NEW EXTENSIONS FOR A THEOREM BY MOCANU

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ABSTRACT. For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ with $f(0) = f'(0) - 1 = f''(0) = 0$, P. T. Mocanu (Mathematica (Cluj), **42**(2000)) has considered some sufficient arguments of $f'(z)+zf''(z)$ for $|\arg(zf'(z)/f(z))| < \pi\mu/2$. The object of the present paper is to discuss those probrems for $f(z)$ with $f''(0) = f'''(0) = \ldots = f^{(n)}(0) = 0$ and $f^{(n+1)}(0) \neq 0$.

1. Introduction

Let $\mathcal{A}_n$ denote the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots \qquad (n = 1, 2, 3, \ldots)$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A} = \mathcal{A}_1$.

Also, let $\mathcal{H}[1, n]$ denote the class of functions $f(z)$ of the form

$$f(z) = 1 + \sum_{k=n}^{\infty} a_k z^k \qquad (n = 1, 2, 3, \ldots)$$

which are analytic in $\mathbb{U}$.

Further, let the class $\mathcal{STS}(\mu)$ of $f(z) \in \mathcal{A}_n$ be defined by

$$\mathcal{STS}(\mu) = \left\{ f(z) \in \mathcal{A}_n : \left| \arg\left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\pi}{2}\mu,\ 0 < \mu \leqq 1 \right\}$$

and $\mathcal{S}^* = \mathcal{STS}(1)$. This class $\mathcal{STS}(\mu)$ was considered by Shiraishi and Owa [4].

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisfying $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$ and $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

2. Lemmas

We need the following lemmas to consider our main results.

Lemma 1. *Let n be a positive integer, $\lambda > 0$, and let $\beta_0 = \beta_0(\lambda, n)$ be the root of the equation*

$$\beta\pi = \frac{3}{2}\pi - \arctan(n\lambda\beta).$$

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In addition, let

$$\alpha = \alpha(\beta, \lambda, n) = \beta + \frac{2}{\pi}\arctan(n\lambda\beta)$$

for $0 < \beta \leqq \beta_0$.

If $p(z) \in \mathcal{H}[1, n]$ *and*

$$p(z) + \lambda z p'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (z \in \mathbb{U}),$$

then

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta} \qquad (z \in \mathbb{U}).$$

Lemma 2. *Let* $q(z)$ *be the convex function in* $\mathbb{U}$, *with* $q(0) = 1$ *and* $\mathrm{Re}(q(z)) > 0$ *for* $\mathbb{U}$. *Let the function* $h(z)$ *be given by*

$$h(z) = (q(z))^2 + nzq'(z) \qquad (z \in \mathbb{U}).$$

If $p(z) \in \mathcal{H}[1, n]$ *and*

$$(p(z))^2 + zp'(z) \prec h(z) \qquad (z \in \mathbb{U}),$$

then $p(z) \prec q(z)$ *and this is sharp.*

The above lemmas were given by Mocanu [3].

Applying Lemma 2, we obtain the following lemma.

Lemma 3. *If* $p(z) \in \mathcal{H}[1, n]$ *satisfies*

$$|\arg((p(z))^2 + zp'(z))| < \phi(\mu) \qquad (z \in \mathbb{U})$$

for

$$\phi(\mu) = \frac{\pi}{2}(\mu + 1) - \arctan\frac{\cos\frac{\mu\pi}{2}}{\sin\frac{\mu\pi}{2} + \frac{n\mu}{1-\mu}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}}$$

and $0 < \mu \leqq 1$, *then*

$$|\arg(p(z))| < \frac{\pi}{2}\mu \qquad (z \in \mathbb{U}).$$

Proof. Let us define the function $q(z)$ by

$$q(z) = \left(\frac{1+z}{1-z}\right)^{\mu} \qquad (z \in \mathbb{U}) \tag{1}$$

for $0 < \mu \leqq 1$ and the function $h(z)$ by

$$h(z) = (q(z))^2 + nzq'(z) \qquad (z \in \mathbb{U}).$$

Then the function $q(z)$ is convex in $\mathbb{U}$ with $\mathrm{Re}(q(z)) > 0$, $h(z)$ is univalent in $\mathbb{U}$ and $h(\mathbb{U})$ is the symmetric domain with respect to the real axis.

