EXTENSIONS OF NUNOKAWA LEMMA FOR ARGUMENT PROPERTIES

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ABSTRACT. Let $\mathcal{H}[a_0, n]$ be the class of functions $p(z) = a_0 + a_n z^n + \cdots$ which are analytic in the open unit disk $\mathbb{U}$. For $p(z) \in \mathcal{H}[1, 2]$, M. Nunokawa, S. Owa, N. Uyanik and H. Shiraishi (Math. Comput. Modelling. **55** (2012), 1245–1250) have shown some theorems for argument properties. The object of the present paper is to discuss some extensions of Nnnokawa lemma and its applications for argument properties.

1. INTRODUCTION

Let $\mathcal{H}[a_0, n]$ denote the class of functions $p(z)$ of the form

$$p(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ for some $a_0 \in \mathbb{C}$ and a positive integer n. If $p(z) \in \mathcal{H}[a_0, n]$ satisfies

$$|\arg(p(z))| < \frac{\pi}{2}\mu \qquad (z \in \mathbb{U})$$

for some real $0 < \mu \leqq 1$, then we say that $p(z)$ belongs to the class $\mathcal{STH}[a_0, n](\mu)$.

The basic tool in proving our results is the following lemma due to S. S. Miller and P. T. Mocanu [1] (also [2]).

Lemma 1. *Let the function $w(z)$ definded by*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots \qquad (n = 1, 2, 3, \cdots)$$

be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $m \geqq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = m.$$

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2. Main results

Applying Lemma 1, we derive the following result.

Theorem 1. *Let $p(z) \in \mathcal{H}[a_0, n]$ for some real $a_0 > 0$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\mathrm{Re}(p(z)) > 0 \quad for \quad |z| < |z_0|$$

and $p(z_0) = \beta i$ is a pure imaginary number for some real $\beta \neq 0$.

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = il$$

where

$$l \geqq \frac{n}{2}\left(\frac{a_0}{\beta} + \frac{\beta}{a_0}\right) \geqq n$$

if $\beta > 0$ and

$$l \leqq \frac{n}{2}\left(\frac{a_0}{\beta} + \frac{\beta}{a_0}\right) \leqq -n$$

if $\beta < 0$.

Proof. Let us put

$$w(z) = \frac{a_0 - p(z)}{a_0 + p(z)} = c_n z^n + c_{n+1} z^{n+1} + c_{n+2} z^{n+2} + \cdots \qquad (z \in \mathbb{U}).$$

Then, we have that $w(z)$ is analytic in $|z| < |z_0|$, $w(0) = 0$, $|w(z)| < 1$ for $|z| < |z_0|$ and

$$|w(z_0)| = \left| \frac{a_0^2 - \beta^2 - 2a_0\beta i}{a_0^2 + \beta^2} \right| = 1.$$

From Lemma 1, we obtain

$$\frac{z_0 w'(z_0)}{w(z_0)} = \frac{-2a_0 z_0 p'(z_0)}{a_0^2 - \{p(z_0)\}^2} = \frac{-2a_0 z_0 p'(z_0)}{a_0^2 + \beta^2} = m \qquad (m \geqq n).$$

This shows that

$$z_0 p'(z_0) = -\frac{m}{2}\left(a_0 + \frac{\beta^2}{a_0}\right) \qquad (m \geqq n).$$

From the fact that $z_0 p'(z_0)$ is a real number and $p(z_0)$ is a pure imaginary number, we can put

$$\frac{z_0 p'(z_0)}{p(z_0)} = il$$

where l is a real number.

