SOME CONFORMAL MAPPING INEQUALITIES FOR STARLIKE AND CONVEX FUNCTIONS

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Introduction

Let $w = f(z) = \sum_1^\infty a_n z^n$ be regular and univalent for $|z| < 1$ and map the disc onto a domain D starlike with respect to $w = 0$ For $0 < r < 1$, $-\pi \leqslant \theta \leqslant \pi$, we write $C(r, \theta)$ for the image in D of the ray joining $z = 0$ to $z = re^{i\theta}$. The length of $C(r, \theta)$ is given by

$$l(r, \theta) = \int_0^r |f'(\rho e^{i\theta})| \, d\rho. \tag{1}$$

It follows from a result of Gehring and Hayman **[1]** that there is an absolute constant A such that

$$l(r, \theta) < A |f(re^{i\theta})|. \tag{2}$$

In this paper we shall show by means of a direct argument (Theorem 1) that (2) holds with

$$2 \leqslant A \leqslant 1 + \log 4 < 2{\cdot}3863. \tag{3}$$

In the case where D is convex we can improve our estimate for A and show that

$$\tfrac{1}{2}\pi \leqslant A \leqslant 1 + \log 2 < 2. \tag{4}$$

We also consider the analogous problem for the inverse of a starlike function. In fact if $w_0 = f(z_0) \in D$ and L is the line segment joining $w = 0$ to $w = w_0$, then $L \subset D$, so $\gamma = f^{-1}(L)$ is an arc in $\{|z| < 1\}$ joining $z = 0$ to $z = z_0$ The length $l(\gamma)$ of γ satisfies

$$l(\gamma) < (1 + \pi) |z_0|, \tag{5}$$

and the constant $1 + \pi$ is the best possible We can improve $1 + \pi$ to $1 + 3\pi/4$ when D is convex.

Statement of results

THEOREM 1. *Let $w = f(z) = \sum_1^\infty a_n z^n$ be a regular, starlike univalent function for $|z| < 1$ Let $C(r, \theta) = \{f(\rho e^{i\theta}), \ 0 \leqslant \rho \leqslant r\}$ and let $l(r, \theta)$ be the length of $C(r, \theta)$. Let $T(r, \theta)$ be the total variation of* arg w *on $C(r, \theta)$, so that*

$$T(r, \theta) = \int_0^r \left| \frac{\partial}{\partial \rho} \arg f(\rho e^{i\theta}) \right| d\rho = \int_0^r \left| \mathrm{Im} \frac{e^{i\theta} f'(\rho e^{i\theta})}{f(\rho e^{i\theta})} \right| d\rho \tag{6}$$

Then

$$l(r, \theta) < (1 + \log 4) |f(re^{i\theta})| \tag{7}$$

and

$$T(r, \theta) < \pi \tag{8}$$

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for $0 < r < 1$, $|\theta| \leqslant \pi$. *The constant* $(1+\log 4)$ *in* (7) *cannot be improved to any constant smaller than* 2. *The constant* π *in* (8) *is the best possible. In the case when* $f(z)$ *is convex univalent we have*

$$l(r,\theta) < (1+\log 2)\,|f(re^{i\theta})| \tag{9}$$

and

$$T(r,\theta) < \tfrac{1}{2}\pi \tag{10}$$

for $0 < r < 1$, $|\theta| \leqslant \pi$ *The constant* $1+\log 2$ *in* (9) *cannot be improved to any constant smaller than* $\frac{1}{2}\pi$. *The constant* $\frac{1}{2}\pi$ *in* (10) *is the best possible.*

THEOREM 2. *Let* $w = f(z) = \sum_{1}^{\infty} a_n z^n$ *be regular and starlike univalent for* $|z| < 1$, *and let* D *be the starlike image domain of* f. *Let* $w_0 = f(z_0) \in D$ *and* γ *be the arc in the unit disc joining* $z = 0$ *to* $z = z_0$ *whose image is the line segment* L *joining* $w = 0$ *to* $w = w_0$. *If* $l(\gamma)$ *is the length of* γ, *then*

$$l(\gamma) < (1+\pi)\,|z_0|. \tag{11}$$

The constant $1+\pi$ *is the best possible. In the case when* $f(z)$ *is convex univalent, we have*

$$l(\gamma) < (1+3\pi/4)\,|z_0|, \tag{12}$$

and $1+3\pi/4$ *cannot be improved to any constant smaller than* $\frac{1}{2}\pi$.

