, C_2), \tag{1.4}$$

where $\gamma_1 = \gamma_1(\dim C_1, \dim C_2)$ *is a constant depending only on dimensions of* C_1 *and* C_2.

Moreover, if $C \in \mathrm{Conv}_F(X)$ *is bounded then*:

(i) $S_X(\tau C + a) = \tau S_X(C) + a$ *for every* $a \in X$ *and* $\tau \in \mathbf{R}$;
(ii) *there is an ellipsoid* $\mathscr{E}_{C,S}$ *centered at* $S_X(C)$ *such that*

$$\mathscr{E}_{C,S} \subset C \subset \gamma_2 \circ \mathscr{E}_{C,S}, \tag{1.5}$$

where $\gamma_2 = \gamma_2(\dim C)$ *is a constant depending only on dimension of* C.

We call S_X a *Steiner-type selector*. In particular, the restriction of this selector to the family $\mathcal{K}_m(X)$ defines the selector $S^{(m)}$ which is used in (1.3). Note that, by property (i), $S_X(C)$ coincides with the center of C for every bounded centrally symmetric set $C \in \mathrm{Conv}_F(X)$. Furthermore, property (ii) implies that the point $S_X(C)$ is located rather "deeply" in the interior of the set C. Note, by way of comparison, that the Steiner point of a set always belongs to the relative interior of the set [Sh], but, as noted in [Sc3, p. 308], an estimate for the position of the Steiner point (e.g., similar to (1.5)) seems to be unknown.

As already mentioned above, the Steiner point map commutes with affine isometries of a Hilbert space. However, this property fails for other affine transformations, and consequently there does not seem to be any obvious way of generalizing the Steiner point construction to the case of a non-Hilbert Banach space. (We refer the reader to [PY1, Section 6], for some partial results which indicate the difficulties of making such a generalization.)

One key element of our approach here is to use barycenters rather than Steiner points. We use the notation $b(C)$ for the barycenter (center of mass) of a convex body $C \in \mathscr{K}(X)$. (The precise definition and some basic properties are given in the next section.) It is known that the barycentric map $b : \mathscr{K}(X) \to X$ is a continuous selector [SW] but (unlike the Steiner point map in Hilbert spaces) it is not a Lipschitz map. However we are able to show that the barycentric map does have a certain "Lipschitz-like" property, where the usual Lipschitz constant is augmented by an additional factor which depends on a certain "geometrical" quantity associated with sets $C \in \mathscr{K}(X)$. To define this quantity, for each $C \in \mathscr{K}(X)$, we first choose some Lebesgue measure λ on affspan(C), the affine span of C. Then we define the *regularity coefficient* of C to be the number

$$\delta_C := \frac{\lambda(B^{(C)} \cap \text{affspan}(C))}{\lambda(C)}, \tag{1.6}$$

where $B^{(C)}$ denotes a ball (with respect to $\|\cdot\|$) of *minimal radius* among all balls which contain C and whose centers lie in affspan(C). (It is easy to see that δ_C does not depend on the choice of the Lebesgue measure λ.)

Theorem 1.3. *For every* $C_1, C_2 \in \mathscr{K}(X)$

$$\|b(C_1) - b(C_2)\| \leqslant \gamma(\delta_{C_1} + \delta_{C_2}) \mathrm{d_H}(C_1, C_2),$$

where γ is a constant depending only on $\dim C_1$ *and* $\dim C_2$.

The proof of this theorem is given in Section 2. It is an important ingredient in the proof of Theorem 1.2.

In Section 5 we present an application of the results of the previous sections which deals with the modified barycentric mapping $S(C) := b(C + B(0,1))$ introduced in [AC, p. 77], for the case $X = \mathbf{R}^n$. It is proved in [AC] that S is a Lipschitz continuous selector on every bounded subset of $\mathscr{K}(\mathbf{R}^n)$. We prove that a slight generalization of this selector, namely the mapping $S^{(X)}(C) := b(C + B(0, \text{diam}\, C))$, provides a Lipschitz continuous selector on $\mathscr{K}(\mathbf{R}^n)$. We also discuss a further generalization of this selector to the case of an arbitrary Minkowski space X. In particular, we consider the following natural question: under what conditions on X does the barycenter $b(C + B(0,1))$ belong to C for every $C \in \mathscr{K}(X)$?

In Section 6 we consider the Grassmannian $G(X)$ of all finite-dimensional subspaces of a Banach space X, equipped with the metric

$$\mathrm{d_G}(L_1, L_2) := \mathrm{d_H}(L_1 \cap B(0,1), L_2 \cap B(0,1)). \tag{1.7}$$

Since every subspace $L \in G(X)$ is finite-dimensional, there is a Lipschitz (even linear) continuous projector $P_L : X \to L$. Clearly, such a projector is not unique. We show that one can choose P_L in such a way that the mapping $L \to P_L$ defined on $G(X)$ will be *Lipschitz continuous* with respect to the metric d_G.

Our interest in Helly-type criteria for the existence of Lipschitz selections was initially motivated by some close and intriguing connections of this problem with the classical Whitney extension problem [W1,W2], namely, the problem of characterizing those functions defined on a closed subset, say $Y \subset \mathbf{R}^n$, which are the restrictions to Y of smooth functions on $\mathbf{R}^n$. In particular, earlier versions of Theorem 1.1 for the special case $X = \mathbf{R}^n$ provided the geometrical tools for an approach to the Whitney problem developed in [S1,BS1,BS2,BS3,S4] which led us to a Helly-type description of the restrictions of smooth functions (in terms of the so-called "finiteness property").

For other applications of Lipschitz continuous selectors, to differential inclusions, metric projections (where the Steiner selector plays an important role) and real interpolation theory, we refer the reader to works [Ar,AC,AF,DLP,M,Al,S2] and references therein.

2. Barycentric selectors. The proof of Theorem 1.3

The proof of Theorem 1.3 is based on a series of auxiliary lemmas. Let Y be an n-dimensional Banach space equipped with the norm $\|\cdot\|_Y$ and an n-dimensional Lebesgue measure λ. Given $a \in Y$ and $r \geqslant 0$ we let $B_Y(a,r)$ denote a ball in Y of radius r centered at a. Recall also that $\mathscr{K}(Y)$ stands for the family of all compact convex subsets of Y.

Lemma 2.1. *Let $C \in \mathscr{K}(Y)$ and let $B_Y(a,R)$ be a ball containing C. Then for every $t > 0$*

$$\lambda(C + B_Y(0,t)) - \lambda(C) \leqslant \left(\left(1 + \frac{t}{R}\right)^n - 1\right)\lambda(B_Y(a,R)).$$

Proof. We will need some basic definitions and results related to the theory of mixed volumes, see, e.g. [BZ, pp. 136–137]; [Sc3, Section 5.1].

We let $V(K_1, K_2, \dots, K_n)$ denote the mixed volume of sets $K_i \in \mathscr{K}(Y),\ i = 1, \dots, n$. For $K_1, K_2 \in \mathscr{K}(Y)$ and non-negative integers n_1, n_2 we denote

$$V(K_1[n_1], K_2[n_2]) := V\Bigg(\underbrace{K_1, \dots, K_1}_{n_1}, \underbrace{K_2, \dots, K_2}_{n_2}\Bigg). \tag{2.1}$$

Then by the Minkowski theorem

$$\lambda(C + B_Y(0,t)) = \lambda(C + tB_Y(0,1)) = \sum_{k=0}^{n} \binom{n}{k} V(C[n-k], B_Y(0,1)[k])t^k. \tag{2.2}$$

Since $V(C, C, \ldots, C) = \lambda(C)$, we have

$$\lambda(C + B_Y(0,t)) - \lambda(C) = \sum_{k=1}^{n} \binom{n}{k} V(C[n-k], B_Y(0,1)[k])\, t^k.$$

Since $C \subset B_Y(a,R)$ and the mixed volume is an increasing function with respect to inclusion (see [BZ, p. 138]), we obtain

$$V(C[n-k], B_Y(0,1)[k]) \leqslant V(B_Y(a,R)[n-k], B_Y(0,1)[k]).$$

Since the mixed volume is positive multilinear and invariant with respect to shifts [BZ, pp. 137–138], we have

$$\begin{aligned} V(B_Y(a,R)[n-k], B_Y(0,1)[k]) &= R^{-k} V(B_Y(a,R)[n-k], B_Y(0,R)[k]) \\ &= R^{-k} V(B_Y(a,R)[n-k], B_Y(a,R)[k]) \\ &= R^{-k} \lambda(B_Y(a,R)), \end{aligned}$$

so that

$$\begin{aligned} \lambda(C + B_Y(0,t)) - \lambda(C) &\leqslant \left(\sum_{k=1}^{n} \binom{n}{k} \left(\frac{t}{R}\right)^k \right) \lambda(B_Y(a,R)) \\ &= \left(\left(1 + \frac{t}{R}\right)^n - 1 \right) \lambda(B_Y(a,R)). \qquad \square \end{aligned}$$

Given subsets $A, B \subset Y$ we denote

$$\operatorname{dist}(A,B) := \inf\{\|a-b\|_Y : a \in A, y \in B\}.$$

Lemma 2.2. *Let L_1, L_2 be two linear subspaces of Y such that* $\dim L_1 = \dim L_2 = m$. *Then there is a linear mapping A from L_1 onto L_2 such that $\gamma^{-1} \leqslant \|A\| \leqslant \gamma$ and for every $x \in L_1$*

$$\|x - Ax\|_Y \leqslant \gamma \operatorname{dist}(x, L_2).$$

Here $\gamma = \gamma(m)$ is a positive constant depending only on m.

Proof. Every $2m$-dimensional Minkowski space is linearly isomorphic to a Euclidean space, and the norms of the isomorphism and its inverse are bounded by constants that depend only on m. Consequently, it is sufficient to prove the lemma for the case of a Euclidean space Y of dimension at most $2m$. By $\langle \cdot, \cdot \rangle$ we will denote the inner product in Y.

We shall use the notation $||\cdot||$ for the norm of Y. Let us show that there is a linear *isometry* $A : L_1 \to L_2$ such that for every $x \in L_1$

$$||x - Ax|| \leqslant 5^m \operatorname{dist}(x, L_2). \tag{2.3}$$

We will prove this assertion by induction on $m := \dim L_1 (= \dim L_2)$. For $m = 0$ nothing to prove. Suppose that the result is true for some $m \geqslant 0$ and prove it for $m + 1$. To this end we let $S := \{y \in Y : ||y|| = 1\}$ denote the unit sphere of Y. Fix subspaces $L_1, L_2 \subset Y$ of dimension $m + 1$ and denote by e_1 a point from $S \cap L_1$ such that

$$\operatorname{dist}(S \cap L_1, L_2) = \operatorname{dist}(e_1, L_2). \tag{2.4}$$

Given a linear subspace $L \subset Y$ we let Pr denote the orthogonal projector on L. Define a point $e_2 \in S \cap L_2$ by letting $e_2 := \Pr_{L_2}(e_1)/||\Pr_{L_2}(e_1)||$ if $\Pr_{L_2}(e_1) \neq 0$ and any point of $S \cap L_2$ otherwise. Then it can be readily shown that

$$||e_1 - e_2|| \leqslant \sqrt{2} \operatorname{dist}(e_1, L_2). \tag{2.5}$$

We note that by (2.4) for every $x \in L_1$, $x \neq 0$ we have

$$\operatorname{dist}(e_1, L_2) \leqslant \operatorname{dist}\left(\frac{x}{||x||}, L_2\right) = ||x||^{-1} \operatorname{dist}(x, L_2).$$

We let E_i define the one-dimensional space generated by $e_i, i = 1, 2$. We denote by $\tilde{L}_i$ the orthogonal complement of E_i in L_i. Then $\dim \tilde{L}_i = \dim L_i - 1 = m$ so that by the induction assumption there is a linear isometry $\tilde{A} : \tilde{L}_1 \to \tilde{L}_2$ such that

$$||x - \tilde{A}x|| \leqslant 5^m \operatorname{dist}(x, \tilde{L}_2), \quad x \in \tilde{L}_1.$$

Now we define the required operator $A : L_1 \to L_2$ by letting $Ae_1 := e_2$ and $A|_{\tilde{L}_1} := \tilde{A}|_{\tilde{L}_1}$.

Clearly, A is an isometry so that it remains to prove (2.3). Since $L_1 = E_1 \oplus \tilde{L}_1$, for every $x \in L_1$ there is a decomposition $x = x' + x''$ where $x' := \Pr_{E_1}(x) \in E_1$ and $x'' := \Pr_{\tilde{L}_1}(x) \in \tilde{L}_1$. Recall that $\dim E_1 = 1$ so that by (2.6)

$$\operatorname{dist}(x', L_2) = ||x'|| \operatorname{dist}(e_1, L_2) \leqslant ||x'|| \operatorname{dist}\left(\frac{x}{||x||}, L_2\right) = \frac{||x'||}{||x||} \operatorname{dist}(x, L_2).$$

Since x' is the orthogonal projection of x on E_1, we have $||x'|| \leqslant ||x||$. Hence

$$\operatorname{dist}(x', L_2) \leqslant \operatorname{dist}(x, L_2),$$

so that by (2.5)

$$\|x' - Ax'\| = \|x'\|\|e_1 - Ae_1\| = \|x'\|\|e_1 - e_2\| \leqslant \sqrt{2}\|x'\| \operatorname{dist}(e_1, L_2)$$
$$= \sqrt{2} \operatorname{dist}(x', L_2) \leqslant \sqrt{2} \operatorname{dist}(x, L_2).$$

On the other hand, by the induction assumption

$$\|x'' - Ax''\| = \|x'' - \tilde{A}x''\| \leqslant 5^m \operatorname{dist}(x'', L_2).$$

We let $v_1, \dots, v_m$ denote an orthonormal basis in $\tilde{L}_2$. Then for every $y \in Y$

$$\operatorname{Pr}_{\tilde{L}_2}(y) = \sum_{i=1}^{m} \langle v_i, y \rangle v_i$$

so that

$$\operatorname{dist}(x'', \tilde{L}_2) = \|x'' - \operatorname{Pr}_{\tilde{L}_2}(x'')\| = \|x'' - \sum_{i=1}^{m} \langle v_i, x'' \rangle v_i\|.$$

Hence

$$\operatorname{dist}(x'', \tilde{L}_2) \leqslant \|x'' - \sum_{i=1}^{m} \langle v_i, x'' \rangle v_i - \langle e_2, x'' \rangle\| + |\langle e_2, x'' \rangle|$$

so that

$$\operatorname{dist}(x'', \tilde{L}_2) \leqslant \operatorname{dist}(x'', L_2) + |\langle e_2, x'' \rangle|.$$

Since $\langle e_1, x'' \rangle = 0$, by (2.5) and (2.6) we have

$$|\langle e_2, x'' \rangle| = |\langle e_1 - e_2, x'' \rangle| \leqslant \|e_1 - e_2\| \cdot \|x''\| \leqslant \sqrt{2}\|x''\| \operatorname{dist}(e_1, L_2)$$
$$\leqslant \sqrt{2} \frac{\|x''\|}{\|x\|} \operatorname{dist}(x, L_2) \leqslant \sqrt{2} \operatorname{dist}(x, L_2).$$

Hence we obtain

$$\operatorname{dist}(x'', \tilde{L}_2) \leqslant \operatorname{dist}(x'', L_2) + \sqrt{2} \operatorname{dist}(x, L_2).$$

Since $x = x' + x''$, we have

$$\operatorname{dist}(x'', L_2) \leqslant \operatorname{dist}(x, L_2) + \operatorname{dist}(x', L_2)$$

so that

$$\operatorname{dist}(x'', L_2) \leqslant 2 \operatorname{dist}(x, L_2).$$

Hence

$$\|x'' - Ax''\| \leqslant (\sqrt{2} + 2) \cdot 5^m \operatorname{dist}(x, L_2) \leqslant 5^{m+1} \operatorname{dist}(x, L_2).$$

Finally,

$$\|x - Ax\| \leqslant \|x' - Ax'\| + \|x'' - Ax''\| \leqslant (\sqrt{2} + 2 \cdot 4^m) \operatorname{dist}(x, L_2) \leqslant 4^{m+1} \operatorname{dist}(x, L_2)$$

and the lemma follows. □

Before presenting a final auxiliary lemma and the proof of Theorem 1.2 we need to recall several notions and result concerning Kolmogorov n-widths (see, e.g. [P] or [Sin]) and the barycenters of convex sets.