If we set $z = \exp(i\theta)$, $0 \leqq \theta < \pi$ and $x = \cot\frac{\theta}{2}$, then $x \geqq 0$, $z = \frac{ix-1}{ix+1}$ and $q(z) = (ix)^{\mu}$. Hence

$$h(e^{i\theta}) = (ix)^{\mu-1}H(x),$$

where

$$H(x) = (ix)^{\mu+1} - \frac{n}{2}\mu(1+x^2).$$

Noting that $\cos\frac{(\mu+1)\pi}{2} = -\sin\frac{\mu\pi}{2}$ and $\sin\frac{(\mu+1)\pi}{2} = \cos\frac{\mu\pi}{2}$, we see that

$$H(x) = P(x) + iQ(x),$$

where

$$\begin{cases} P(x) = -\sin\frac{\mu\pi}{2}x^{\mu+1} - \frac{n}{2}\mu(1+x^2) \\ Q(x) = \cos\frac{\mu\pi}{2}x^{\mu+1}. \end{cases}$$

Let

$$\varphi(\mu) = \min\{\arg(H(x)) : x \geqq 0\}$$

and

$$\phi(\mu) = \varphi(\mu) + \frac{\pi}{2}(\mu - 1). \tag{2}$$

From (1) we deduce

$$\arg(h(e^{i\theta})) \geqq \phi(\mu).$$

Since

$$G(x) = Q'(x)P(x) - P'(x)Q(x) = \frac{n}{2}\mu\cos\frac{\mu\pi}{2}x^{\mu}((1-\mu)x^2 - (1+\mu)) = 0$$

has the root $x_0 = \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}}$ and

$$\frac{Q(x_0)}{P(x_0)} = -\frac{\cos\frac{\mu\pi}{2}}{\sin\frac{\mu\pi}{2} + \frac{n\mu}{1-\mu}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}},$$

we deduce

$$\varphi(\mu) = \pi - \arctan\frac{\cos\frac{\mu\pi}{2}}{\sin\frac{\mu\pi}{2} + \frac{n\mu}{1-\mu}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}}. \tag{3}$$

Hence

$$|\arg(h(e^{i\theta}))| \geqq \phi(\mu) \qquad (-\pi < \theta < \pi)$$

where $\phi(\mu)$ is given by (2) and (3).

From the assumption,

$$(p(z))^2 + zp'(z) \prec h(z) \qquad (z \in \mathbb{U}).$$

Hence by Lemma 2 we deduce $p(z) \prec q(z)$.

So, we obtain

$$|\arg(p(z))| < \frac{\pi}{2}\mu \qquad (z \in \mathbb{U}).$$

□

3. Main results

Using Lemma 1 and Lemma 3, we get the following result.

Theorem 1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$|\arg(f'(z) + zf''(z))| < \frac{\pi}{2}\alpha \qquad (z \in \mathbb{U})$$

for

$$\alpha = \beta + \frac{2}{\pi}\arctan(n\beta),$$
$$\beta = \gamma + \frac{2}{\pi}\arctan(n\gamma),$$
$$\frac{\pi}{2}(\alpha + \gamma) \leqq \phi(\mu)$$

and

$$\phi(\mu) = \frac{\pi}{2}(\mu + 1) - \arctan \frac{\cos\frac{\mu\pi}{2}}{\sin\frac{\mu\pi}{2} + \frac{n\mu}{1-\mu}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}}$$

with some real $\alpha, \gamma > 0$ and $0 < \mu \leqq 1$, then $f(z) \in \mathcal{STS}(\mu)$.

Proof. By using Lemma 1 with $\lambda = 1$ we deduce

$$|\arg(f'(z))| < \frac{\pi}{2}\beta \qquad (z \in \mathbb{U}).$$

with

$$\alpha = \beta + \frac{2}{\pi}\arctan(n\beta).$$

Using again Lemma 1, we get

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| < \frac{\pi}{2}\gamma \qquad (z \in \mathbb{U}),$$

where γ is the solution of the equation

$$\beta = \gamma + \frac{2}{\pi}\arctan(n\gamma).$$

If we set $p(z) = \dfrac{zf'(z)}{f(z)}$ and $P(z) = \dfrac{f(z)}{z}$, then we have $p(z) \in \mathcal{H}[1, n]$ and

$$f'(z) + zf''(z) = P(z)((p(z))^2 + zp'(z)) \qquad (z \in \mathbb{U}),$$

where

$$|\arg(P(z))| < \frac{\pi}{2}\gamma \qquad (z \in \mathbb{U}).$$

It follows that

$$|\arg((p(z))^2 + zp'(z))| \leqq |\arg(f'(z) + zf''(z))| + |\arg(P(z))| < \frac{\pi}{2}(\alpha + \gamma).$$

For the condition of $\phi(\mu)$, we deduce that

$$|\arg((p(z))^2 + zp'(z))| < \phi(\mu) \qquad (z \in \mathbb{U})$$

implies by means of Lemma 3.

$$|\arg(p(z))| < \frac{\pi}{2}\mu \qquad (z \in \mathbb{U}).$$

□

We consider an example for Theorem 1.