For the case $\beta > 0$, we have

$$\begin{aligned} l &= \operatorname{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= \operatorname{Im}\left(-z_0 p'(z_0)\frac{1}{\beta}i\right) \\ &= \frac{m}{2}\left(a_0 + \frac{\beta^2}{a_0}\right)\frac{1}{\beta} \\ &\geqq \frac{n}{2}\left(a_0 + \frac{\beta^2}{a_0}\right)\frac{1}{\beta} \\ &= \frac{n}{2}\left(\frac{a_0}{\beta} + \frac{\beta}{a_0}\right) \geqq n \end{aligned}$$

and for the case $\beta < 0$, we get

$$\begin{aligned} l &= \operatorname{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= \operatorname{Im}\left(-z_0 p'(z_0)\frac{1}{\beta}i\right) \\ &= \frac{m}{2}\left(a_0 + \frac{\beta^2}{a_0}\right)\frac{1}{\beta} \\ &\leqq \frac{n}{2}\left(a_0 + \frac{\beta^2}{a_0}\right)\frac{1}{\beta} \\ &= \frac{n}{2}\left(\frac{a_0}{\beta} + \frac{\beta}{a_0}\right) \leqq -n. \end{aligned}$$

This completes our proof. □

Putting $a_0 = 1$ in Theorem 1, we have Corollary 1.

Corollary 1. *Let $p(z) \in \mathcal{H}[1,n]$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\operatorname{Re}(p(z)) > 0 \quad for \quad |z| < |z_0|,$$

$\operatorname{Re}(p(z_0)) = 0$ *and* $p(z_0) \neq 0$.

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = il$$

where l is a real and $|l| \geqq n$.

From Theorem 1, we get Theorem 2.

Theorem 2. *Let $p(z) \in \mathcal{H}[a_0,n]$ for some real $a_0 < 0$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\operatorname{Re}(p(z)) < 0 \quad for \quad |z| < |z_0|$$

and $p(z_0) = \beta i$ is a pure imagenary number for some real $\beta \neq 0$.

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = il$$

where

$$l \geqq \frac{n}{2}\left(\frac{a_0}{\beta} + \frac{\beta}{a_0}\right) \geqq n$$

if $\beta < 0$ *and*

$$l \leqq \frac{n}{2}\left(\frac{a_0}{\beta} + \frac{\beta}{a_0}\right) \leqq -n$$

if $\beta > 0$.

Proof. We put the function

$$q(z) = -p(z) \qquad (z \in \mathbb{U}),$$

$q(z)$ satisfies the assumption of Theorem 1 and using Theorem 1, we get the result of Theorem 2. □

3. Applications of Theorem 1

Using Theorem 1, we obtain following result.

Theorem 3. *If* $p(z) \in \mathcal{H}[a_0, n]$ *for some real* $a_0 > 0$ *and a positive integer* $n \geqq 2$ *satisfies* $p(z) \neq 0$ *for* $z \in \mathbb{U}$ *and*

$$|\arg(p(z) - zp'(z))| < \arctan(n\mu) - \frac{\pi}{2}\mu \qquad (z \in \mathbb{U})$$

for some real number $0 < \mu \leqq \sqrt{\dfrac{2n-\pi}{n^2\pi}} < 1$*, then* $p(z) \in \mathcal{STH}[a_0, n](\mu)$.

Proof. Let us consider

$$q(z) = (p(z))^{\frac{1}{\mu}} = a_0^{\frac{1}{\mu}} + c_n z^n + c_{n+1} z^{n+1} + \cdots \qquad (z \in \mathbb{U})$$

suppose that $p(z)$ satisfies

$$|\arg(p(z))| < \frac{\pi}{2}\mu \qquad (|z| < |z_0|)$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\mu \qquad (z_0 \in \mathbb{U}).$$

Then, the function $q(z)$ satisfies

$$\mathrm{Re}(q(z)) > 0 \qquad (|z| < |z_0|)$$

and $\mathrm{Re}(q(z_0)) = 0$ with $q(z_0) \neq 0$.