Let C be a convex compact subset of X of affine dimension m (i.e., $m = \dim C := \dim(\operatorname{affspan}(C))$) and let $\operatorname{Aff}_n(X)$ be the family of all n-dimensional *affine* subsets of X. We define the Kolmogorov n-width of C in X by letting

$$\begin{aligned} d_n(C; X) &:= \inf_{L \in \operatorname{Aff}_n(X)} \inf\{\varepsilon > 0 : L + B(0, \varepsilon) \supset C\} \\ &= \inf_{L \in \operatorname{Aff}_n(X)} \sup_{x \in C} \operatorname{dist}(x, L). \end{aligned} \tag{2.6}$$

It can be readily seen that

$$d_n(C; X) = \inf\{\mathrm{d_H}(C, A) : A \in \mathscr{K}(X), \dim A = n\}. \tag{2.7}$$

For the reader's convenience we give a proof of this equality. Let $L \in \operatorname{Aff}_n(X)$ and $\varepsilon > 0$ be such that $C \subset L + B(0, \varepsilon)$. For arbitrary $\varepsilon_1 > \varepsilon$ put $A := L \cap \{C + B(0, \varepsilon_1)\}$. Clearly, $A \in \mathscr{K}(X)$ and $\dim A = n$. Moreover, by the definition of L and ε for every $c \in C$ there is $x \in L$ such that $\|c - x\| \leqslant \varepsilon$ which implies that $x \in A$. Hence $C \subset A + B(0, \varepsilon_1)$ which together with the definition of A gives $\mathrm{d_H}(C, A) \leqslant \varepsilon_1$. Since $\varepsilon_1 > \varepsilon$ is arbitrary, we obtain the required inequality

$$d_n(C; X) \leqslant \inf\{\mathrm{d_H}(C, A) : A \in \mathscr{K}(X), \dim A = n\}.$$

The converse inequality is obvious. There is one last property of the Kolmogorov n-width which we shall need: By (2.7) the n-width is a distance function in the metric space $(\mathscr{K}(X), \mathrm{d_H})$ so that it satisfies the Lipschitz condition with respect to $\mathrm{d_H}$. Thus for every $C_1, C_2 \in \mathscr{K}(X)$

$$|d_n(C_1; X) - d_n(C_2; X)| \leqslant \mathrm{d_H}(C_1, C_2). \tag{2.8}$$

Next, we recall some basic properties of the barycenter of a convex set. Let $C \in \mathscr{K}(X)$ and let λ be an m-dimensional Lebesgue measure on $\operatorname{affspan}(C)$ calculated with respect to some basis of $\operatorname{affspan}(C)$. We recall that the *barycenter* of C is defined by letting

$$b(C) := \frac{1}{\lambda(C)} \int_C x \, \mathrm{d}\lambda(x). \tag{2.9}$$

Clearly, $b(C)$ is well defined and $b(C) \in C$ for every $C \in \mathscr{K}(X)$. Moreover, $b(\cdot)$ commutes with affine transformations, i.e., for every affine mapping A from $\operatorname{affspan}(C)$ onto an affine subspace L with $\dim L = \dim C$ we have

$$b(AC) = Ab(C). \tag{2.10}$$

Note that a result going back to Minkowski [Min] (see also [BF, p. 58]; [Sc3, p. 308]) implies that, for every convex body C in an m-dimensional Euclidean space E_m,

$$B\left(b(C), \frac{1}{m+1} \Delta(C)\right) \subset C \subset B\left(b(C), \frac{m}{m+1} \operatorname{diam} C\right). \tag{2.11}$$

Here $\Delta(C) := \min\{h_C(u) + h_C(-u) : ||u|| = 1\}$ is the width of C. Since $\Delta(C) = 2d_{m-1}(C; X)$, we have

$$B\left(b(C), \frac{2}{m+1} d_{m-1}(C; X)\right) \subset C. \tag{2.12}$$

Now suppose that $C \in \mathscr{K}(X)$ and $\dim C = m$. Then, by passing to an equivalent Euclidean norm on $\operatorname{affspan}(C)$, we obtain from (2.12) that

$$B(b(C), \alpha^{-1} d_{m-1}(C; X)) \cap \operatorname{affspan}(C) \subset C, \tag{2.13}$$

where $\alpha = \alpha(m) \geqslant 1$ is a constant depending only on m.

Note one more useful inclusion which follows from (2.11): for every m-dimensional subset $C \in \mathscr{K}(X)$ there exists an ellipsoid $\mathscr{E}_{C,b}$ *centered at* $b(C)$ such that

$$\mathscr{E}_{C,b} \subset C \subset \beta \circ \mathscr{E}_{C,b}, \tag{2.14}$$

where $\beta = \beta(m)$ is a constant depending only on m. (Recall that $\beta \circ \mathscr{E}_{C,b}$ denotes the dilation of $\mathscr{E}_{C,b}$ with respect to its center by a factor of β.)

As usual we define an m-dimensional ellipsoid in X as an affine image of a ball in $\mathbf{R}^m$. We recall that for every m-dimensional subset $C \in \mathscr{K}(X)$ there is an ellipsoid $\mathscr{E}_C$ such that

$$\mathscr{E}_C \subset C \subset \gamma(m) \circ \mathscr{E}_C. \tag{2.15}$$

In particular, the Löwner–John ellipsoid of C, i.e., the (unique) ellipsoid of maximal volume contained in C, satisfies (2.15) with $\gamma(m) = m$, see [John].

To prove the existence of the ellipsoid $\mathscr{E}_{C,b}$ satisfying (2.14) assume that the ellipsoid $\mathscr{E}_C$ from (2.15) is centered at the origin so that we can take $\mathscr{E}_C$ as the unit ball in the space $E_m := \operatorname{affspan}(C)$. Then we define $\mathscr{E}_{C,b}$ by letting

$$\mathscr{E}_{C,b} := (m+1)^{-1}\Delta(C) \circ \mathscr{E}_C + b(C).$$

Let us note that by (2.15) $\operatorname{diam} C \sim \Delta(C)$ $(\sim \gamma(m))$ so that the required inclusion (2.14) follows from (2.11).

Several times in the proofs to follow we will use the following simple property of the regularity coefficient, see (1.6).

Claim 2.3. *Let $L \subset X$ be a finite-dimensional affine subset and let $B(a,r)$ be a ball in X such that $L \cap B(a,r) \neq \emptyset$. If $C := L \cap B(a,2r)$, then*

$$\delta_C \leqslant 7^{\dim L}. \tag{2.16}$$

Proof. Fix a point $c \in C$ such that $||c-a|| \leqslant \frac{3r}{2}$. Then for every $y \in L$ satisfying $||y-c|| \leqslant \frac{r}{2}$ we have

$$||y-a|| \leqslant ||y-c|| + ||c-a|| \leqslant 2r$$

so that $L \cap B(c, r/2) \subset C$. On the other hand,

$$C := L \cap B(a,2r) \subset B\left(c, \frac{3r}{2}\right) + B(0,2r) = B\left(c, \frac{7r}{2}\right).$$

Hence $\delta_C \leqslant \lambda(L \cap B(c, \frac{7r}{2}))/\lambda(L \cap B(c, \frac{r}{2}))$ and (2.16) follows. $\square$

Throughout the paper the symbol "$\sim$" will denote equivalence up to a constant depending only on m.

Lemma 2.4. *For every m-dimensional subset $C \in \mathscr{K}(X)$*

$$\frac{1}{\gamma}\delta_C \leqslant \frac{(\operatorname{diam} C)^m}{\prod_{n=0}^{m-1} d_n(C;X)} \leqslant \gamma \delta_C,$$

where $\gamma = \gamma(m)$ is a positive constant depending only on m.

Proof. Let $\mathscr{E}_C$ be an ellipsoid satisfying (2.15). Clearly, $\operatorname{diam} C \sim \operatorname{diam} \mathscr{E}_C$, $\lambda(C) \sim \lambda(\mathscr{E}_C)$, $\delta(C) \sim \delta(\mathscr{E}_C)$, and $d_n(C;X) \sim d_n(\mathscr{E}_C;X)$.

This observation shows that it suffices to prove the lemma for an m-dimensional ellipsoid $\mathscr{E} \subset X$. It can be easily seen that in this case the infimum in (2.6) can be taken over all $L \in \operatorname{Aff}_n(X)$ which contain the center of $\mathscr{E}$. Moreover, we obtain an equivalent value of $d_n(\mathscr{E};X)$ (up to a constant depending only on m) if the infimum in (2.6) is taken over all $L \in \operatorname{Aff}_n(X)$ contained in $\operatorname{affspan}(\mathscr{E})$. Note also that we can replace the norm $||\cdot||$ on $\operatorname{affspan}(\mathscr{E})$ by a Euclidean norm which is equivalent to $||\cdot||$ up to a constant depending on m. Then, with the help of a suitable shift and rotation, we can reduce the lemma's assertion to the case of an ellipsoid $\mathscr{E} \subset \mathbf{R}^m$ of the form

$$\mathscr{E} := \left\{ x = (x_1, \dots, x_m) \in \mathbf{R}^m : \sum_{n=1}^{m} \left(\frac{x_n}{s_n} \right)^2 \leqslant 1 \right\} \quad \text{where } s_1 \geqslant s_2 \geqslant \dots \geqslant s_m.$$

In this case, see, e.g. [T, Chapter 4, Section 4.4], $d_n(\mathscr{E}; \mathbf{R}^m) = s_{n+1}, \ 0 \leqslant n \leqslant m-1$. On the other hand, the ball $B^{(\mathscr{E})}$ of minimal radius containing $\mathscr{E}$ is $B(0, s_1)$ so that

$$\delta_{\mathscr{E}} := \frac{\lambda(B^{(\mathscr{E})})}{\lambda(\mathscr{E})} = \frac{s_1^m \lambda(B(0,1))}{(\prod_{n=1}^m s_n)\lambda(B(0,1))}$$

$$= \frac{(\frac{1}{2} \operatorname{diam} \mathscr{E})^m}{\prod_{n=1}^m s_n} = \frac{1}{2^m} \frac{(\operatorname{diam} \mathscr{E})^m}{\prod_{n=1}^m d_n(\mathscr{E}; \mathbf{R}^m)}. \qquad \square$$

Proof of Theorem 1.3. We prove the theorem into four steps.

Step 1: $\operatorname{affspan}(C_1) = \operatorname{affspan}(C_2) =: Y$.

Without loss of generality, we may assume that Y is a linear subspace of X of dimension n. Given $c \in Y$ and $s \geqslant 0$ we denote $B_Y(c, s) := B(c, s) \cap Y$. Assume that $\operatorname{diam} C_1 \leqslant \operatorname{diam} C_2$ and put $r := \mathrm{d_H}(C_1, C_2)$.

Clearly, $\|b(C_1) - b(C_2)\| \leqslant \operatorname{diam}(C_1 \cup C_2)$ and

$$C_2 \subset C_1 + B_Y(0, r), \tag{2.17}$$

so that if $\operatorname{diam} C_1 \leqslant r$ we have

$$\|b(C_1) - b(C_2)\| \leqslant \operatorname{diam}(C_1 + B_Y(0, r)) \leqslant \operatorname{diam} C_1 + 2r \leqslant 3r. \tag{2.18}$$

Now suppose that

$$r := \mathrm{d_H}(C_1, C_2) < \operatorname{diam} C_1 \tag{2.19}$$

and $C_1 \subset C_2$. Fix a point $x_0 \in C_1$. Then

$$\|b(C_1) - b(C_2)\| = \left\| \frac{1}{\lambda(C_1)} \int_{C_1} (x - x_0)\, d\lambda(x) - \frac{1}{\lambda(C_2)} \int_{C_2} (x - x_0)\, d\lambda(x) \right\|$$

$$\leqslant \left\| \frac{1}{\lambda(C_1)} \int_{C_1} (x - x_0)\, d\lambda(x) - \frac{1}{\lambda(C_1)} \int_{C_2} (x - x_0)\, d\lambda(x) \right\|$$

$$+ \left\| \frac{1}{\lambda(C_1)} \int_{C_2} (x - x_0)\, d\lambda(x) - \frac{1}{\lambda(C_2)} \int_{C_2} (x - x_0)\, d\lambda(x) \right\|.$$

Hence

$$\begin{aligned}||b(C_1)-b(C_2)|| &\leqslant \frac{1}{\lambda(C_1)}\int_{C_2\setminus C_1}||x-x_0||\,d\lambda(x)\\ &\quad+\left(\frac{1}{\lambda(C_1)}-\frac{1}{\lambda(C_2)}\right)\int_{C_2}||x-x_0||\,d\lambda(x)\\ &\leqslant 2\operatorname{diam} C_2\,\frac{\lambda(C_2)-\lambda(C_1)}{\lambda(C_1)}\end{aligned}$$

so that by (2.17)

$$||b(C_1)-b(C_2)||\leqslant 2\operatorname{diam} C_2\,\frac{\lambda(C_1+B_Y(0,r))-\lambda(C_1)}{\lambda(C_1)}.$$

Recall that $B^{(C)}$ stands for a ball of minimal radius with center in affspan(C) such that $C\subset B^{(C)}$. We denote $B_Y^{(C)}:=B^{(C)}\cap\operatorname{affspan}(C)$ so that by (1.6) $\lambda(B_Y^{(C)})\leqslant\delta_C\lambda(C)$.