Example 1. Let us consider the function

$$f(z) = z + \sin\frac{\pi\alpha}{2} z^{n+1} \qquad (z \in \mathbb{U})$$

with $0 < \alpha \leqq 1$. If we put

$$\mu = \frac{2}{\pi}\arcsin\frac{n(n+1)\sin\frac{\pi\alpha}{2}}{(n+1)^3 - \sin^2\frac{\pi\alpha}{2}},$$

the function $f(z)$ satisfies the condition of Theorem 1.

Because differenciating the function $f(z)$, we obtain

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{(n+1)(n+1+\sin\frac{\pi\alpha}{2}z^n)}{(n+1)^2 + \sin\frac{\pi\alpha}{2}z^n} \\ &= n+1-\frac{n}{1+\frac{\sin\frac{\pi\alpha}{2}}{(n+1)^2}z^n} \qquad (z \in \mathbb{U})\end{aligned}$$

and therefore,

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \arcsin\frac{n(n+1)\sin\frac{\pi\alpha}{2}}{(n+1)^3 - \sin^2\frac{\pi\alpha}{2}} \qquad (z \in \mathbb{U}).$$

If we fix one of the values for α, β or γ in Theorem 1, then we can obtain others. For example, if we put $n = 2$, $\alpha = 1$ and $\mu = \frac{1}{2}$, then we get $\beta = \frac{1}{2}$, $\gamma = 0.227\ldots$ and the following result due to Mocanu [3].

Corollary 1. *If $f(z) \in \mathcal{A}_2$ satisfies*

$$\operatorname{Re}(f'(z) + zf''(z)) > 0 \qquad (z \in \mathbb{U}),$$

then

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{4} \qquad (z \in \mathbb{U}).$$

Moreover, putting $n = 2$, $\alpha = \frac{3}{2}$ and $\mu = 1$, we have Corollary 2 given by Mocanu [3].

Corollary 2. *If $f(z) \in \mathcal{A}_2$ satisfies*

$$|\arg(f'(z) + zf''(z))| < \frac{3}{4}\pi \qquad (z \in \mathbb{U}),$$

then

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

Futhermore, putting $n = 1$, we derive Corollary 3 and Corollary 4 which were showed by Mocanu [2].

Corollary 3. *If $f(z) \in \mathcal{A}$ satisfies*

$$\mathrm{Re}(f'(z) + zf''(z)) > 0 \qquad (z \in \mathbb{U}),$$

then

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{3} \qquad (z \in \mathbb{U}).$$

Corollary 4. *If $f(z) \in \mathcal{A}$ satisfies*

$$|\arg(f'(z) + zf''(z))| < \frac{2}{3}\pi \qquad (z \in \mathbb{U}),$$

then

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

4. Integral version of the results

Let us define the function

$$F(z) = \int_0^z \frac{f(t)}{t} dt \qquad (z \in \mathbb{U})$$

for $f(z) \in \mathcal{A}_n$. This integral operator $F(z)$ is given by Alexander [1] and is said to be Alexander integral operator.

For this Alexander integral operator for $f(z)$, we derive

Theorem 2. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$|\arg(f'(z))| < \frac{\pi}{2}\alpha \qquad (z \in \mathbb{U})$$

for

$$\alpha = \beta + \frac{2}{\pi}\arctan(n\beta),$$

$$\beta = \gamma + \frac{2}{\pi}\arctan(n\gamma),$$

$$\frac{\pi}{2}(\alpha + \gamma) \leqq \phi(\mu)$$

and

$$\phi(\mu) = \frac{\pi}{2}(\mu + 1) - \arctan \frac{\cos\frac{\mu\pi}{2}}{\sin\frac{\mu\pi}{2} + \frac{n\mu}{1-\mu}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1+\mu}{2}}}$$

with some real $\alpha, \gamma > 0$ and $0 < \mu \leqq 1$, then the Alexander integral operator $F(z)$ of $f(z)$ belongs to the class $\mathcal{STS}(\mu)$.

The proof of Theorem 2 follows by replacing $f(z)$ with $F(z)$ in Theorem 1.

References

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