Applying Theorem 1, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{1}{\mu}\frac{z_0 p'(z_0)}{p(z_0)} = il \tag{1}$$

where

$$l \geqq \frac{n}{2}\left(\frac{\beta}{a_0^{\frac{1}{\mu}}}+\frac{a_0^{\frac{1}{\mu}}}{\beta}\right) \geqq n \qquad (q(z_0)=i\beta) \tag{2}$$

and

$$l \leqq -\frac{n}{2}\left(\frac{\beta}{a_0^{\frac{1}{\mu}}}+\frac{a_0^{\frac{1}{\mu}}}{\beta}\right) \leqq -n \qquad (q(z_0)=-i\beta) \tag{3}$$

for $\beta > 0$ which satisfies

$$q(z_0) = (p(z_0))^{\frac{1}{\mu}} = \pm i\beta.$$

For such $p(z)$, if $\arg(p(z_0)) = \frac{\pi}{2}\mu$, we have that

$$\begin{aligned}
\arg(p(z_0) - z_0p'(z_0)) &= \arg\left(p(z_0)\left(1-\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\\
&= \frac{\pi}{2}\mu + \arg(1-il\mu)\\
&\leqq \frac{\pi}{2}\mu + \arg(1-in\mu)\\
&= \frac{\pi}{2}\mu - \arctan(n\mu)\\
&= -\left(\arctan(n\mu) - \frac{\pi}{2}\mu\right)
\end{aligned}$$

which contradicts the condition in the theorem.

If $\arg(p(z_0)) = -\frac{\pi}{2}\mu$, we get

$$\begin{aligned}
\arg(p(z_0) - z_0p'(z_0)) &= \arg(1-il\mu) - \frac{\pi}{2}\mu\\
&\geqq \arg(1+in\mu) - \frac{\pi}{2}\mu\\
&= \arctan(n\mu) - \frac{\pi}{2}\mu
\end{aligned}$$

which contradicts the condition in the theorem.

This implies that there is no $z_0 \in \mathbb{U}$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\mu \qquad (|z| < |z_0|)$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\mu.$$

Thus $p(z)$ satisfies

$$|\arg(p(z))| < \frac{\pi}{2}\mu$$

for all $z \in \mathbb{U}$, that is, $p(z) \in \mathcal{STH}[a_0, n](\mu)$. □

Remark 1. Consider the function

$$g(x) = \arctan(nx) - \frac{\pi}{2}x$$

for some real number $0 < x \leqq \sqrt{\dfrac{2n-\pi}{n^2\pi}} \leqq \sqrt{\dfrac{4-\pi}{4\pi}}$ and some positive integer $n \geqq 2$, we get

$$g'(x) = \frac{2n-\pi-n^2\pi x^2}{2(1+n^2x^2)} \geqq 0 \qquad \left(0 < x \leqq \sqrt{\frac{2n-\pi}{n^2\pi}}\right).$$

From $g(0) = 0$, we know that $g(z)$ is a simple increasing function and $g(x) > 0$.

When $n = 2$ and $a_0 = 1$ in Theorem 3, we have Corollary 2 due to M. Nunokawa, S. Owa, N. Uyanik and H. Shiraishi [5].

Corollary 2. *If $p(z) \in \mathcal{H}[1,2]$ satisfies $p(z) \neq 0$ for $z \in \mathbb{U}$ and*

$$|\arg(p(z) - zp'(z))| < \arctan(2\mu) - \frac{\pi}{2}\mu \qquad (z \in \mathbb{U})$$

for some real number $0 < \mu \leqq \sqrt{\dfrac{4-\pi}{4\pi}}$, then $p(z) \in \mathcal{STH}[1,2](\mu)$.

Also, using Theorem 1, we obtain the following theorem.

Theorem 4. *If $p(z) \in \mathcal{H}[a_0,n]$ for some real $a_0 > 0$ and a positive integer $n \geqq 1$ satisfies $p(z) \neq 0$ for $z \in \mathbb{U}$ and*

$$\left|\arg\left(p(z) - \frac{zp'(z)}{p(z)}\right)\right| < \arctan\left(\frac{\rho(\mu)\cos\frac{\pi\mu}{2}}{1-\rho(\mu)\sin\frac{\pi\mu}{2}}\right) - \frac{\pi}{2}\mu \qquad (z \in \mathbb{U}) \tag{4}$$

for some real number μ $(0 < \mu < \mu_0)$, where

$$\rho(\mu) = \frac{n\mu}{2a_0}\left(\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1-\mu}{2}} + \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}\right) \tag{5}$$

and some real number $0 < \mu_0 < 1$ satisfies

$$\rho(\mu_0) = \sin\frac{\pi\mu_0}{2},$$

then $p(z) \in \mathcal{STH}[a_0,n](\mu)$.