We let R denote the radius of $B^{(C_1)}$. Since $C_1\subset B_Y^{(C_1)}$, by Lemma 2.1 we obtain

$$\lambda(C_1+B_Y(0,r))-\lambda(C_1)\leqslant\left(\left(1+\frac{r}{R}\right)^n-1\right)\lambda(B_Y^{(C_1)})\leqslant\left(\left(1+\frac{r}{R}\right)^n-1\right)\delta_{C_1}\lambda(C_1),$$

so that

$$||b(C_1)-b(C_2)||\leqslant 2\left(\left(1+\frac{r}{R}\right)^n-1\right)\delta_{C_1}\operatorname{diam} C_2.$$

Clearly, $\operatorname{diam} C_1\leqslant\operatorname{diam} B^{(C_1)}=2R$ so that by (2.19) $r\leqslant 2R$. Hence

$$\left(1+\frac{r}{R}\right)^n-1\leqslant n\frac{r}{R}\left(1+\frac{r}{R}\right)^{n-1}\leqslant 2n3^{n-1}\,\frac{r}{\operatorname{diam} C_1}.$$

By (2.17) and (2.19)

$$\operatorname{diam} C_2\leqslant\operatorname{diam} C_1+2r\leqslant 3\operatorname{diam} C_1,$$

so that

$$||b(C_1)-b(C_2)||\leqslant 4n3^{n-1}\delta_{C_1}r\frac{\operatorname{diam} C_2}{\operatorname{diam} C_1}\leqslant 12n3^{n-1}\delta_{C_1}r=\gamma_1(n)\delta_{C_1}r. \qquad (2.20)$$

Now suppose that (2.19) holds and C_1, C_2 are two arbitrary sets from $\mathscr{K}(X)$ such that $\operatorname{affspan}(C_1)=\operatorname{affspan}(C_2)$. Clearly, by (2.17)

$$C_1\cup C_2\subset C_3:=C_1+B_Y(0,r)$$

and $d_H(C_1, C_3) \leqslant r$, $d_H(C_2, C_3) \leqslant 2r$. Then by (2.20)

$$\|b(C_1) - b(C_2)\| \leqslant \|b(C_1) - b(C_3)\| + \|b(C_3) - b(C_2)\| \leqslant \gamma_1(n)(\delta_{C_1} r + \delta_{C_2} 2r).$$

This proves the theorem for the case $\operatorname{affspan}(C_1) = \operatorname{affspan}(C_2)$.

Step 2: $\dim C_1 = \dim C_2 (= m)$ and

$$\operatorname{affspan}(C_1) \cap \operatorname{affspan}(C_2) \neq \emptyset.$$

In this case we may suppose that $L_1 := \operatorname{affspan}(C_1)$ and $L_2 := \operatorname{affspan}(C_2)$ are *linear subspaces* of X. Since $\dim L_1 = \dim L_2 = m$, by Lemma 2.2 there is a linear operator A mapping L_1 onto L_2 such that $\gamma^{-1} \leqslant \|A\| \leqslant \gamma$ and

$$\|x - Ax\| \leqslant \gamma \operatorname{dist}(x, L_2), \quad x \in L_1,$$

where $\gamma = \gamma(m)$ is a positive constant. Hence for every $x \in C_1$ we have

$$\|x - Ax\| \leqslant \gamma \operatorname{dist}(x, L_2) \leqslant \gamma \operatorname{dist}(x, C_2) \leqslant \gamma d_H(C_1, C_2). \tag{2.21}$$

Since A is a one-to-one mapping, this implies $d_H(C_1, AC_1) \leqslant \gamma d_H(C_1, C_2)$ so that

$$d_H(C_2, AC_1) \leqslant d_H(C_1, C_2) + d_H(C_1, AC_1) \leqslant (\gamma + 1) d_H(C_1, C_2). \tag{2.22}$$

Moreover, by (2.10) $b(AC_1) = Ab(C_1)$ which together with (2.21) yields

$$\|b(C_1) - b(AC_1)\| = \|b(C_1) - Ab(C_1)\| \leqslant \gamma d_H(C_1, C_2).$$

Since $\operatorname{affspan}(C_2) = \operatorname{affspan}(AC_1)$, by the result of Step 1 and (2.22) we obtain

$$\|b(C_2) - b(AC_1)\| \leqslant \gamma_1(\delta_{C_2} + \delta_{AC_1}) d_H(C_2, AC_1) \leqslant \gamma_1(\gamma + 1)(\delta_{C_2} + \delta_{AC_1}) d_H(C_1, C_2).$$

Hence

$$\begin{aligned} \|b(C_1) - b(C_1)\| &\leqslant \|b(C_1) - b(AC_1)\| + \|b(AC_1) - b(C_2)\| \\ &\leqslant (\gamma + \gamma_1(\gamma + 1)(\delta_{C_2} + \delta_{AC_1}))\, d_H(C_1, C_2). \end{aligned}$$

It remains to note that A is a quasi-isometric mapping so that $\delta_{AC_1} \sim \delta_{C_1}$. This finishes the proof of Step 2.

Step 3: $\dim C_1 = \dim C_2 (= m)$ and $\operatorname{affspan}(C_1) \cap \operatorname{affspan}(C_2) = \emptyset$.

Choose $a_1 \in C_1$ and $a_2 \in C_2$ such that $\|a_1 - a_2\| \leqslant d_H(C_1, C_2)$ and put $\widetilde{C}_2 := C_2 + a_1 - a_2$. Then

$$d_H(C_1, \widetilde{C}_2) \leqslant d_H(C_1, C_2) + \|a_1 - a_2\| \leqslant 2\, d_H(C_1, C_2) \tag{2.23}$$

and

$$d_H(\widetilde{C}_2, C_2) \leqslant \|a_1 - a_2\| \leqslant d_H(C_1, C_2).$$

It is also clear that $b(\widetilde{C}_2) = b(C_2) + a_1 - a_2$ so that

$$||b(\widetilde{C}_2) - b(C_2)|| \leqslant \mathrm{d_H}(C_1, C_2).$$

Since $\operatorname{affspan}(C_1) \cap \operatorname{affspan}(\widetilde{C}_2) \neq \emptyset$, by the result of Step 2 we have

$$||b(C_1) - b(\widetilde{C}_2)|| \leqslant \gamma_2(\delta_{C_1} + \delta_{\tilde{C}_2})\mathrm{d_H}(C_1, \widetilde{C}_2)$$

with $\gamma_2 = \gamma_2(m)$. Since $\delta_{\tilde{C}_2} = \delta_{C_2}$, this inequality and (2.23) imply

$$||b(C_1) - b(\widetilde{C}_2)|| \leqslant 2\gamma_2(\delta_{C_1} + \delta_{C_2})\mathrm{d_H}(C_1, C_2).$$

Hence

$$\begin{aligned} ||b(C_1) - b(C_2)|| &\leqslant ||b(C_1) - b(\widetilde{C}_2)|| + ||b(\widetilde{C}_2) - b(C_2)|| \\ &\leqslant (2\gamma_2(\delta_{C_1} + \delta_{C_2}) + 1)\mathrm{d_H}(C_1, C_2). \end{aligned}$$

Step 4: $l \coloneqq \dim C_2 < m \coloneqq \dim C_1$. Then

$$\begin{aligned} \mathrm{d_H}(C_1, C_2) &\geqslant \inf\{\varepsilon > 0 : C_1 \subset C_2 + B(0, \varepsilon)\} \\ &\geqslant \inf\{\varepsilon > 0 : C_1 \subset \operatorname{affspan}(C_2) + B(0, \varepsilon)\} \\ &\geqslant d_l(C_1; X) \geqslant d_{m-1}(C_1; X). \end{aligned}$$

Hence

$$\operatorname{diam} C_1 \leqslant \frac{\operatorname{diam} C_1}{d_{m-1}(C_1; X)} \mathrm{d_H}(C_1, C_2) \leqslant \frac{(\operatorname{diam} C_1)^m}{\prod_{n=0}^{m-1} d_n(C_1; X)} \mathrm{d_H}(C_1, C_2),$$

so that by Lemma 2.4

$$\operatorname{diam} C_1 \leqslant \gamma(m)\delta_{C_1}\mathrm{d_H}(C_1, C_2).$$

Finally, by (2.18)

$$\begin{aligned} ||b(C_1) - b(C_2)|| &\leqslant \operatorname{diam} C_1 + 2\,\mathrm{d_H}(C_1, C_2) \leqslant (\gamma(m)\delta_{C_1} + 2)\mathrm{d_H}(C_1, C_2) \\ &\leqslant 2\gamma(m)(\delta_{C_1} + \delta_{C_2})\mathrm{d_H}(C_1, C_2). \end{aligned}$$

Theorem 1.3 is completely proved. $\square$

3. A Steiner-type point of a convex body. Proof of Theorem 1.2

Without loss of generality, we may suppose that X is a space $\ell_\infty(U)$ of bounded functions defined on a set U. As usual we equip this space with the uniform norm

$$\|x\| := \sup_{u \in U} |x(u)|.$$

First, we will prove the theorem for the family $\mathcal{K}(X)$ of all compact convex finite-dimensional subsets of X. We will construct the required selector $S_X : \mathcal{K}(X) \to X$ by induction on dimension of subsets from $\mathcal{K}(X)$. For the family $\mathcal{K}_0(X)$ of all one point subset of X we define S_X by letting $S_X(\{x\}) := x, x \in X$. Clearly, in this case S_X satisfies all conditions of Theorem 1.2 with constants $\gamma_1 = \gamma_2 = 1$.

Let us suppose that for an integer $m \geqslant 0$ the mapping S_X is defined on the family

$$\mathcal{K}_m(X) := \{C \in \mathcal{K}(X) : \dim C \leqslant m\}$$

and satisfies the following conditions:

(i) for every $C_1, C_2 \in \mathcal{K}_m(X)$

$$\|S_X(C_1) - S_X(C_2)\| \leqslant \gamma_1 \mathrm{d_H}(C_1, C_2); \tag{3.1}$$

(ii) for every $C \in \mathcal{K}_m(X)$ there is an ellipsoid $\mathcal{E}_{C,S}$ centered at $S_X(C)$ such that

$$\mathcal{E}_{C,S} \subset C \subset \gamma_2 \circ \mathcal{E}_{C,S}; \tag{3.2}$$

(iii) for every $C \in \mathcal{K}_m(X)$, $a \in X$ and $\tau \in \mathbf{R}$

$$S_X(\tau C + a) = \tau S_X(C) + a. \tag{3.3}$$

Here γ_1 and γ_2 are constants depending only on m.

Our aim is to extend S_X from $\mathcal{K}_m(X)$ to the family $\mathcal{K}_{m+1}(X)$ while preserving properties (3.1)–(3.3). We will do this via a series of auxiliary lemmas.

A mapping $T : \mathcal{K}_n(X) \to X$ is said to be DS-equivariant (dilation-shift equivariant) if T commutes with all dilations and shifts of X, i.e., for every $a \in X$, $\tau \in \mathbf{R}$ and $C \in \mathcal{K}_n(X)$

$$T(\tau C + a) = \tau T(C) + a.$$

Lemma 3.1. *Every Lipschitz DS-equivariant mapping $T : \mathcal{K}_m(X) \to X$ can be extended to a Lipschitz DS-equivariant mapping $\tilde{T} : \mathcal{K}_{m+1}(X) \to X$ such that*

$$|\tilde{T}|_{\mathrm{Lip}((\mathcal{K}_{m+1}(X), \mathrm{d_H}); X)} \leqslant |T|_{\mathrm{Lip}((\mathcal{K}_m(X), \mathrm{d_H}); X)}. \tag{3.4}$$

Proof. We denote $\theta := |T|_{\mathrm{Lip}((\mathscr{K}_m(X),\mathrm{d_H});X)}$. We also recall that $X = \ell_\infty(U)$, and so every element $x \in X$ can be identified with a function $x = x(u)$, $u \in U$ such that $\|x\| = \sup\{|x(u)| : u \in U\}$. Given $C \in \mathscr{K}_{m+1}(X)$ and $u \in U$ we set

$$[\tilde{T}(C)](u) := \inf_{K \in \mathscr{K}_m(X)} \{[T(K)](u) + \theta \mathrm{d_H}(K, C)\}.$$

This is an instance of a well-known extension formula for mappings from a metric space into $l_\infty(U)$.

It can be readily seen (see e.g. [BL, p. 12]) that $\tilde{T}(C) = T(C)$ for every $C \in \mathscr{K}_m(X)$ and

$$|[\tilde{T}(C_1)](u) - [\tilde{T}(C_2)](u)| \leqslant \theta \mathrm{d_H}(C_1, C_2), \quad C_1, C_2 \in \mathscr{K}_{m+1}(X).$$

Thus $\tilde{T}$ is an extension of T satisfying (3.4). Let us prove that $\tilde{T}$ is DS-equivariant whenever T is. In fact, for every $a \in X, \tau \in \mathbf{R}$ and $u \in U$ we have

$$\begin{aligned}[\tilde{T}(\tau C + a)](u) &= \inf_{K \in \mathscr{K}_m(X)} \{[T(K)](u) + \theta \mathrm{d_H}(K, \tau C + a)\} \\ &= \inf_{K \in \mathscr{K}_m(X)} \{[T(\tau K + a)](u) + \theta \mathrm{d_H}(\tau K + a, \tau C + a)\}.\end{aligned}$$

Since T is DS-equivariant on $\mathscr{K}_m(X)$ and $\mathrm{d_H}(\tau K + a, \tau C + a) = \tau \mathrm{d_H}(K, C)$ we obtain

$$\begin{aligned}[\tilde{T}(\tau C + a)](u) &= \inf_{K \in \mathscr{K}_m(X)} \{\tau[T(K)](u) + a(u) + \tau\theta \mathrm{d_H}(K, C)\} \\ &= \tau \inf_{K \in \mathscr{K}_m(X)} \{[T(K)](u) + \theta \mathrm{d_H}(K, C)\} + a(u) \\ &= \tau[\tilde{T}(C)](u) + a(u). \qquad \square\end{aligned}$$

To define the required extension of S_X to $\mathscr{K}_{m+1}(X)$ we need the following property of ellipsoids. Let $\mathscr{E}$ be a k-dimensional ellipsoid centered at a point $c \in X$. Then there is a simplex $\mathrm{Sim}(\mathscr{E}) = \mathrm{conv}\{a_0, a_1, \dots, a_k\}$ with vertices $\{a_i \in \mathscr{E} : i = 0, \dots, k\}$ such that

$$c = \frac{1}{k+1}(a_0 + a_1 + \cdots + a_k)$$

and

$$\frac{1}{\sigma} \circ \mathscr{E} \subset \mathrm{Sim}(\mathscr{E}) \subset \mathscr{E}, \tag{3.5}$$

where $\sigma = \sigma(k) \geqslant 1$ is a positive constant depending only on k. Clearly, it suffices to prove this assertion for the case when $\mathscr{E}$ is the unit ball of $\mathbf{R}^k$. Then in fact we can obtain (3.5), choosing $\mathrm{Sim}(\mathscr{E})$ to be *any* right simplex inscribed in $\mathscr{E}$.

