Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2012,**32B**(2):717–723
Acta Mathematica Scientia 数学物理学报
http://actams.wipm.ac.cn

RESULTS ON A QUESTION OF ZHANG AND YANG*

Li Sheng (李升)
LMIB & School of Mathematics and Systems Science, Beihang University, Beijing 100191, China
College of Science, Guangdong Ocean University, Zhanjiang 524088, China
E-mail: lish_ls@sina.com
Gao Zongsheng (高宗升)
LMIB & School of Mathematics and Systems Science, Beihang University, Beijing 100191, China
E-mail: zshgao@buaa.edu.cn

Abstract For a meromorphic function f, let $N_{(l+1}(r,\frac{1}{f})$ denote the counting function of zeros of f of order l at least. Let f be a nonconstant meromorphic function, such that $\overline{N}(r,f)=S(r,f)$. Denote $F=f^n$. Suppose that F and F' share 1 CM. If (1) $n\geq 3$, or (2) $n=2$ and $N(r,\frac{1}{f})=O(N_{(3}(r,\frac{1}{f}))$, then, $F=F'$, and f assumes the form

$$f(z)=ce^{\frac{1}{n}z},$$

where c is a nonzero constant. This main result of this article gives a positive answer to a question raised by Zhang and Yang [1] for the meromorphic functions case in some sense. And a relative result is proved.

Key words Nevanlinna theory; shared value; derivative

2000 MR Subject Classification 30D35; 30D45

1 Introduction

In this article, we assume that the reader is familiar with the standard notations of the Nevanlinna theory [2–5]. For any nonconstant meromorphic function f, we define the order of growth of f by

$$\rho(f):=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

And we denote by $S(r,f)$ any quantity satisfying

$$\lim_{r\to\infty}\frac{S(r,f)}{T(r,f)}=0,\quad r\notin E,$$

where $E\subset(0,\infty)$ is of finite linear measure. A meromorphic function a is said to be a small function of f if $T(r,a)=S(r,f)$.

*Received January 1, 2010; revised March 3, 2011. This work is supported by NNSF of China (11171013) and Fundamental Research Funds for the Central Universities NO. 300414. The first author is also supported by the Innovation Foundation of BUAA for Ph.D. Candidates.

We say that two meromorphic functions f and g share a small function a IM when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

Recently, many articles gave lots of results on the sharing value problems. In particular, a subtopic that a meromorphic function f and its derivative $f^{(k)}$ share some value a CM is well investigated (see [1, 6–13]). An interesting problem still open is the following conjecture proposed by Brück [6]:

Conjecture Let f be a nonconstant entire function. Suppose that

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then,

$$\frac{f'-a}{f-a} = c$$

holds for some nonzero constant c.

The case that $a=0$ and that $N(r,\frac{1}{f'}) = S(r,f)$ had been proved by Brück himself [6] while the case that f is of finite order had been proved by Gundersen and Yang [8]. However, the corresponding conjecture for meromorphic functions fails in general (see [8]). In 2009, Zhang and Yang [1] proved the following result.

Theorem A Let f be a nonconstant entire (respectively meromorphic) function and $n \geq 3$ (respectively $n \geq 4$) be an integer. Denote $F = f^n$. If F and F' share 1 CM, then, $F = F'$, and f assumes the form

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

As examples given by Gundersen and Yang [8] show that the conclusion of Theorem A may fail when $n=1$, Zhang and Yang asked a question: Can n in Theorem A be reduced? Lü, Xu, and Chen [9] answered this question for the entire functions case by proving the following result.

Theorem B Let Q be a polynomial and $n \geq 2$ be an integer. Let f be a transcendental entire function, and let $F = f^n$. If F and F' share Q CM, then,

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Theorem B is a corollary of Theorem 1.1 of [9] whose proof mainly depends on the famous L. Zalcman Lemma of theory of normal family. However, the meromorphic functions case is still unsolved. In the following, we will use Nevanlinna theory to investigate the question of Zhang and Yang. And we will use the notation $N_{(l+1}(r,\frac{1}{f})$ to denote the counting function of zeros of f of order l at least. We prove the following theorem.