Section 1

Proof of Theorem 1. We write

$$F(z) = \frac{zf'(z)}{f(z)}, \quad (|z| < 1), \tag{13}$$

so that, since $f(z)$ is starlike univalent,

$$\operatorname{Re} F(z) > 0, \quad (|z| < 1), \tag{14}$$

and $F(0) = 1$. Then

$$\begin{aligned} l(r,\theta) &= \int_0^r |F(\rho e^{i\theta})|\,|f(\rho e^{i\theta})|\,\rho^{-1}\,d\rho \\ &= \int_0^r \operatorname{Re} F(\rho e^{i\theta})\,|f(\rho e^{i\theta})|\,\rho^{-1}\,d\rho + \int_0^r \{|F(\rho e^{i\theta})| - \operatorname{Re} F(\rho e^{i\theta})\}|\,f(\rho e^{i\theta})|\,\rho^{-1}\,d\rho \\ &= I_1 + I_2 \quad \text{say.} \end{aligned}$$

Now

$$I_1 = \int_0^r \frac{\partial}{\partial\rho}|f(\rho e^{i\theta})|\,d\rho = |f(re^{i\theta})|.$$

Also (14) implies that

$$\frac{\partial}{\partial\rho}|f(\rho e^{i\theta})| > 0, \tag{15}$$

so that $|f(\rho e^{i\theta})|$ is increasing with ρ for each fixed θ. Hence

$$I_2 \leqslant |f(re^{i\theta})| \int_0^r \{|F(\rho e^{i\theta})| - \operatorname{Re} F(\rho e^{i\theta})\} \rho^{-1} \, d\rho.$$

To prove (7) it remains to show that

$$I_3 = \int_0^r \{|F(\rho e^{i\theta})| - \operatorname{Re} F(\rho e^{i\theta})\} \rho^{-1} \, d\rho < \log 4. \tag{16}$$

According to the Integral Representation formula for functions with positive real part we can write

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}} \, dV(t), \tag{17}$$

where $V(t)$ is increasing, $V(t)-t$ has period 2π and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dV(t) = 1.$$

Without loss of generality we can assume that $\theta = 0$, so that

$$I_3 = \int_0^r \left(\frac{1}{2\pi} \left| \int_{-\pi}^{\pi} \frac{1+\rho e^{-it}}{1-\rho e^{-it}} \, dV(t) \right| - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-\rho^2}{|1-\rho e^{-it}|^2} \, dV(t) \right) \rho^{-1} \, d\rho$$

$$\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \int_0^r \left(\left| \frac{1+\rho e^{-it}}{1-\rho e^{-it}} \right| - \frac{1-\rho^2}{|1-\rho e^{-it}|^2} \right) \rho^{-1} \, d\rho \right\} dV(t).$$

It is clear therefore, that it will be sufficient to prove that

$$I_4 = \int_0^r \left\{ \left| \frac{1+\rho e^{+it}}{1-\rho e^{+it}} \right| - \frac{1-\rho^2}{|1-\rho e^{+it}|^2} \right\} \rho^{-1} \, d\rho < \log 4 \tag{18}$$

for each fixed t, $0 < t < \pi$.

Suppose, then, that t is fixed, $0 < t < \pi$, and consider the function

$$\lambda(\phi) = \frac{\sin \phi}{\sin(\phi + t)}, \quad 0 \leqslant \phi \leqslant \tfrac{1}{2}(\pi - t). \tag{19}$$

Then $\lambda(\phi)$ is a $1:1$ mapping of $[0, \frac{1}{2}(\pi - t)]$ onto $[0, 1]$. Hence we can change the variable in I_4 by means of the substitution

$$\rho = \frac{\sin \phi}{\sin(\phi + t)}, \quad 0 \leqslant \phi \leqslant \phi_0, \tag{20}$$

where

$$r = \frac{\sin \phi_0}{\sin(\phi_0 + t)} \tag{21}$$

and

$$0 < \phi_0 < \tfrac{1}{2}(\pi - t). \tag{22}$$

We note that

$$\lambda'(\phi) = \frac{\sin t}{\sin^2(\phi+t)} \tag{23}$$

and

$$|1-\lambda e^{it}| = \frac{\sin t}{\sin(\phi+t)}. \tag{24}$$

Thus

$$I_4 \leqslant \int_0^r \left(1+\frac{1}{\rho}\right)\left\{\frac{1}{|1-\rho e^{it}|} - \frac{1-\rho}{|1-\rho e^{it}|^2}\right\} d\rho$$

$$= \int_0^{\phi_0} \left(1+\frac{\sin(\phi+t)}{\sin\phi}\right)\left\{\frac{\sin(\phi+t)}{\sin t} - \frac{\sin^2(\phi+t)}{\sin^2 t}\left(1-\frac{\sin\phi}{\sin(\phi+t)}\right)\right\}\frac{\sin t}{\sin^2(\phi+t)}\,d\phi$$

$$= \int_0^{\phi_0}\left(\tan\frac{\phi}{2}+\tan\frac{\phi+t}{2}\right)d\phi$$

$$= 2\left[\log\frac{1}{\cos\frac{\phi}{2}\cos\frac{\phi+t}{2}}\right]_0^{\phi_0} = 2\log\frac{\cos\frac{t}{2}}{\cos\frac{\phi_0}{2}\cos\frac{\phi_0+t}{2}}.$$