We now define the extension of S_X to $\mathcal{K}_{m+1}(X)$ as follows. Since the mapping $S_X : \mathcal{K}_m(X) \to X$ is DS-equivariant, by Lemma 3.1 there is a DS-equivariant mapping $\tilde{S} : \mathcal{K}_{m+1}(X) \to X$ such that $\tilde{S}|_{\mathcal{K}_m(X)} = S_X$ and

$$\|\tilde{S}(C_1) - \tilde{S}(C_2)\| \leqslant \gamma_1 \mathrm{d_H}(C_1, C_2) \tag{3.6}$$

for all $C_1, C_2 \in \mathcal{K}_{m+1}(X)$.

We denote

$$\eta = \eta(m) := 8\gamma_2(m)\sigma(m) \tag{3.7}$$

and

$$\widetilde{C} := C \cap B(\tilde{S}(C), R_m(C)), \tag{3.8}$$

where

$$R_m(C) := 8\eta^{m+1}\gamma_1 d_m(C). \tag{3.9}$$

Here γ_1 and γ_1 are constants from (3.1) and (3.2). (We recall that d_m stands for the m-width of a convex set. Hereafter for the sake of brevity we will write $d_m(C)$ instead of $d_m(C; X)$.)

Finally, we put

$$S_X(C) := b(\widetilde{C}), \quad C \in \mathcal{K}_{m+1}(X). \tag{3.10}$$

Since $d_m(C) = 0$ for every $C \in \mathcal{K}_m(X)$, the mapping defined by (3.10) is an extension of S_X from $\mathcal{K}_m(X)$ to $\mathcal{K}_{m+1}(X)$. It is also clear that $S_X(C) = b(\widetilde{C}) \in \widetilde{C} \subset C$ so that S_X is a selector. To complete the proof of Theorem 1.2 it remains to verify that the mapping S_X satisfies on $\mathcal{K}_{m+1}(X)$ conditions (3.1)–(3.3). The following lemma will take care of (3.3).

Lemma 3.2. *The mapping S_X is DS-equivariant on $\mathcal{K}_{m+1}(X)$.*

Proof. Since $\tilde{S}$ is DS-equivariant, for every $\tau \in \mathbf{R}$ and $a \in X$ we have $\tilde{S}(\tau C + a) = \tau \tilde{S}(C) + a$. On the other hand, definition (2.6) of the Kolmogorov m-width implies $d_m(\dot{\tau} C + a) = \tau d_m(C)$ so that $R_m(\tau C + a) = \tau R_m(C)$ as well. Hence

$$\begin{aligned} (\tau C + a)^{\sim} &:= (\tau C + a) \cap B(\tilde{S}(\tau C + a), R_m(\tau C + a)) \\ &= (\tau C + a) \cap (\tau B(\tilde{S}(C), R_m(C)) + a) \\ &= \tau(C \cap B(\tilde{S}(C), R_m(C)) + a) = \tau \widetilde{C} + a. \end{aligned}$$

Since the barycentric map commutes with affine transformations, we obtain

$$S_X(\tau C + a) = b((\tau C + a)^{\sim}) = b(\tau \widetilde{C} + a) = \tau b(\widetilde{C}) + a = \tau S_X(C) + a. \qquad \square$$

Hereafter in this section, we let C denote a subset from the family $\mathscr{K}_{m+1}(X)$ of dimension $m+1$.

Lemma 3.3. *Suppose that $d_0(C) \leqslant \eta^m d_m(C)$. Then*

$$C \subset B(\tilde{S}(C), \frac{1}{2} R_m(C)) \tag{3.11}$$

and $\delta_{\tilde{C}}$ is bounded by a constant depending only on m. Moreover, there is an ellipsoid $\mathscr{E}_{C,S}$ centered at $S_X(C)$ which satisfies (3.2).

Proof. Let us apply (3.6) to $C_1 = C$ and $C_2 = \{x\}$ where $x \in C$ is arbitrary. Then

$$\|\tilde{S}(C) - x\| = \|\tilde{S}(C) - \tilde{S}(\{x\})\| \leqslant \gamma_1 \mathrm{d_H}(C, \{x\}) \leqslant \gamma_1 \operatorname{diam} C.$$

Since $\operatorname{diam} C \leqslant 2d_0(C)$, we have $C \subset B(\tilde{S}(C), 2\gamma_1 d_0(C))$. Since $d_0(C) \leqslant \eta^m d_m(C)$, this implies (3.11). Moreover, from (3.11) and (3.8) it follows that $\widetilde{C} = C$. By lemma's hypothesis $d_k(C) \sim d_m(C)$, $1 \leqslant k \leqslant m-1$, so that by Lemma 2.4 $\delta_{\tilde{C}} = \delta_C$ does not exceed a constant depending only on m.

Finally, $S_X(C) \coloneqq b(\widetilde{C}) = b(C)$ so that the existence of an ellipsoid $\mathscr{E}_{C,S}$ centered at $S_X(C)$ and satisfying (3.2) follows from (2.14). $\square$

Let us consider the case

$$d_0(C) > \eta^m d_m(C). \tag{3.12}$$

Put

$$n \coloneqq \max\left\{ i : \frac{d_i(C)}{d_{i+1}(C)} > \eta, \quad i = 0, 1, \ldots, m-1 \right\}.$$

By (3.12) n is well defined. Moreover, definition of n implies

$$d_k(C) > \eta d_{n+1}(C), \quad k = 0, \ldots, n \tag{3.13}$$

and

$$d_k(C) \leqslant \eta^m d_m(C), \quad k = n+1, \ldots, m-1. \tag{3.14}$$

Lemma 3.4. *There is an* $(n+1)$*-dimensional ellipsoid* $\widetilde{\mathscr{E}} \subset C$ *centered at a point* $\tilde{c}$ *such that*

$$||\tilde{c} - \tilde{S}_X(C)|| \leqslant \frac{1}{2} R_m(C) \tag{3.15}$$

and

$$C \subset \lambda_1 \circ \widetilde{\mathscr{E}} + B(0, \lambda_2 d_m(C)), \tag{3.16}$$

where $\lambda_1 := \gamma_2 \sigma \beta$ *and* $\lambda_2 := \eta^{m+1}$.

Proof. We recall that the constants β and σ are defined by (2.14) and (3.5), respectively.

By (2.7) there is a set $A \in \mathscr{K}_{n+1}(X)$, $\dim A = n+1$ such that

$$\mathrm{d_H}(C, A) \leqslant 2 d_{n+1}(C). \tag{3.17}$$

Then by (3.13) and (2.8) we have

$$d_k(A) \geqslant d_k(C) - \mathrm{d_H}(C, A) \geqslant d_k(C) - 2d_{n+1}(C) \geqslant d_k(C) - \frac{2}{\eta} d_k(C), \quad k = 0, \dots, n.$$

Since $\eta \geqslant 4$, this implies

$$d_k(C) \leqslant 2 d_k(A), \quad k = 0, \dots, n. \tag{3.18}$$

Since $n+1 \leqslant m$ and $A \in \mathscr{K}_{n+1}(X) \subset \mathscr{K}_m(X)$, we have $\tilde{S}_X(A) = S_X(A)$ so that by (3.1) and (3.17)

$$||S_X(A) - \tilde{S}_X(C)|| \leqslant \gamma_1 \mathrm{d_H}(C, A) \leqslant 2\gamma_1 d_{n+1}(C). \tag{3.19}$$

Moreover, by (3.2) there is an ellipsoid $\mathscr{E}_{A,S}$ centered at $S_X(A)$ such that

$$\mathscr{E}_{A,S} \subset A \subset \gamma_2 \circ \mathscr{E}_{A,S}. \tag{3.20}$$

By (3.5) there is an $(n+1)$-dimensional simplex

$$\Delta_A := \mathrm{Sim}(\mathscr{E}_{A,S}) = \mathrm{conv}\{a_0, a_1, \dots, a_{n+1}\}$$

with vertices $a_i \in A$, $i = 0, \dots, n+1$ such that

$$S_X(A) = \frac{1}{n+2} \sum_{i=0}^{n+1} a_i \tag{3.21}$$

and

$$\frac{1}{\sigma} \circ \mathscr{E}_{A,S} \subset \Delta_A \subset \mathscr{E}_{A,S}. \tag{3.22}$$

This inclusion, (3.18) and (3.20) imply

$$d_k(C) \leqslant 2d_k(A) \leqslant 2\gamma_2 d_k(\mathscr{E}_{A,S}) \leqslant 2\gamma_2\sigma d_k(\Delta_A), \quad k = 0, \dots, n. \tag{3.23}$$

By (3.17) there are points $c_i \in C$, $i = 0, \dots, n+1$ such that

$$\|a_i - c_i\| \leqslant \mathrm{d_H}(C, A) \leqslant 2d_{n+1}(C). \tag{3.24}$$

Denote $\Delta_C := \mathrm{conv}\{c_0, c_1, \dots, c_{n+1}\}$. Then

$$\mathrm{d_H}(\Delta_C, \Delta_A) \leqslant \max_{i=0,\dots,n+1} \|a_i - c_i\| \leqslant 2d_{n+1}(C)$$

so that

$$\Delta_A \subset \Delta_C + B(0, 2d_{n+1}(C)). \tag{3.25}$$

We recall that by (3.7) $\eta = 8\gamma_2\sigma$ so that by (2.8), (3.23) and (3.13) we have

$$\begin{aligned} d_n(\Delta_C) &\geqslant d_n(\Delta_A) - \mathrm{d_H}(\Delta_C, \Delta_A) \geqslant d_n(\Delta_A) - 2d_{n+1}(C) \\ &\geqslant \frac{1}{2\gamma_2\sigma} d_n(C) - \frac{2}{\eta} d_n(C) = \frac{1}{4\gamma_2\sigma} d_n(C) \geqslant \frac{1}{4\gamma_2\sigma} d_m(C). \end{aligned}$$

Since $\dim C = m+1$, $d_m(C) > 0$ so that $d_n(\Delta_C) > 0$ as well and therefore $\dim \Delta_C = n+1$. Thus Δ_C is *an* $(n+1)$*-dimensional simplex* so that the point

$$\tilde{c} := \frac{1}{n+2} \sum_{i=0}^{n+1} c_i$$

is the barycenter of Δ_C, i.e., $\tilde{c} = b(\Delta_C)$. Then by (3.19) and (3.21)

$$\|\tilde{c} - \tilde{S}_X(C)\| \leqslant \|\tilde{c} - S_X(A)\| + \|S_X(A) - \tilde{S}_X(C)\| \leqslant \frac{1}{n+2} \sum_{i=0}^{n+1} \|a_i - c_i\| + 2\gamma_1 d_{n+1}(C)$$

so that by (3.24) and (3.14)

$$\|\tilde{c} - \tilde{S}_X(C)\| \leqslant 2(\gamma_1 + 1) d_{n+1}(C) \leqslant 2(\gamma_1 + 1)\eta^m d_m(C) \leqslant 4\gamma_1 \eta^m d_m(C).$$

Since $R_m(C) := 8\eta^{m+1}\gamma_1 d_m(C)$, (3.15) follows. Let us prove (3.16).

By (2.14) there is an $(n+1)$-dimensional ellipsoid $\widetilde{\mathscr{E}} = \mathscr{E}_{\Delta_C, b}$ centered at the point $\tilde{c} = b(\Delta_C)$ such that

$$\widetilde{\mathscr{E}} \subset \Delta_C \subset \beta \circ \widetilde{\mathscr{E}}. \tag{3.26}$$

Finally, by (3.17) $C \subset A + B(0, 2d_{n+1}(C))$ which together with inclusions (3.20), (3.22), (3.25) and (3.26) imply

$$\begin{aligned} C &\subset (\gamma_2\sigma\beta) \circ \widetilde{\mathscr{E}} + B(0, 2d_{n+1}(C)) + B(0, 2\gamma_2\sigma d_{n+1}(C)) \\ &= (\gamma_2\sigma\beta) \circ \widetilde{\mathscr{E}} + B(0, (2\gamma_2\sigma + 2)d_{n+1}(C)) \\ &\subset (\gamma_2\sigma\beta) \circ \widetilde{\mathscr{E}} + B(0, 8\gamma_2\sigma d_{n+1}(C)). \end{aligned}$$

Since $\eta = 8\gamma_2\sigma$ and by (3.14) $d_{n+1}(C) \leqslant \eta^m d_m(C)$, (3.16) follows. □

Lemma 3.5. *Let $\tilde{c}$ be the point from Lemma* 3.4. *Then there is a point $w \in C$ such that*

$$\|w - \tilde{c}\| \leqslant \lambda_2 d_m(C) \tag{3.27}$$

and

$$B(w, \lambda_3^{-1} d_m(C)) \cap \operatorname{affspan}(C) \subset C. \tag{3.28}$$

Here $\lambda_3 = \lambda_3(m) \geqslant 1$ is a constant depending only on m.

Proof. We recall that $\dim C = m + 1$ so that by (2.13)

$$B(b(C), \alpha^{-1} d_m(C)) \cap \operatorname{affspan}(C) \subset C, \tag{3.29}$$

where $\alpha = \alpha(m) \geqslant 1$. Let $\widetilde{\mathscr{E}} \subset C$ be the ellipsoid from Lemma 3.4 centered at $\tilde{c}$. Since $b(C) \in C$, by (3.16) there is a point $u \in \lambda_1 \circ \widetilde{\mathscr{E}}$ such that

$$\|u - b(C)\| \leqslant \lambda_2 d_m(C). \tag{3.30}$$

Let ℓ be the straight line which passes through u and $\tilde{c}$. This line intersects the boundary of the ellipsoid $\widetilde{\mathscr{E}}$ at two points which we will denote by v and v'. More specifically, we will suppose that v' is nearer than v to u, i.e., the line segment $[u, v]$ contains $\tilde{c}$.