Theorem 1.1 Let f be a nonconstant meromorphic function, such that $\overline{N}(r,f) = S(r,f)$. Denote $F = f^n$. Suppose that F and F' share 1 CM. If (1) $n \geq 3$, or (2) $n = 2$ and $N(r,\frac{1}{f}) = O(N_{(3}(r,\frac{1}{f}))$, then $F = F'$, and f assumes the form

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

For the case sharing the small function a IM, Zhang and Yang [1] proved the following result.

Theorem C Let f be a nonconstant entire (respectively meromorphic) function, n, k be positive integers, and $a(z)$ be a small meromorphic funtion of f, such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and $n > 2k+3$ (respectively $n > 2k+3+\sqrt{(2k+3)(k+3)}$), then $f^n = (f^n)^{(k)}$, and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

As a continuation of Theorem 1.1 and Theorem C, we consider the special case when $k = 1$ and $a(z)$ is a rational function, and prove the following result.

Theorem 1.2 Let f be a nonconstant entire (respectively meromorphic) function and $a(z) \not\equiv 0$ be a rational function. If $f^n - a$ and $(f^n)' - a$ share the value 0 IM and $n > 4$ (respectively $n > 4 + 2\sqrt{3}$), then $f^n = (f^n)'$, and f assumes the form

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

2 Lemmas

To prove Theorem 1.1, we need two lemmas. The following Hadamard's theorem for entire function of infinite order is well known (see, for example, [14]).

Lemma 2.1 [14] Let f be a transcendental entire function of infinite order, then f can be represented by

$$f(z) = U(z)e^{V(z)},$$

where U and V are entire functions with

$$\lambda(f) = \lambda(U) = \rho(U), \quad \lambda_1(f) = \lambda_1(U) = \rho_1(U),$$

$$\rho_1(f) = \max\{\rho_1(U), \rho_1(e^V)\},$$

where $\lambda_1(f)$ is defined by

$$\lambda_1(f) := \limsup_{r\to\infty} \frac{\log\log N(r, \frac{1}{f})}{\log r}.$$

Lemma 2.2 [4] Let f be a meromorphic function, and k be a positive integer. Then,

$$N(r, \frac{1}{f^{(k)}}) \le N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f).$$

The following Lemma 2.3 is the key for us to prove Theorem 1.2. Here, we need another notation. Let F and G be two non-constant meromorphic functions, such that F and G share the value 1 IM. Let z_0 be a 1-point of F of order p, a 1-point of G of order q. We denote by $N_L(r, \frac{1}{F-1})$ the counting function of those 1-points of F where $p > q$. Here, each point in this counting function is also counted only once.

Lemma 2.3 Let f be a non-constant meromorphic function and $a(z) \not\equiv 0$ be a rational function. If $f^n - a$ and $(f^n)' - a$ share the value 0 IM, then, for $F = \frac{f^n}{a}$, we have

$$N_L(r, \frac{1}{F-1}) = S(r, f), \tag{2.1}$$

If $a(z) \equiv c \in \mathbb{C}$, then,

$$N_L(r, \frac{1}{F-1}) = 0.$$

Proof Notice that each point in the counting function $N_L(r, \frac{1}{F-1})$ is a 1-point of F of order $p \geq 2$. Let z_0 be a 1-point of F of order p_0. Then, we can write

$$F(z) - 1 = (z - z_0)^{p_0} U(z),$$

where $U(z)$ is a meromorphic function such that $U(z_0), U'(z_0), a(z_0), a'(z_0) \neq 0, \infty$. If $p_0 \geq 2$, then for $G = \frac{(f^n)'}{a}$, we obtain

$$G(z) - 1 = \frac{(z - z_0)^{p_0 - 1}[p_0 a(z) U(z) + (z - z_0)(a(z)U(z))']}{a(z)} - \frac{a(z) - a'(z)}{a(z)},$$

which yields $a(z_0) - a'(z_0) = 0$. As $a(z) \not\equiv 0$ is a rational function, $a(z) - a'(z) = 0$ has only finitely many zeros. Therefore, we see that

$$N_L(r, \frac{1}{F-1}) = O(N(r, \frac{1}{a(z) - a'(z)})) = O(\log r) = S(r, f).$$

3 Proof of Theorem 1.1

As $F - 1$ and $F' - 1$ share the value 0 CM, we see that

$$G = \frac{F-1}{F'-1}$$

has no poles and all its zeroes are simple so that $G(z) = 0$ if and only if $F(z) = f^n(z) = \infty$. Thus, by Lemma 2.1 and the standard Hadamard's theorem for entire function, we can write

$$\frac{F-1}{F'-1} = G = Pe^g, \tag{3.1}$$

where P, g are entire functions such that

$$N(r, \frac{1}{P}) = \overline{N}(r, F) = \overline{N}(r, f) = S(r, f) = S(r, F). \tag{3.2}$$

If $P(z) \equiv C \neq 0$, then, $F = f^n$ is an entire function. By Theorem B, our conclusion holds.