Proof. The condition (4) implies, in particular, that $p(z) \neq 0$ for $z \in \mathbb{U}$. We consider that there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\mu \qquad (|z| < |z_0|)$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\mu.$$

By using the same prosess of Theorem 3, we obtain the equation (1) and we can write

$$\frac{z_0p'(z_0)}{p(z_0)} = i\mu l,$$

where l is a real number which satisfies the equation (2) and (3).

For the case $\arg(p(z_0)) = -\dfrac{\pi}{2}\mu$, applying the same method as the proof of Theorem 3, we have that

$$\begin{aligned}
&\arg\left(p(z_0) - \frac{z_0p'(z_0)}{p(z_0)}\right)\\
&= \arg(p(z_0)) + \arg\left(1 - \frac{z_0p'(z_0)}{p(z_0)}\frac{1}{p(z_0)}\right)\\
&= -\frac{\pi}{2}\mu + \arg\left(1 - \frac{i\mu l}{(-i\beta)^{\mu}}\right)\\
&= -\frac{\pi}{2}\mu + \arg\left(1 - e^{i\frac{1+\mu}{2}\pi}\frac{\mu}{\beta^{\mu}}l\right)\\
&\geqq -\frac{\pi}{2}\mu + \arg\left(1 + e^{i\frac{1+\mu}{2}\pi}\frac{n\mu}{2\beta^{\mu}}\left(\frac{\beta}{a_0^{\frac{1}{\mu}}} + \frac{a_0^{\frac{1}{\mu}}}{\beta}\right)\right)\\
&= -\frac{\pi}{2}\mu + \arg\left(1 + e^{i\frac{1+\mu}{2}\pi}\frac{n\mu}{2a_0}\left(\left(\frac{\beta}{a_0^{\frac{1}{\mu}}}\right)^{1-\mu} + \left(\frac{\beta}{a_0^{\frac{1}{\mu}}}\right)^{-1-\mu}\right)\right).
\end{aligned}$$

On the other hand, let us put

$$g(x) = x^{1-\mu} + x^{-1-\mu} \qquad \left(x = \frac{\beta}{a_0^{\frac{1}{\mu}}} > 0\right).$$

Then, by easy calculation, we have

$$g'(x) = (1-\mu)x^{-\mu} + (-1-\mu)x^{-2-\mu} \qquad (x > 0)$$

and $g(x)$ takes the minimum value at

$$x = \sqrt{\frac{1+\mu}{1-\mu}}.$$

Therefore, we have

$$\begin{aligned}
&\arg\left(p(z_0) - \frac{z_0p'(z_0)}{p(z_0)}\right)\\
&\geqq -\frac{\pi}{2}\mu + \arg\left(1 + e^{i\frac{1+\mu}{2}\pi}\frac{n\mu}{2a_0}\left(\left(\frac{\beta}{a_0^{\frac{1}{\mu}}}\right)^{1-\mu} + \left(\frac{\beta}{a_0^{\frac{1}{\mu}}}\right)^{-1-\mu}\right)\right)\\
&\geqq -\frac{\pi}{2}\mu + \arg\left(1 + e^{i\frac{1+\mu}{2}\pi}\frac{n\mu}{2a_0}\left(\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1-\mu}{2}} + \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}\right)\right)\\
&= \arctan\left(\frac{\rho(\mu)\sin\left(\frac{1+\mu}{2}\pi\right)}{1 + \rho(\mu)\cos\left(\frac{1+\mu}{2}\pi\right)}\right) - \frac{\pi}{2}\mu\\
&= \arctan\left(\frac{\rho(\mu)\cos\frac{\pi\mu}{2}}{1 - \rho(\mu)\sin\frac{\pi\mu}{2}}\right) - \frac{\pi}{2}\mu
\end{aligned}$$

for the equation (5), which contradicts the condition (4).