Let $\theta := \|v - \tilde{c}\| / \|v - u\|$ so that $\tilde{c} = (1 - \theta)v + \theta u$. Since $\|v - \tilde{c}\| \leqslant \|v - u\|$ and $u \in \lambda_1 \circ \widetilde{\mathscr{E}}$, we have $\lambda_1^{-1} \leqslant \theta \leqslant 1$.

We define the point w by setting $w := (1 - \theta)v + \theta b(C)$. Then by (3.30)

$$\|w - \tilde{c}\| = \theta \|u - b(C)\| \leqslant \|u - b(C)\| \leqslant \lambda_2 d_m(C)$$

which proves (3.27). Let us prove (3.28). Since C is convex and $v \in C$, by (3.29)

$$B(w, \theta\alpha^{-1} d_m(C)) \cap \operatorname{affspan}(C) = [(1 - \theta)v + \theta B(b(C), \alpha^{-1} d_m(C))] \cap \operatorname{affspan}(C) \subset C.$$

Since $\lambda_1^{-1} \leqslant \theta$, this implies the required inclusion (3.28) with $\lambda_3 := \alpha\lambda_1$. □

Lemma 3.6. *Let* $\widetilde{C}$ *be the set defined by* (3.8). *Then* $\dim \widetilde{C} = (m+1)$ *and* $\delta_{\tilde{C}}$ *is bounded by a constant depending only on m. Moreover,*

$$d_m(\widetilde{C}) \sim \operatorname{diam} \widetilde{C} \sim d_m(C). \tag{3.31}$$

Proof. We recall that "$\sim$" stands for equivalence up to a constant depending only on m. We denote $A := B(w, r_m(C)) \cap \operatorname{affspan}(C)$ where $r_m(C) := \lambda_3^{-1} d_m(C)$. Then by (3.27) and (3.15) for every $x \in A$ we have

$$\begin{aligned} \|x - \tilde{S}_X(C)\| &\leqslant \|x - w\| + \|w - \tilde{c}\| + \|\tilde{c} - \tilde{S}_X(C)\| \\ &\leqslant \lambda_3^{-1} d_m(C) + \lambda_2 d_m(C) + \frac{1}{2} R_m(C). \end{aligned}$$

Since $\lambda_3 \geqslant 1, \lambda_2 = \eta^{m+1}$ (see Lemma 3.4) and

$$R_m(C) := 8\gamma_1 \eta^{m+1} d_m(C) \geqslant 8\eta^{m+1} d_m(C),$$

we obtain

$$\|x - \tilde{S}_X(C)\| \leqslant R_m(C)$$

which implies $A \subset B(\tilde{S}_X(C), R_m(C))$. By (3.28) $A \subset C$ so that

$$A \subset C \cap B(\tilde{S}_X(C), R_m(C)) =: \widetilde{C}. \tag{3.32}$$

Clearly, $\dim A = \dim \operatorname{affspan}(C) = m+1$. Since $\dim A \leqslant \dim \widetilde{C} \leqslant \dim C = m+1$, this, in particular, implies that $\dim \widetilde{C} = m+1$. On the other hand, by (1.6),

$$\begin{aligned} \delta_{\tilde{C}} &\leqslant \frac{\lambda(B(\tilde{S}_X(C), R_m(C)) \cap \operatorname{affspan}(C))}{\lambda(A)} \\ &\leqslant \frac{\lambda(B(\tilde{S}_X(C), R_m(C)) \cap \operatorname{affspan}(C))}{\lambda(B(w, r_m(C)) \cap \operatorname{affspan}(C))} \\ &= \left(\frac{R_m(C)}{r_m(C)}\right)^{m+1} \leqslant \gamma(m). \end{aligned}$$

It remains to note that by (3.32) $2r_m(C) \leqslant \operatorname{diam} \widetilde{C} \leqslant 2R_m(C)$ so that $\operatorname{diam} \widetilde{C} \sim d_m(C)$. Since $\delta_{\tilde{C}} \leqslant \gamma(m)$, by Lemma 2.4 $\operatorname{diam} \widetilde{C} \sim d_m(\widetilde{C})$ and the lemma follows. $\square$

Lemma 3.7. *There is an ellipsoid* $\mathscr{E}_{C,S}$ *centered at* $S_X(C)$ *such that*

$$\mathscr{E}_{C,S} \subset C \subset \gamma \circ \mathscr{E}_{C,S}, \tag{3.33}$$

where γ *is a constant depending only on m.*

Proof. Since all the sets appearing in (3.33) are contained in the affine span of C, we may assume, without loss of generality, that $X = \mathrm{affspan}(C)$.

Since $S_X(C) = b(\widetilde{C}) \in \widetilde{C}$, by (3.8) $\|S_X(C) - \tilde{S}_X(C)\| \leqslant R_m(C)$. Let $\widetilde{\mathscr{E}}$ be the ellipsoid from Lemma 3.4 centered at the point $\tilde{c}$. Then by (3.15)

$$\|S_X(C) - \tilde{c}\| \leqslant \|S_X(C) - \tilde{S}_X(C)\| + \|\tilde{S}_X(C) - \tilde{c}\| \leqslant R_m(C) + \frac{1}{2} R_m(C),$$

so that

$$\|S_X(C) - \tilde{c}\| \leqslant \gamma d_m(C) \tag{3.34}$$

for some $\gamma = \gamma(m)$. By Lemma 3.6 $\dim \widetilde{C} = m + 1$ so that by (2.13)

$$B(b(\widetilde{C}), \alpha^{-1} d_m(\widetilde{C})) \subset \widetilde{C}.$$

(Of course $\mathrm{affspan}(C) = \mathrm{affspan}(\widetilde{C})$, and recall that we are assuming that $X = \mathrm{affspan}(C)$.) Since $b(\widetilde{C}) = S_X(C)$ and by Lemma 3.6 $d_m(\widetilde{C}) \sim d_m(C)$, we have

$$B(S_X(C), \alpha_1^{-1} d_m(C)) \subset \widetilde{C}, \tag{3.35}$$

where $\alpha_1 = \alpha_1(m) \geqslant 1$.

Let ℓ be the straight line which passes through $\tilde{c}$ and $S_X(C)$. We denote $r := \alpha_1^{-1} d_m(C)$ and $B := B(S_X(C), r)$. Then ℓ intersects the boundary of the ball B at two points which we will denote by v and v'. Let us suppose that v' is nearer than v to $S_X(C)$, i.e., the line segment $[v, \tilde{c}]$ contains $S_X(C)$. We denote $\theta := \|v - S_X(C)\| / \|v - \tilde{c}\|$. Then

$$S_X(C) = (1 - \theta)v + \theta \tilde{c}. \tag{3.36}$$

Since v lies on the boundary of the ball B, $\|v - S_X(C)\| = r$. Hence by (3.34)

$$1 \geqslant \theta = \frac{1}{1 + \|S_X(C) - \tilde{c}\| / \|v - S_X(C)\|} \geqslant \frac{1}{\alpha_2}, \tag{3.37}$$

where $\alpha_2 = 1 + \gamma a_1$.

Then we define an ellipsoid $\mathscr{E}'$ by letting

$$\mathscr{E}' = (1 - \theta)v + \theta \widetilde{\mathscr{E}}. \tag{3.38}$$

By (3.36) $\mathscr{E}'$ is centered at $S_X(C)$. Since $v \in \widetilde{C} \subset C$ and $\widetilde{\mathscr{E}} \subset C$, $\mathscr{E}' \subset C$ as well.

We denote

$$A := \frac{1}{2} \mathscr{E}' + \frac{1}{2} B. \tag{3.39}$$

Since by (3.35) $B \subset \widetilde{C} \subset C$ and $\mathscr{E}' \subset C$, we have $A \subset C$. Clearly, A is symmetric with respect to $S_X(C)$ so that $b(A) = S_X(C)$. Then by (2.14) there is an ellipsoid

$\mathscr{E}_{C,S} := \mathscr{E}_{A,b}$ centered at $b(A)(= S_X(C))$ such that

$$\mathscr{E}_{C,S} \subset A \subset \beta \circ \mathscr{E}_{C,S} \tag{3.40}$$

with $\beta = \beta(m)$. Let us prove that $\mathscr{E}_{C,S}$ satisfies (3.33).

Clearly, $\mathscr{E}_{C,S} \subset A \subset C$. To prove the inverse inclusion we note that by (3.39) for every $t > 0$ we have

$$t \circ A = \frac{t}{2} \circ \mathscr{E}' + B\left(0, \frac{t}{2} r\right). \tag{3.41}$$

On the other hand, by (3.38) and (3.36)

$$\begin{aligned}\widetilde{\mathscr{E}} &= \theta^{-1}(\mathscr{E}' - (1-\theta)v) = \theta^{-1}(\mathscr{E}' - S_X(C)) + \tilde{c} \\ &= (\theta^{-1}(\mathscr{E}' - S_X(C)) + S_X(C)) + (\tilde{c} - S_X(C)) \\ &= \theta^{-1} \circ \mathscr{E}' + (\tilde{c} - S_X(C)).\end{aligned}$$

Then by (3.16)

$$C \subset \lambda_1 \circ \widetilde{\mathscr{E}} + B(0, \lambda_2 d_m(C)) = (\lambda_1 \theta^{-1}) \circ \mathscr{E}' + (\tilde{c} - S_X(C)) + B(0, \lambda_2 d_m(C)).$$

But by (3.34) $\tilde{c} - S_X(C) \in B(0, \gamma d_m(C))$ and by (3.37) $\theta^{-1} \leqslant \alpha_2$ so that

$$\begin{aligned}C &\subset (\lambda_1 \alpha_2) \circ \mathscr{E}' + B(0, \gamma d_m(C)) + B(0, \lambda_2 d_m(C)) \\ &= (\lambda_1 \alpha_2) \circ \mathscr{E}' + B(0, (\gamma + \lambda_2) d_m(C)).\end{aligned}$$

Since $r := \alpha_1^{-1} d_m(c)$, we obtain

$$C \subset (\lambda_1 \alpha_2) \circ \mathscr{E}' + B(0, (\gamma + \lambda_2)\alpha_1 r) \subset \frac{\tau}{2} \circ \mathscr{E}' + B\left(0, \frac{\tau}{2} r\right),$$

where $\tau := 2\lambda_1\alpha_2 + 2(\gamma + \lambda_2)\alpha_1$. From this and (3.41) it follows $C \subset \tau \circ A$ so that by (3.40) $C \subset (\tau\beta) \circ \mathscr{E}_{C,S}$. $\square$

Let us finish the proof of Theorem 1.2. Inclusion (3.2) for the mapping

$$S_X : \mathscr{K}_{m+1}(X) \to X$$

follows from Lemma 3.3 (for the case $d_0(C) \leqslant \eta^m d_m(C)$) and Lemma 3.7 ($d_0(C) > \eta^m d_m(C)$). In turn, equality (3.3) for S_X on $\mathscr{K}_{m+1}(X)$ follows from Lemma 3.2. Let us prove inequality (3.1).

The following lemma (in a slightly general form) is proved in [PR]. For similar results and ideas we refer the reader to [Ar,AF, p. 369]; [PY2,BL, p. 26], and references therein.

Lemma 3.8. *Suppose that a convex set $C \subset X$ and a ball $B(a,r), a \in X, r > 0$ have a common point. Then for every $s > 0$*

$$(C + B(0,s)) \cap (B(a,2r) + B(0,s)) \subset C \cap B(a,2r) + B(0,7s).$$

Lemma 3.9. *Let $C_i \subset X$ be a convex set and let $B(a_i, r_i), a_i \in X, r_i > 0$, be a ball in X such that $C_i \cap B(a_i, r_i) \neq \emptyset$, $i = 1,2$. Then*

$$\mathrm{d_H}(C_1 \cap B(a_1, 2r_1), C_2 \cap B(a_2, 2r_2)) \leqslant 14(\mathrm{d_H}(C_1, C_2) + \|a_1 - a_2\| + |r_1 - r_2|).$$

Proof. We denote

$$s \coloneqq \mathrm{d_H}(C_1, C_2) + \|a_1 - a_2\| + 2|r_1 - r_2|.$$

Then

$$C_1 + B(0,s) \supset C_1 + B(0, \mathrm{d_H}(C_1, C_2)) \supset C_2.$$

On the other hand, for every $x \in B(a_2, 2r_2)$ we have

$$\begin{aligned} \|x - a_1\| &\leqslant \|x - a_2\| + \|a_1 - a_2\| \leqslant 2r_2 + \|a_1 - a_2\| \\ &\leqslant 2r_1 + 2|r_1 - r_2| + \|a_1 - a_2\| \leqslant 2r_1 + s \end{aligned}$$

so that

$$B(a_2, 2r_2) \subset B(a_1, 2r_1 + s) = B(a_1, 2r_1) + B(0,s).$$

Applying Lemma 3.8 to C_1 and $B(a_1, r_1)$ we obtain

$$\begin{aligned} C_1 \cap B(a_1, 2r_1) + B(0,7s) &\supset (C_1 + B(0,s)) \cap (B(a_1, 2r_1) + B(0,s)) \\ &\supset C_2 \cap B(a_2, 2r_2). \end{aligned}$$

Similarly we get

$$C_2 \cap B(a_2, 2r_2) + B(0,7s) \supset C_1 \cap B(a_1, 2r_1).$$

Hence

$$\mathrm{d_H}(C_1 \cap B(a_1, 2r_1), C_2 \cap B(a_2, 2r_2)) \leqslant 7s. \qquad \square$$

Now let us prove that for every $C_1, C_2 \in \mathscr{K}_{m+1}(X)$

$$\mathrm{d_H}(\widetilde{C}_1, \widetilde{C}_2) \leqslant \tilde{\gamma}\, \mathrm{d_H}(C_1, C_2), \tag{3.42}$$

where $\tilde{\gamma} = 14\gamma_1(8\eta^{m+1} + 2)$. In fact, by Lemma 3.3 (for the case $d_0(C) \leqslant \eta^m d_m(C)$) and by Lemma 3.4 ($d_0(C) > \eta^m d_m(C)$) we have

$$B(\tilde{S}(C), \frac{1}{2}R_m(C)) \cap C \neq \emptyset$$

so that by (3.8) and Lemma 3.9

$$\begin{aligned} \mathrm{d_H}(\widetilde{C}_1, \widetilde{C}_2) &= \mathrm{d_H}(C_1 \cap B(\tilde{S}(C_1), R_m(C_1)), C_2 \cap B(\tilde{S}(C_2), R_m(C_2))) \\ &\leqslant 14(\mathrm{d_H}(C_1, C_2) + \|\tilde{S}(C_1) - \tilde{S}(C_2)\| + |R_m(C_1) - R_m(C_2)|). \end{aligned}$$

But by (3.6)

$$\|\tilde{S}(C_1) - \tilde{S}(C_2)\| \leqslant \gamma_1 \, \mathrm{d_H}(C_1, C_2)$$

and by (3.9) and (2.8)

$$|R_m(C_1) - R_m(C_2)| = 8\eta^{m+1}\gamma_1 |d_m(C_1) - d_m(C_2)| \leqslant 8\eta^{m+1}\gamma_1 \, \mathrm{d_H}(C_1, C_2),$$

and (3.42) follows. Then by Theorem 1.3

$$\|S_X(C_1) - S_X(C_2)\| = \|b(\widetilde{C}_1) - b(\widetilde{C}_2)\| \leqslant \gamma(m)(\delta_{\tilde{C}_1} + \delta_{\tilde{C}_2}) \, \mathrm{d_H}(\widetilde{C}_1, \widetilde{C}_2)$$

so that by (3.42)

$$\|S_X(C_1) - S_X(C_2)\| \leqslant \gamma(m)\tilde{\gamma}(\delta_{\tilde{C}_1} + \delta_{\tilde{C}_2}) \, \mathrm{d_H}(C_1, C_2).$$

It remains to note that by Lemmas 3.3 and 3.6 $\delta_{\tilde{C}} \leqslant \xi$ where $\xi = \xi(m)$ is a constant depending only on m. Hence

$$\|S_X(C_1) - S_X(C_2)\| \leqslant \gamma_1 \, \mathrm{d_H}(C_1, C_2)$$

with $\gamma_1 = \gamma_1(m+1) := 14\gamma(m)\tilde{\gamma}\xi$ and the required inequality (3.1) is proved.