If $P(z) \not\equiv C$, there are two cases: case 1: f has no zeros; case 2: f has zeros.

Case 1 f has no zeros. So does $F = f^n$. By Lemma 2.2, we have

$$N(r, \frac{1}{F'}) = O(\overline{N}(r, F)) = O(\overline{N}(r, f)) = S(r, f) = S(r, F). \tag{3.3}$$

Case 2 f has zeros. We are going to discuss by two subcases: subcase 1: $n \geq 3$; subcase 2: $n = 2$ and $N(r, \frac{1}{f}) = O(N_{(3}(r, \frac{1}{f}))$.

Subcase 1 $n \geq 3$. As $F = f^n$, then, $F''(z_0) = F'(z_0) = 0$ provided that $F(z_0) = 0$. From (3.1), we have

$$((P' + Pg')e^g - 1)F' - Pe^g F'' = (P' + Pg')e^g. \tag{3.4}$$

Thus, from (3.4), we see that all zeros of F are zeros of $P' + Pg'$. Particulary, if z_0 is a zero of F of order k, then z_0 is a zero of $P' + Pg'$ of order $d = k - 2 \geq 1$. Thus, by applying Lemma 2.2 with (3.2), we see that

$$\begin{aligned} N(r, \frac{1}{F}) &\leq 2N(r, \frac{1}{P' + Pg'}) \\ &= 2N(r, \frac{1}{(Pe^g)'}) \\ &\leq 2N(r, \frac{1}{Pe^g}) + 2\overline{N}(r, Pe^g) + S(r, Pe^g) \\ &= 2N(r, \frac{1}{P}) + S(r, \frac{F-1}{F'-1}) \\ &= S(r, F). \end{aligned}$$

This and Lemma 2.2 imply that (3.3) holds again.

Subcase 2 $n = 2$ and $N(r, \frac{1}{f}) = O(N_{(3}(r, \frac{1}{f}))$. Let z_1 be a zero of f of order $l \geq 2$, then, $F''(z_1) = F'(z_1) = F(z_1) = 0$. Therefore, with a similar arguing as in the Subcase 1, we can also deduce that

$$N_{(3}(r, \frac{1}{F}) \leq 2N(r, \frac{1}{P' + Pg'}) = 2N(r, \frac{1}{(Pe^g)'}) = S(r, F).$$

This and our assumption yield that

$$N(r, \frac{1}{F}) = O(N_{(3}(r, \frac{1}{F})) = S(r, F).$$

And hence (3.3) holds for this case.

Now, we see that (3.3) always holds. Set

$$H = \frac{F'''}{F''} - \frac{F''}{F'} - 2\frac{F''}{F'-1} + 2\frac{F'}{F-1}, \tag{3.5}$$

Then, we obtain that H is meromorphic and hence from the fundamental estimate of the logarithmic derivative, we have

$$m(r, H) = S(r, F). \tag{3.6}$$

As the poles of H coincide with the zeros of F' and F'', and $\overline{N}(r, F') = \overline{N}(r, F) = S(r, F)$, from (3.3) and Lemma 2.2, we see that

$$\begin{aligned} N(r, H) &\leq N(r, \frac{1}{F'}) + N(r, \frac{1}{F''}) \\ &\leq 2N(r, \frac{1}{F'}) + \overline{N}(r, F') + S(r, F) \\ &= S(r, F). \end{aligned}$$