If $\arg(p(z_0)) = \frac{\pi}{2}\mu$, then we see that

$$\begin{aligned} &\arg\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= \frac{\pi}{2}\mu + \arg\left(1 - \frac{i\mu l}{(i\beta)^{\mu}}\right) \\ &\leqq \frac{\pi}{2}\mu - \arctan\left(\frac{\rho(\mu)\sin\left(\frac{1+\mu}{2}\pi\right)}{1+\rho(\mu)\cos\left(\frac{1+\mu}{2}\pi\right)}\right) \\ &= \frac{\pi}{2}\mu - \arctan\left(\frac{\rho(\mu)\cos\frac{\pi\mu}{2}}{1-\rho(\mu)\sin\frac{\pi\mu}{2}}\right) \end{aligned}$$

for the equation (5), which also contradicts the condition (4).

This shows that there is no $z_0 \in \mathbb{U}$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\mu \qquad (|z| < |z_0|)$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\mu.$$

Therefore, $p(z_0)$ satisfies $|\arg(p(z))| < \frac{\pi}{2}\mu$ for all $z \in \mathbb{U}$, completing the proof. □

Remark 2. Let the function

$$g(x) = \arctan\left(\frac{\rho(x)\cos\frac{\pi x}{2}}{1-\rho(x)\sin\frac{\pi x}{2}}\right) - \frac{\pi}{2}x$$

where $\rho(x)$ satisfies (5) for some positive integer $n \geqq 2$ and some real $a_0 > 0$.

We know $g(x) > 0$ for all x $(0 < x \leqq \mu_0)$ where some real number $0 < \mu_0 < 1$ satisfies

$$\rho(\mu_0) = \sin\frac{\pi\mu_0}{2}.$$

Because, we have the following figure for the function $g(x)$.

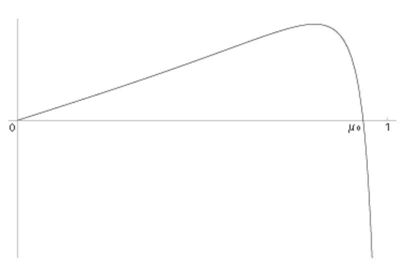

FIGURE 1. The image of $g(x)$.

Considering $n = 2$ and $a_0 = 1$ in Theorem 4, we have the following corollary due to M. Nunokawa, S. Owa, N. Uyanik and H. Shiraishi [5].

Corollary 3. *If $p(z) \in \mathcal{H}[1,2]$ satisfies $p(z) \neq 0$ for $z \in \mathbb{U}$ and*

$$\left|\arg\left(p(z) - \frac{zp'(z)}{p(z)}\right)\right| < \arctan\left(\frac{\rho(\mu)\cos\frac{\pi\mu}{2}}{1-\rho(\mu)\sin\frac{\pi\mu}{2}}\right) - \frac{\pi}{2}\mu$$

$$= \arctan\left(\frac{\rho(\mu)\sin\left(\frac{1+\mu}{2}\pi\right)}{1+\rho(\mu)\cos\left(\frac{1+\mu}{2}\pi\right)}\right) - \frac{\pi}{2}\mu \qquad (z \in \mathbb{U})$$

for some real number μ $(0 < \mu < \mu_0)$, where

$$\rho(\mu) = \mu\left(\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1-\mu}{2}} + \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}\right)$$

and some real number $0 < \mu_0 < 1$ satisfies

$$\rho(\mu_0) = \sin\frac{\pi\mu_0}{2},$$

then $p(z) \in \mathcal{STH}[1,2](\mu)$.

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