Thus we have proved Theorem 1.2 for the family $\mathscr{K}(X)$ of all compact finite-dimensional subsets of X. Now, using an argument based on an idea of [AF, p. 372], we will extend this result to the family $\mathrm{Conv_{cl}}(X)$ of all *closed* finite-dimensional subsets of X.

To this end given $C \subset \mathrm{Conv_{cl}}(X)$ we define a subset $\widetilde{C} \in \mathscr{K}(X)$ by letting $\widetilde{C} := C$ if $C \in \mathscr{K}(X)$ and $\widetilde{C} := C \cap B(0, 2\,\mathrm{dist}(0, C))$ if C is *unbounded*. Then we define S_X on the family $\mathrm{Conv_{cl}}(X)$ by setting $S_X(C) := S_X(\widetilde{C})$. Clearly, $S_X(C) \in C$. Let us prove that $S_X : \mathrm{Conv_{cl}}(X) \to X$ satisfies inequality (1.4).

Since for every $C_1, C_2 \in \mathrm{Conv}_{\mathrm{cl}}(X)$

$$|\mathrm{dist}(0, C_1) - \mathrm{dist}(0, C_2)| \leqslant \mathrm{d_H}(C_1, C_2),$$

by Lemma 3.9

$$\mathrm{d_H}(\widetilde{C}_1, \widetilde{C}_2) = \mathrm{d_H}(C_1 \cap B(0, 2\,\mathrm{dist}(0, C_1)), C_2 \cap B(0, 2\,\mathrm{dist}(0, C_2)))$$
$$\leqslant 14(\mathrm{d_H}(C_1, C_2) + 2|\mathrm{dist}(0, C_1) - \mathrm{dist}(0, C_2)|) \leqslant 42\,\mathrm{d_H}(C_1, C_2).$$

Since $\widetilde{C}_i \in \mathscr{K}(X),\ i = 1, 2$, and S_X is Lipschitz continuous on $\mathscr{K}(X)$, we obtain

$$||S_X(C_1) - S_X(C_2)|| = ||S_X(\widetilde{C}_1) - S_X(\widetilde{C}_2)|| \leqslant \gamma_1\,\mathrm{d_H}(\widetilde{C}_1, \widetilde{C}_2) \leqslant 42\gamma_1\,\mathrm{d_H}(C_1, C_2).$$

Thus to finish the proof of Theorem 1.2 it remains to extend the mapping S_X from $\mathrm{Conv}_{\mathrm{cl}}(X)$ to $\mathrm{Conv}_{\mathcal{F}}(X)$. We define this extension by letting $S_X(C) := S_X(\mathrm{cl}(C))$ where cl stands for the closure of C in X. Since the restriction of S_X to $\mathrm{Conv}_{\mathrm{cl}}(X)$ satisfies the conditions of Theorem 1.2, it can be readily seen that the same is true for S_X defined on all of the family $\mathrm{Conv}_{\mathcal{F}}(X)$.

Theorem 1.2 is completely proved. $\square$

4. Proof of Theorem 1.1

As in the previous section we assume that X is a space $\ell_\infty(U)$ of bounded functions defined on a set U.

We prove the theorem by induction on m. For $m = 0$ nothing to prove. Let us assume that the result is true for given $m \geqslant 0$ and prove it for $m + 1$.

Let $F : \mathscr{M} \to \mathscr{A}_{m+1}(X)$ be a set-valued mapping satisfying the theorem's hypotheses, i.e., the restriction $F|_{\mathscr{M}'}$ of F to every subset $\mathscr{M}' \subset \mathscr{M}$ consisting of at most 2^{m+2} points has a Lipschitz selection $f_{\mathscr{M}'}$ such that $|f_{\mathscr{M}'}|_{\mathrm{Lip}(\mathscr{M}';X)} \leqslant 1$. We will construct the required Lipschitz selection of F in three steps. In the first two steps we will mainly follow the scheme of the proof suggested in [S3] for the case of a Hilbert space.

Step 1. Let $\{x_1, x_2\} \subset \mathscr{M},\ x_1 \neq x_2$. Then $F|_{\{x_1,x_2\}}$ has a Lipschitz selection with the Lipschitz seminorm at most 1, so that there are points $a^i(x_1, x_2) \in F(x_i),\ i = 1, 2$, satisfying

$$||a^1(x_1, x_2) - a^2(x_1, x_2)|| \leqslant \rho(x_1, x_2). \tag{4.1}$$

We denote

$$a(x_1, x_2) := a^1(x_1, x_2) - a^2(x_1, x_2) \tag{4.2}$$

and

$$C(x_1, x_2) := F(x_1) \cap \{F(x_2) + B(0, 2\rho(x_1, x_2)) + a(x_1, x_2)\}. \tag{4.3}$$

Then $C(x_1, x_2)$ is non-empty and symmetric with respect to the point $a^1(x_1, x_2)$. This set is a convex closed subset of the finite-dimensional affine manifold $F(x_1)$ so that by the separation theorem $C(x_1, x_2)$ can be represented as intersection of layers in $F(x_1)$ symmetric with respect to $a^1(x_1, x_2)$. In other words, there exists a family of layers $F(x_1) \bigcap \{L_i + B(0, r_i)\}$ where i belongs to some family of indexes $\mathscr{I}(x_1, x_2)$, which satisfy the following conditions:

(a) L_i is an affine subset of $F(x_1)$ of dimension at most m passing through $a^1(x_1, x_2)$;

(b)
$$C(x_1, x_2) = \bigcap_{i \in \mathscr{I}(x_1, x_2)} (F(x_1) \cap \{L_i + B(0, r_i)\}). \tag{4.4}$$

Note that r_i may assign the value $+\infty$ (e.g., if $F(x_1)$ is parallel to $F(x_2)$).

Now we define a pseudometric space $(\widetilde{\mathscr{M}}, \tilde{\rho})$ by letting

$$\widetilde{\mathscr{M}} := \{\tilde{x} = (x_1, x_2, i) : x_1, x_2 \in \mathscr{M},\ i \in \mathscr{I}(x_1, x_2)\}.$$

We equip this set with a pseudometric $\tilde{\rho}$ defined by the formula

$$\tilde{\rho}(\tilde{x}, \tilde{x}') := \rho(x_1, {x'}_1) + r_i + r_{i'}$$

for $\tilde{x} = (x_1, x_2, i) \neq \tilde{x}' = ({x'}_1, {x'}_2, i')$ and $\tilde{\rho}(\tilde{x}, \tilde{x}) := 0$.

Finally, we define a set-valued mapping $\tilde{F} : \widetilde{\mathscr{M}} \to \mathscr{A}_m(X)$ by setting

$$\tilde{F}(\tilde{x}) := L_i, \quad \tilde{x} = (x_1, x_2, i) \in \widetilde{\mathscr{M}}.$$

Lemma 4.1. *There is a Lipschitz selection $\tilde{f}$ of $\tilde{F}$ satisfying*

$$|\tilde{f}|_{\mathrm{Lip}((\widetilde{\mathscr{M}}, \tilde{\rho}); X)} \leqslant \gamma(m). \tag{4.5}$$

Proof. Basing on the induction assumption we have to show that for every $S \subset \widetilde{\mathscr{M}}$ consisting of at most 2^{m+1} points there is a Lipschitz selection $\tilde{f}_S$ of the restriction $\tilde{F}|_S$ satisfying $|\tilde{f}_S|_{\mathrm{Lip}(S;X)} \leqslant 1$. To prove this given $\tilde{x} = (x_1, x_2, i) \in \widetilde{\mathscr{M}}$ we denote

$$\mathrm{pr}_k(\tilde{x}) := x_k, \quad k = 1, 2, \quad \mathrm{pr}_3(\tilde{x}) := i. \tag{4.6}$$

Sometimes we also write $x_k(\tilde{x})$ for $\mathrm{pr}_k(\tilde{x}), k = 1, 2$ and $i(\tilde{x})$ for $\mathrm{pr}_3(\tilde{x})$.

Then we introduce a subset $\mathscr{M}'$ of $\mathscr{M}$ by setting $\mathscr{M}' := \mathrm{pr}_1(S) \cup \mathrm{pr}_2(S)$. Clearly, $\mathscr{M}'$ consists of at most $2\,\mathrm{card}\ S \leqslant 2^{m+2}$ points. Then by the induction assumption there exists a Lipschitz selection $f_{\mathscr{M}'} : \mathscr{M}' \to X$ of the restriction $F|_{\mathscr{M}'}$ with the seminorm

$|f_{\mathcal{M}'}|_{\mathrm{Lip}(\mathcal{M}';X)} \leqslant 1$. Prove that for every $x_1, x_2 \in \mathcal{M}'$

$$f_{\mathcal{M}'}(x_1) \in C(x_1, x_2). \tag{4.7}$$

In fact, $f_{\mathcal{M}'}(x_1)$ belongs to $F(x_1)$ and

$$\begin{aligned}\|f_{\mathcal{M}'}(x_1) - f_{\mathcal{M}'}(x_2) - a(x_1,x_2)\| &\leqslant \|f_{\mathcal{M}'}(x_1) - f_{\mathcal{M}'}(x_2)\| \\ &\quad + \|a^1(x_1,x_2) - a^2(x_1,x_2)\| \leqslant 2\rho(x_1,x_2),\end{aligned}$$

see (4.1). Since $f_{\mathcal{M}'}(x_2) \in F(x_2)$, $f_{\mathcal{M}'}(x_1)$ belongs to the layer $F(x_2) + B(0, 2\rho(x_1,x_2))$ shifted by $a(x_1,x_2)$, and (4.7) follows.

We define now the required Lipschitz selection $\tilde{f}_S : S \to X$ by letting $\tilde{f}_S(\tilde{x})$ to be a point of $\tilde{F}(\tilde{x})$ *nearest* to $f_{\mathcal{M}'}(x_1(\tilde{x}))$. Then by (4.7) and (4.4) we have

$$f_{\mathcal{M}'}(x_1(\tilde{x})) \in L_{i(\tilde{x})} + B(0, r_i(\tilde{x})).$$

Since $\tilde{f}_S(\tilde{x})$ is the nearest from points of $\tilde{F}(\tilde{x})$ to $f_{\mathcal{M}'}(x_1(\tilde{x}))$, we obtain

$$\|\tilde{f}_S(\tilde{x}) - f_{\mathcal{M}'}(x_1(\tilde{x}))\| \leqslant r_{i(\tilde{x})}.$$

Hence

$$\begin{aligned}\|\tilde{f}_S(\tilde{x}) - \tilde{f}_S(\tilde{x}')\| &\leqslant r_{i(\tilde{x})} + r_{i(\tilde{x}')} + \|f_{\mathcal{M}'}(x_1(\tilde{x})) - f_{\mathcal{M}'}(x_1(\tilde{x}'))\| \\ &\leqslant r_{i(\tilde{x})} + r_{i(\tilde{x}')} + \rho(x_1(\tilde{x}), x_1(\tilde{x}')) = \tilde{\rho}(\tilde{x}, \tilde{x}')\end{aligned}$$

which is equivalent to the required inequality $|\tilde{f}_S|_{\mathrm{Lip}(S;X)} \leqslant 1$. □

Step 2:

Lemma 4.2. *Let K be a set-valued mapping from $(\mathcal{M}, \rho)$ into the family of all balls (i.e., infinite-dimensional cubes) of the space $X = \ell_\infty(U)$. Suppose that for every $x, y \in \mathcal{M}$ the restriction $K|_{\{x,y\}}$ has a Lipschitz selection whose Lipschitz seminorm does not exceed* 1. *Then K also has a Lipschitz selection $\kappa : \mathcal{M} \to X$ with the Lipschitz seminorm $|\kappa|_{\mathrm{Lip}(\mathcal{M})} \leqslant 1$.*

Proof. Since $X = \ell_\infty(U)$, it suffices to prove the lemma for $X = \mathbf{R}$. For the proof of the result in this case we refer the reader to [L]; see also [S4]. □

Lemma 4.3. *There is a set-valued mapping $g : \mathcal{M} \to X$ such that*

$$|g|_{\mathrm{Lip}((\mathcal{M},\rho);X)} \leqslant \gamma(m) \tag{4.8}$$

and

$$||g(x) - \tilde{f}(\tilde{x})|| \leqslant \gamma(m) r_{i(\tilde{x})} \tag{4.9}$$

for every $\tilde{x} \in \widetilde{\mathscr{M}}$ *with* $\mathrm{pr}_1(\tilde{x}) = x$.