From this and (3.6), we have

$$T(r, H) = S(r, F). \tag{3.7}$$

We assume that $H \not\equiv 0$. Notice that all zeros of $F-1$ and $F'-1$ are simple. Let z_1 be a zero of $F-1$ and $F'-1$, then $F''(z_1) \neq 0$, and it is seen that H is holomorphic at z_1, and $H(z_1)=0$ which with (3.7) yields that

$$N(r, \frac{1}{F-1}) \leq N(r, \frac{1}{H}) = S(r, F). \tag{3.8}$$

As F and F' share the value 1 CM, by the second fundamental theorem of Nevanlinna theory and with (3.8), we have

$$T(r, F') \leq N(r, \frac{1}{F'}) + N(r, \frac{1}{F-1}) + \overline{N}(r, F') + S(r, F') = S(r, F).$$

It follows that

$$m(r, \frac{1}{F-1}) \leq m(r, \frac{1}{F'}) + S(r, F) = S(r, F). \tag{3.9}$$

From (3.8) and (3.9), we conclude a contradiction that $T(r, F) = S(r, F)$.

Thus, we have $H \equiv 0$, and integration of (3.5) yields that

$$A\frac{F''}{F'} = (\frac{F'-1}{F-1})^2,$$

where A is a nonzero constant. As F and F' share 1 CM, we see that $A = F''(z_2)$, where $F(z_2) = F'(z_2) = 1$. Thus, by assuming that $\frac{F''}{F'}$ is not a constant, we see that

$$N(r, \frac{1}{F-1}) = O(N(r, \frac{F''}{F'})) = S(r, F),$$

and so (3.9) holds and again gives the contradiction $T(r, F) = S(r, F)$. Therefore, we now see that $\frac{F''}{F'}$ is a constant and obtain that, for a nonzero constant C,

$$\frac{F'-1}{F-1} = C,$$

which equals to

$$f^{n-1}(f - nCf') = 1 - C. \tag{3.10}$$

If $C = 1$, then from (3.10), we have $f - nf' \equiv 0$, which yields that $f = ce^{\frac{1}{n}z}$, where c is a nonzero constant.

If $C \neq 1$, then from (3.10), we see that f has no zeros and hence $f = e^h$, where h is an entire function. Now, it follows from (3.10) that

$$nCh' = 1 - (1 - C)e^{-nh},$$

which is impossible. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

We can use the same idea as in the proof Theorem 1.6 in [1] to prove Theorem 1.2. What we need to do is to apply Lemma 1.3 using (2.1) instead of (2.14) in [1]. Thus, we omit all those details.

References

[1] Zhang J L, Yang L Z. A power of a meromorphic function sharing a small funtion with its derivative. Ann Acad Sci Fenn Math, 2009, **34**: 249–260

[2] Hayman W. Meromorphic Function. Oxford: Clarendon Press, 1964

[3] Laine I. Nevanlinna Theory and Complex Differential Equations. Berlin: W. de Gruyter, 1993

[4] Yang C C, Yi H X. Uniqueness Theory of Meromorphic Functions. The Newtherlands: Kluwer Academic publishers, 2003

[5] Yang L. Value Distribution Theory and New Research. Beijing: Science Press, 1982 (in Chinese)

[6] Brück R. On entire functions which share one value CM with their first derivative. Results in Math, 1996, **30**: 21–24

[7] Gundersen G G. Meromorphic functions that share finite value with their derivative. J Math Anal Appl, 1980, **75**: 441–446

[8] Gundersen G G, Yang L Z. Entire functions that share one value with one or two of their derivatives. J Math Anal Appl, 1998, **223**: 88–95

[9] Lin W C, Yi H X. Meromorphic functions sharing two finite sets. Acta Mathematica Scientia, 2007, **27B**(4): 845–851

[10] Lü F, Xu J F, Chen A. Entire functions sharing polynomials with their first derivatives. Arch Math, 2009, **92**: 593–601

[11] Rubel L A, Yang C C. Values share by an entire function and its derivative. Lecture Notes in Math. Berlin: Springger-Vaerlag, 1977, **599**: 101–103

[12] Yang L Z. Entire functions that share finite value with their derivatives. Bull Austral Math Soc, 1990, **41**: 337–342

[13] Zhang J L, Yang L Z. Some results related to a conjecture of R. Brück. J Inequal Pure Appl Math, 2007, 8:1, Art 18

[14] Jank G, Volkmann L. Meromorphe Funktionen und Differentialgleichungen. Basel-Boston: Birkhäuser, 1985