Proof. Let us introduce a pseudometric space $(\widetilde{\mathscr{M}}, \hat{\rho})$ by setting

$$\hat{\rho}(\tilde{x}, \tilde{x}') := \gamma\rho(\mathrm{pr}_1(\tilde{x}), \mathrm{pr}_1(\tilde{x}')),$$

where $\gamma = \gamma(m)$ is the constant from inequality (4.5). (We recall that pr_i is defined in (4.6)). We define a set-valued mapping K from $\widetilde{\mathscr{M}}$ into the family of all balls of X by setting

$$K(\tilde{x}) := B(\tilde{f}(\tilde{x}), \gamma r_{i(\tilde{x})}). \tag{4.10}$$

Let us prove that K has a Lipschitz (with respect to $\hat{\rho}$) selection. In fact, by Lemma 4.1 the centers $\tilde{f}(\tilde{x})$ and $\tilde{f}(\tilde{x}')$ of the balls $K(\tilde{x})$ and $K(\tilde{x}')$ satisfy the inequality

$$||\tilde{f}(\tilde{x}) - \tilde{f}(\tilde{x}')|| \leqslant \gamma\tilde{\rho}(\tilde{x}, \tilde{x}') := \gamma(r_{i(\tilde{x})} + r_{i(\tilde{x}')} + \rho(x_1(\tilde{x}), x_1(\tilde{x}'))).$$

Since $\gamma r_{i(\tilde{x})}$ and $\gamma r_{i(\tilde{x}')}$ are radii of these balls, there are points $b \in K(\tilde{x}),\ b' \in K(\tilde{x}')$ such that

$$||b - b'|| \leqslant \gamma\rho(x_1(\tilde{x}), x_1(\tilde{x}')) =: \hat{\rho}(\tilde{x}, \tilde{x}'). \tag{4.11}$$

Thus K satisfies on $(\widetilde{\mathscr{M}}, \hat{\rho})$ conditions of Lemma 4.2 so that there is a Lipschitz selection $\tilde{g}$ of K satisfying $|\tilde{g}|_{\mathrm{Lip}((\widetilde{\mathscr{M}},\hat{\rho});X)} \leqslant 1$. Hence

$$||\tilde{g}(\tilde{x}) - \tilde{f}(\tilde{x})|| \leqslant \gamma r_{i(\tilde{x})}, \quad \tilde{x} \in \widetilde{\mathscr{M}} \tag{4.12}$$

and

$$||\tilde{g}(\tilde{x}) - \tilde{g}(\tilde{x}')|| \leqslant \hat{\rho}(\tilde{x}, \tilde{x}') = \gamma\rho(\mathrm{pr}_1(\tilde{x}), \mathrm{pr}_1(\tilde{x}')), \quad \tilde{x}, \tilde{x}' \in \widetilde{\mathscr{M}}. \tag{4.13}$$

Thus for every $\tilde{x} = (x, x_2, i),\ \tilde{x}' = (x, x'_2, i') \in \widetilde{\mathscr{M}}$ we have

$$||\tilde{g}(\tilde{x}) - \tilde{g}(\tilde{x}')|| \leqslant \gamma\rho(\mathrm{pr}_1(\tilde{x}), \mathrm{pr}_1(\tilde{x}')) = 0$$

so that $\tilde{g}(\tilde{x}) = \tilde{g}(\tilde{x}')$ whenever $\mathrm{pr}_1(\tilde{x}) = \mathrm{pr}_1(\tilde{x}')$. It remains to put $g(x) := \tilde{g}(\tilde{x})$, $\tilde{x} = (x, x_2, i) \in \widetilde{\mathscr{M}}$ and the required inequalities (4.8) and (4.9) will follow from (4.13) and (4.12). $\square$

Step 3: We define a set-valued mapping G on $(\mathscr{M}, \rho)$ by setting

$$G(x) := F(x) \cap B(g(x), 2\,\mathrm{dist}(g(x), F(x))). \tag{4.14}$$

(We recall that $\operatorname{dist}(a, A) := \inf\{\|a - a'\| : a' \in A\}$.) Then we apply to G the result of Theorem 1.2 to determine the required Lipschitz selection $f : \mathcal{M} \to X$ of F by setting

$$f(x) := S_X(G(x)). \tag{4.15}$$

We recall that $S_X : \mathcal{K}(X) \to X$ is the Lipschitz continuous selector from Theorem 1.2. Since $\dim G(x), \dim G(y) \leqslant m+1$ for every $x, y \in \mathcal{M}$, inequality (1.4) implies

$$\|f(x) - f(y)\| = \|S_X(G(x)) - S_X(G(y))\| \leqslant \gamma_1(m)\, \mathrm{d_H}(G(x), G(y)). \tag{4.16}$$

Thus it remains to prove that $G : (\mathcal{M}, \rho) \to (\mathcal{K}(X), \mathrm{d_H})$ is Lipschitz continuous. Some of the steps for doing this are contained in the next lemma.

Lemma 4.4. *For every* $x, y \in \mathcal{M}$

$$|\operatorname{dist}(g(x), F(x)) - \operatorname{dist}(g(y), F(y))| \leqslant 12\gamma\rho(x, y) \tag{4.17}$$

and

$$G(x) \subset F(y) + B(0, 11\gamma\rho(x, y)). \tag{4.18}$$

Proof. We denote by $h(x)$ a point nearest to $g(x)$ on $F(x)$. Thus

$$\|g(x) - h(x)\| = \operatorname{dist}(g(x), F(x)).$$

We also denote

$$r(x) := \inf\{r_{i(\tilde{x})} : \ \mathrm{pr}_1(\tilde{x}) = x, \tilde{x} \in \widetilde{\mathcal{M}}\}.$$

Since $g(x) = g(\tilde{x})$ for every $\tilde{x} = (x, x', i) \in \widetilde{\mathcal{M}}, i \in \mathcal{T}(x, x'), \tilde{f}(\tilde{x}) \in L_{i(\tilde{x})} \subset F(x)$ and

$$\|g(\tilde{x}) - \tilde{f}(\tilde{x})\| \leqslant \gamma r_{i(\tilde{x})}, \tag{4.19}$$

we have

$$\begin{aligned} \|g(x) - h(x)\| &= \operatorname{dist}(g(x), F(x)) = \operatorname{dist}(g(\tilde{x}), F(x)) \\ &\leqslant \|g(\tilde{x}) - \tilde{f}(\tilde{x})\| \leqslant \gamma r_{i(\tilde{x})} \end{aligned} \tag{4.20}$$

so that

$$\|g(x) - h(x)\| = \operatorname{dist}(g(x), F(x)) \leqslant \gamma r(x).$$

Hence

$$B(g(x), 2\operatorname{dist}(g(x), F(x))) \subset B(g(x), 2\gamma r(x)) \subset B(h(x), 3\gamma r(x)),$$

so that

$$G(x) := F(x) \cap B(g(x), 2\,\mathrm{dist}(g(x), F(x))) \subset F(x) \cap B(h(x), 3\gamma r(x)). \tag{4.21}$$

Let us prove that

$$F(x) \cap B(h(x), 3\gamma r(x)) \subset F(y) + B(0, 11\gamma\rho(x,y)). \tag{4.22}$$

(We recall that $a(x,y) := a^1(x,y) - a^2(x,y)$ is defined in (4.1).) By (4.19), (4.20) and equality $g(x) = g(\tilde{x}), \tilde{x} = (x,y,i) \in \widetilde{\mathscr{M}},\ i \in \mathscr{I}(x,y)$, we have

$$||h(x) - \tilde{f}(\tilde{x})|| \leqslant 2\gamma r_i. \tag{4.23}$$

Now let $u \in B(h(x), 3\gamma r(x))$. Since $r(x) \leqslant r_i$ for every $i \in \mathscr{I}(x,y)$, we have $||u - h(x)|| \leqslant 3\gamma r_i$, which together with (4.23) implies

$$||u - \tilde{f}(\tilde{x})|| \leqslant 5\gamma r_i.$$

Since $\tilde{f}(\tilde{x}) \in \tilde{F}(\tilde{x}) := L_i$, this inequality yields

$$B(h(x), 3\gamma r(x)) \subset \bigcap_{i \in \mathscr{I}(x,y)} B(\tilde{f}(\tilde{x}), 5\gamma r_i) \subset \bigcap_{i \in \mathscr{I}(x,y)} (L_i + B(0, 5\gamma r_i)).$$

Now suppose that $a^1(x,y) = 0$ (changing the coordinate system in X, if any). Then $F(x) \supset L_i \ni a^1(x,y) = 0$ whenever $i \in \mathscr{I}(x,y)$ so that $F(x)$ and L_i are linear subspaces of X. Hence we have

$$B(h(x), 3\gamma r(x)) \cap F(x) \subset 5\gamma\left(\left[\bigcap_{i \in \mathscr{I}(x,y)} (L_i + B(0, r_i))\right] \cap F(x)\right).$$

so that by (4.4)

$$B(h(x), 3\gamma r(x)) \cap F(x) \subset 5\gamma C(x,y).$$

(Here as usual λC stands for the dilation of a set with respect to 0 by a factor of λ.) Hence by (4.3) we obtain

$$B(h(x), 3\gamma r(x)) \cap F(x) \subset 5\gamma[(F(y) + a(x,y) + B(0, 2\rho(x,y))) \cap F(x)].$$

Since $a^1(x,y) \in F(y) + a(x,y)$ and $a^1(x,y) = 0$, $F(y) + a(x,y)$ is a linear space. Hence

$$B(h(x), 3\gamma r(x)) \cap F(x) \subset (F(y) + a(x,y) + B(0, 10\gamma\rho(x,y))) \cap F(x).$$

But by (4.1) and (4.2) $||a(x,y)|| \leqslant \rho(x,y)$ so that

$$B(h(x), 3\gamma r(x)) \cap F(x) \subset F(y) + B(0, (10\gamma + 1)\rho(x,y)).$$

Since $\gamma \geqslant 1$, (4.22) follows.

From (4.21) and (4.22) we have (4.18). Let us prove (4.17). Since $h(x) \in F(x)$, by (4.22) there is an element $v \in F(y)$ such that $\|h(x) - v\| \leqslant 11\gamma\rho(x, y)$. Hence

$$\begin{aligned}\operatorname{dist}(g(y), F(y)) \leqslant \|g(y) - v\| &\leqslant \|g(y) - g(x)\| + \|g(x) - h(x)\| + \|h(x) - v\| \\ &\leqslant \gamma\rho(x, y) + \operatorname{dist}(g(x), F(x)) + 11\gamma\rho(x, y) \\ &= \operatorname{dist}(g(x), F(x)) + 12\gamma\rho(x, y).\end{aligned}$$

Similarly we have

$$\operatorname{dist}(g(x), F(x)) \leqslant \operatorname{dist}(g(y), F(y)) + 12\gamma\rho(x, y)$$

and (4.17) is proved. □

We are in a position to finish the proof of Theorem 1.1. To this end we denote $s := 25\gamma\rho(x, y)$. Then by (4.14) and Lemma 3.8 we have

$$\begin{aligned}G(x) + B(0, 7s) &= F(x) \cap B(g(x), 2\operatorname{dist}(g(x), F(x))) + B(0, 7s) \\ &\supset (F(x) + B(0, s)) \cap (B(g(x), 2\operatorname{dist}(g(x), F(x))) + B(0, s)).\end{aligned}$$

On the other hand, by (4.18)

$$G(y) \subset F(x) + B(0, 11\gamma\rho(x, y)) \subset F(x) + B(0, s),$$

and by (4.8) and (4.17)

$$\begin{aligned}G(y) &\subset B(g(y), 2\operatorname{dist}(g(y), F(y))) \\ &\subset B(g(x), 2\operatorname{dist}(g(x), F(x))) + (\gamma\rho(x, y) + 2 \cdot 12\gamma\rho(x, y))B(0, 1) \\ &= B(g(x), 2\operatorname{dist}(g(x), F(x))) + B(0, s).\end{aligned}$$

Hence

$$G(y) \subset G(x) + B(0, 7s).$$

Similarly we prove that $G(x) \subset G(y) + B(0, 7s)$ so that

$$\mathrm{d_H}(G(x), G(y)) \leqslant 7s = 175\gamma\rho(x, y).$$

Thus we have proved that the mapping $G : (\mathcal{M}, \rho) \to (\mathcal{K}(X), \mathrm{d_H})$ is Lipschitz continuous. This completes the proof of Theorem 1.1. □

5. Barycentric selectors and the centroid of a parallel body

Let X be a Minkowski space (i.e., *a finite-dimensional* Banach space) of dimension n. Given $a \in X,\ r \geqslant 0$ we let $B_X(a,r)$ denote a ball in X of radius r centered at a. We also denote $B_X := B_X(0,1)$. Following an idea of Aubin and Cellina in [AC, p. 77], we define a mapping

$$S^{(X)}(C) := b(C + B_X(0, \operatorname{diam} C)). \tag{5.1}$$

Proposition 5.1. *The mapping* $S^{(X)} : (\mathscr{K}(X), \mathrm{d_H}) \to X$ *is Lipschitz continuous with* $|S^{(X)}|_{\mathrm{Lip}(\mathscr{K}(X);X)} \leqslant \gamma(n)$. *Moreover,* $S^{(X)}$ *commutes with affine isometries and dilations of* X.

Proof. Let us estimate the regularity coefficient $\delta_{\tilde{C}}$ of the set

$$\widetilde{C} := C + B_X(0, \operatorname{diam} C).$$

Clearly, for every $a \in C$ we have

$$B_X(a, \operatorname{diam} C) \subset \widetilde{C} \subset B_X(a, 2 \operatorname{diam} C)$$

so that by (1.6) $\delta_{\tilde{C}} \leqslant 2^n$. Hence by Theorem 1.3 for every $C_1, C_2 \in \mathscr{K}(X)$ we have

$$\begin{aligned} \|S^{(X)}(C_1) - S^{(X)}(C_2)\| &= \|b(\widetilde{C}_1) - b(\widetilde{C}_2)\| \leqslant \gamma(n)(\delta_{\tilde{C}_1} + \delta_{\tilde{C}_2}) \mathrm{d_H}(\widetilde{C}_1, \widetilde{C}_2) \\ &\leqslant 2^{n+1} \gamma(n) \mathrm{d_H}(\widetilde{C}_1, \widetilde{C}_2). \end{aligned}$$

Since $|\operatorname{diam} C_1 - \operatorname{diam} C_2| \leqslant \mathrm{d_H}(C_1, C_2)$, we obtain

$$\mathrm{d_H}(\widetilde{C}_1, \widetilde{C}_2) \leqslant \mathrm{d_H}(C_1, C_2) + |\operatorname{diam} C_1 - \operatorname{diam} C_2| \leqslant 2\, \mathrm{d_H}(C_1, C_2).$$

Hence

$$\|S^{(X)}(C_1) - S^{(X)}(C_2)\| \leqslant 2^{n+2} \gamma(n) \mathrm{d_H}(C_1, C_2)$$

which proves that $S^{(X)}$ is Lipschitz continuous.

Since the barycentric map commutes with affine transformations of X, for every linear isometry A of X and every $C \in \mathscr{K}(X)$ we have $AS^{(X)}(C) = b(AC + (\operatorname{diam} C) AB_X)$. But $AB_X = B_X$ and $\operatorname{diam} C = \operatorname{diam} AC$ so that $AS^{(X)}(C) = S^{(X)}(AC)$. In the same way we prove that $S^{(X)}$ commutes with shifts and dilations of X. □

Theorem 5.2. *The mapping* $S^{(\mathbf{R}^n)}$ *provides a Lipschitz continuous selector on* $(\mathscr{K}(\mathbf{R}^n), \mathrm{d_H})$ *which commutes with affine isometries and dilations of* $\mathbf{R}^n$. *Its Lipschitz seminorm* $|S^{(\mathbf{R}^n)}|_{\mathrm{Lip}(\mathscr{K}(\mathbf{R}^n);\mathbf{R}^n)}$ *does not exceed a constant depending only on* n.

Proof. By Proposition 5.1 we only have to verify that $S^{(\mathbf{R}^n)}(C)$ is a selector. This property follows from a result of [AC, p. 78], which states that $b(C + B_{\mathbf{R}^n}) \in C$ for every $C \in \mathcal{K}(\mathbf{R}^n)$. Then by dilation we obtain $b(C + B_{\mathbf{R}^n}(0,r)) \in C,\ r \geqslant 0$ so that by (5.1) $S^{(\mathbf{R}^n)}(C) \in C$. $\square$

Let us consider the *selection* properties of $S^{(X)}$ for the case of an arbitrary Minkowski space X. Clearly, $S^{(X)}$ provides a selector on $\mathcal{K}(X)$ if and only if

$$b(C + B_X) \in C \quad \text{for every } C \in \mathcal{K}(X). \tag{5.2}$$

Following [BF, p. 58, Sc3, p. 306], we refer to the set $C + B_X$ as *a parallel body* and call $b(C + B_X)$ the *centroid of the parallel body*. Thus the following natural question arises:

Question 5.3. *Given a compact convex set $C \subset X$, does the centroid of the parallel body $C + B_X$ belong to C?*

As we have noted above the answer to this question is positive whenever $X = \mathbf{R}^n$. Moreover, the proof given in [AC] shows that $b(A + B_{\mathbf{R}^n}) \in \operatorname{conv}(A)$ for every bounded subset $A \subset \mathbf{R}^n$. However this proof cannot be adapted to the case of an arbitrary Minkowski space, because it is based on the following specific property of the ball $B_{\mathbf{R}^n}$: Let H be a hyperplane which cuts $B_{\mathbf{R}^n}$ into two sets $B^{(1)}$ and $B^{(2)}$ where $B^{(2)} \ni 0$. Then the mirror reflection of $B^{(1)}$ with respect to H is contained in $B^{(2)}$.

There is another approach to locating of the centroid of a parallel body in $\mathbf{R}^n$ due to Schneider [Sc2,Sc3, Section 5.4] (see also there references related to the case $n = 2$.) This approach leads us to a formula for the position of $b(C + B_{\mathbf{R}^n})$ with respect to the family $p_0(C), p_1(C), \dots, p_n(C)$ of so-called *curvature centroids* of C. (See formula (5.4.14) in [Sc3] for the case $r = 0$; in notations of Section 2, this result is also presented in [BZ, 25.7.5, p. 174].) We recall that $p_i(C) \in C,\ i = 0, \dots, n$ (see [Sc3, (5.4.11), p. 305]). Moreover, $p_0(C) = b(C)$ is the ordinary centroid of C, $p_1(C)$ is the *surface-area* centroid of C, and $p_n(C) = s(C)$ is the centroid of the Gaussian curvature measure, i.e., the Steiner point of C. Then, in the notation we adopted here, Schneider's formula can be written as

$$b(C + B_{\mathbf{R}^n}) = \sum_{i=0}^{n} \alpha_i p_i(C), \tag{5.3}$$

where

$$\alpha_i := \binom{n}{i} \frac{V(C[n-k], B_{\mathbf{R}^n}[k])}{\lambda(C + B_{\mathbf{R}^n})}.$$

(We recall that the mixed volumes $V(\cdot,\cdot)$ are defined by (2.1).) By formula (2.2) (for the case $Y = \mathbf{R}^n$) we have $\sum_{i=0}^{n} \alpha_i = 1$ so that $b(C + B_{\mathbf{R}^n})$ *is a convex combination of* $p_0(C), p_1(C), \dots, p_n(C)$. For more details on Eq. (5.3) we refer the reader to [Sc3, pp. 303–308], and references therein.

Of course, (5.3) again implies the property $b(C + B_{\mathbf{R}^n}) \in C$ for every compact convex $C \subset \mathbf{R}^n$. However further generalizations of the representation (5.3) to arbitrary Minkowski spaces seem to be unknown.

In a recent paper of Gaifullin [G] a complete description is obtained of those Minkowski spaces X for which

$$b(A + B_X) \in \operatorname{conv}(A) \quad \text{for every bounded } A \subset X. \tag{5.4}$$

It is proved that (5.4) is true for every *two-dimensional* Minkowski space X. In particular, this implies property (5.2) for such X and thus shows that $S^{(X)}$ is a Lipschitz continuous selector for every X of $\dim X = 2$.

Another result proved in [G] states that a Minkowski space X with $\dim X > 2$ satisfies (5.4) iff X is a *Euclidean space*, i.e., its unit ball B_X is an ellipsoid.

The paper [G] also contains an example of a (convex) triangle C in the space $X = l_1^3$ such that $b(C + B_X) \notin \operatorname{affspan}(C)$. This shows that in general the answer to Question 5.3 is negative whenever $\dim X > 2$.

These results and observations raise the following problem: describe the position, relative to $\operatorname{conv}(C)$, of the centroid of the parallel body $C + B_X$ in a Minkowski space X. In particular, it would be interesting to find an analog of the representation (5.3) for an *arbitrary* Minkowski space X.

6. Lipschitz projectors on finite-dimensional subspaces of a Banach space

We recall that $G(X)$ stands for the family of all finite-dimensional linear subspace of X equipped with the metric d_G, see (1.7). (For definition (1.7) and other facts concerning distances between subspaces of a Banach space we refer the reader to the paper [O] and references therein). Given a linear subspace $L \in G(X)$ we let $\mathscr{P}(L)$ denote the family of all *Lipschitz homogeneous projectors onto* L, i.e.,

$$\mathscr{P}(L) := \{P \in \operatorname{Lip}(X; L) : P|_L = \operatorname{Id}_L, P(\lambda x) = \lambda P(x) \quad \text{for every } x \in X, \lambda \in \mathbf{R}\}.$$

For a homogeneous operator $T: X \to X$ we denote its operator norm $\|T\|_{\text{op}}$ by letting

$$\|T\|_{\text{op}} := \sup_{x \in X} \frac{\|Tx\|}{\|x\|}.$$

It is well known that for every $L \in G(X)$ of dimension m there is a *linear* (and, consequently, Lipschitz) projector $\widetilde{\Pr}_L$ onto L such that $\|\widetilde{\Pr}_L\|_{\operatorname{Lip}(X;L)} \leqslant \gamma(m)$. This shows that $\mathscr{P}(L) \neq \emptyset$ for each $L \in G(X)$. In this section we consider the following problem: Can one assign a projector $P_L \in \mathscr{P}(L)$ to every $L \in G(X)$, such that

$||P_L||_{\mathrm{Lip}(X;L)} \leqslant \gamma_1(\dim L)$ and

$$||P_{L_1} - P_{L_2}||_{\mathrm{op}} \leqslant \gamma_2 \mathrm{d_G}(L_1, L_2), \quad L_1, L_2 \in G(X), \tag{6.1}$$

where γ_2 is a constant depending only on $\dim L_1$ and $\dim L_2$?

In particular, the *orthogonal projector* $P_L^{\perp}$ on a subspace $L \in G(H)$ solves this problem for the case of a Hilbert space H. Clearly, $||P_L^{\perp}||_{\mathrm{Lip}(H;L)} = 1$. Moreover, $||P_{L_1}^{\perp} - P_{L_2}^{\perp}||_{\mathrm{op}} = \Theta(L_1, L_2)$, see [O, p. 263], where $\Theta(L_1, L_2)$ denotes so-called *the geometric opening* between L_1 and L_2:

$$\Theta(L_1, L_2) := \max\left\{ \sup_{x' \in L_1 \cap B(0,1)} \mathrm{dist}(x', L_2), \sup_{x'' \in L_2 \cap B(0,1)} \mathrm{dist}(x'', L_1) \right\}.$$

Since $\Theta(L_1, L_2) \leqslant \mathrm{d_G}(L_1, L_2)$, the orthogonal projector $P_L^{\perp}$ satisfies (6.1) with $\gamma_2 = 1$.

The main result of this section, Theorem 6.1, shows that the mapping

$$P_L(x) := b(L \cap B(x, 2\ \mathrm{dist}(x, L))) \tag{6.2}$$

provides the required projector for an *arbitrary* Banach space X. (We note that P_L coincides with the orthogonal projector on L whenever X is a Hilbert space. Moreover, P_L is linear if X is a two-dimensional Minkowski space.)

Theorem 6.1. *For every $L \in G(X)$ the mapping $P_L : X \to L$ is a homogeneous projector onto L satisfying the following conditions*:

(i) $||P_L||_{\mathrm{Lip}(X;L)} \leqslant \gamma_1$ *where γ_1 is a constant depending only on dimension of L*;

(ii) *for every $L_1, L_2 \in G(X)$*

$$||P_{L_1} - P_{L_2}||_{\mathrm{op}} \leqslant \gamma_2 \mathrm{d_G}(L_1, L_2), \tag{6.3}$$

where γ_2 depends only on dimensions of L_1 and L_2.

Moreover, P_L is additive modulo L, i.e., $P_L(x + y) = P_L(x) + y$ for every $x \in X$ and $y \in L$.

Proof. Clearly, formula (6.2) implies homogeneity and additivity (modulo L) of P_L.

Let us prove properties (i) and (ii). To this end given $x \in X$ and a subspace $L \in G(X)$ we denote $C(x; L) := L \cap B(x, 2\ \mathrm{dist}(x, L))$. Then for every $y \in C(x; L)$ we have

$$||y|| \leqslant ||y - x|| + ||x|| \leqslant 2\ \mathrm{dist}(x, L) + ||x|| \leqslant 3||x||$$

so that for every $r \geqslant 3||x||$

$$C(x; L) = [B(0, r) \cap L] \cap B(x, 2\ \mathrm{dist}(x, L)).$$

Now we fix subspaces L_i of dimension m_i and points $x_i \in X$, $i = 1, 2$ and put $r := 3(\|x_1\| + \|x_2\|)$. Then by Lemma 3.9

$$\begin{aligned}
&d_H(C(x; L_1), C(y; L_2)) \\
&\quad \leqslant 14(d_H(B(0,r) \cap L_1, B(0,r) \cap L_2) + \|x_1 - x_2\| + 2|\operatorname{dist}(x_1, L_1) - \operatorname{dist}(x_2, L_2)|).
\end{aligned}$$

Since

$$|\operatorname{dist}(x_1, L_1) - \operatorname{dist}(x_2, L_1)| \leqslant \|x_1 - x_2\|,$$

we have

$$\begin{aligned}
|\operatorname{dist}(x_1, L_1) - \operatorname{dist}(x_2, L_2)| &\leqslant |\operatorname{dist}(x_1, L_1) - \operatorname{dist}(x_2, L_1)| \\
&\quad + |\operatorname{dist}(x_2, L_1) - \operatorname{dist}(x_2, L_2)| \\
&\leqslant \|x_1 - x_2\| + |\operatorname{dist}(x_2, L_1) - \operatorname{dist}(x_2, L_2)|.
\end{aligned}$$

On the other hand,

$$|\operatorname{dist}(x_2, L_1) - \operatorname{dist}(x_2, L_2)| = \|x_2\| \, |\operatorname{dist}(a, L_1) - \operatorname{dist}(a, L_2)|,$$

where $a := x_2/\|x_2\|$. We assume that $\operatorname{dist}(a, L_1) \leqslant \operatorname{dist}(a, L_2)$ and denote by b a point of L_1 such that $\operatorname{dist}(a, L_1) = \|a - b\|$. Then

$$|\operatorname{dist}(a, L_2) - \operatorname{dist}(a, L_1)| = \operatorname{dist}(a, L_2) - \|a - b\| \leqslant \operatorname{dist}(b, L_2).$$

But $\|b\| \leqslant \|a\| + \operatorname{dist}(a, L_1) \leqslant 2$ so that

$$\operatorname{dist}(b, L_2) = 2 \operatorname{dist}(b/2, L_2) \leqslant 2\, d_G(L_1, L_2).$$

Hence

$$|\operatorname{dist}(x_1, L_1) - \operatorname{dist}(x_2, L_2) \leqslant \|x_1 - x_2\| + 2\|x_2\| d_G(L_1, L_2).$$

In turn, by (1.7)

$$d_H(B(0,r) \cap L_1, B(0,r) \cap L_2) = r\, d_G(L_1, L_2) = 3(\|x_1\| + \|x_2\|) d_G(L_1, L_2).$$

Combining these estimates, we obtain

$$d_H(C(x; L_1), C(y; L_2)) \leqslant 98((\|x_1\| + \|x_2\|) d_G(L_1, L_2) + \|x_1 - x_2\|).$$

It remains to note that by (2.16) $\delta_{C(x;L)} \leqslant 7^{\dim L}$ so that by Theorem 1.3

$$\begin{aligned} ||P_{L_1}(x_1) - P_{L_2}(x_2)|| &= ||b(C(x_1;L_1)) - b(C(x_2;L_2))|| \\ &\leqslant \gamma(m_1,m_2)(\delta_{C(x_1;L_1)} + \delta_{C(x_2;L_2)})\mathrm{d_H}(C(x_1;L_1),C(x_2;L_2)) \\ &\leqslant \gamma(m_1,m_2)(7^{m_1} + 7^{m_2})\mathrm{d_H}(C(x_1;L_1),C(x_2;L_2)). \end{aligned}$$

Finally, we obtain

$$||P_{L_1}(x_1) - P_{L_2}(x_2)|| \leqslant \tilde{\gamma}((||x_1|| + ||x_2||)\mathrm{d_G}(L_1,L_2) + ||x_1 - x_2||)$$

with $\tilde{\gamma} := 98\gamma(m_1,m_2)(7^{m_1} + 7^{m_2})$. This inequality implies (i) $L_1 = L_2 = L$ and (ii), if $x_1 = x_2 = x$. $\square$

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