Now

$$0<\frac{\phi_0}{2}<\frac{\pi}{2} \quad\text{and}\quad 0<\frac{\phi_0+t}{2}<\frac{\pi}{2}.$$

Hence

$$A(\phi_0) = \frac{1}{\cos\frac{\phi_0}{2}\cos\frac{\phi_0+t}{2}}$$

is increasing with ϕ_0, so

$$A(\phi_0) < A\left(\frac{\pi-t}{2}\right) = \frac{2}{\cos\frac{t}{2}}.$$

Thus

$$I_4 < 2\log 2 = \log 4$$

as required.

This completes the proof of (7). To prove (8) we note that

$$\left|\operatorname{Im}\frac{e^{i\theta}f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2\sin(\theta-t)}{|1-\rho e^{i(\theta-t)}|^2}\,dV(t)\right|$$

$$\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2|\sin(\theta-t)|}{|1-\rho e^{i(\theta-t)}|^2}\,dV(t).$$

Hence

$$T(r,\theta) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\int_0^r\frac{2|\sin(\theta-t)|}{|1-\rho e^{i(\theta-t)}|^2}\,d\rho\right\}dV(t).$$

It only remains to show that

$$I_5 = \int_0^r \frac{2\,|\sin(\theta - t)|}{|1-\rho e^{i(\theta-t)}|^2}\,d\rho < \pi,$$

for each fixed t. We may assume that $\theta = 0$ and t is fixed, $0 < t < \pi$. So

$$I_5 = \int_0^r \frac{2\sin t}{|1-\rho e^{it}|^2}\,d\rho.$$

Writing

$$\rho = \frac{\sin\phi}{\sin(\phi + t)},$$

we obtain

$$I_5 = \int_0^{\phi_0} 2\sin t\,\frac{\sin^2(\phi+t)}{\sin^2 t}\,\frac{\sin t}{\sin^2(\phi+t)}\,d\phi$$

$$= 2\phi_0 < \pi - t < \pi.$$

Thus (8) follows. To complete the proof of the first part of Theorem 1 we consider the function

$$f(z) = \frac{z}{(1-z)^2}, \quad (|z| < 1), \tag{25}$$

which maps $|z| < 1$ onto the whole plane cut along the negative real axis from $-\frac{1}{4}$ to $-\infty$. If $0 < \theta < \pi$, then it is easy to see that

$$\lim_{r\to 1} T(r,\theta) = \pi - \theta,$$

so that, if $\theta > 0$ is small, $T(r,\theta)$ can be made to approach π as closely as we please We will also show that

$$\lim_{\theta\to 0+}\lim_{r\to 1} \frac{l(r,\theta)}{|f(re^{i\theta})|} = 2,$$

and this will complete the proof of the first part of the theorem. If $0 < \theta < \pi$,

$$\lim_{r\to 1} l(r,\theta) = \int_0^1 \frac{|1+\rho e^{i\theta}|}{|1-\rho e^{i\theta}|^3}\,d\rho.$$

Writing

$$\rho = \frac{\sin\phi}{\sin(\phi+\theta)},$$

this becomes

$$\int_0^{(\pi-\theta)/2} |1+\rho(\phi)\,e^{i\theta}|\,\frac{\sin^3(\phi+\theta)}{\sin^3\theta}\,\frac{\sin\theta}{\sin^2(\phi+\theta)}\,d\phi = \int_0^{(\pi-\theta)/2} |1+\rho(\phi)\,e^{i\theta}|\,\frac{\sin(\phi+\theta)}{\sin^2\theta}\,d\phi.$$

Also

$$|f(e^{i\theta})| = \frac{1}{|1-e^{i\theta}|^2} = \frac{1}{4\sin^2\frac{1}{2}\theta}.$$

Hence

$$\lim_{r\to 1}\frac{l(r,\theta)}{|f(re^{i\theta})|} = \int_0^{(\pi-\theta)/2} |1+\rho(\phi)\,e^{i\theta}|\,\frac{\sin(\phi+\theta)}{\cos^2\frac{1}{2}\theta}\,d\phi.$$

As $\theta \to 0$, $\rho(\phi) \to 1$, so the integrand converges to $2\sin\phi$. We can apply the Lebesgue Bounded Convergence theorem so that

$$\lim_{\theta\to 0} \int_0^{(\pi-\theta)/2} |1+\rho(\phi)e^{i\theta}| \frac{\sin(\phi+\theta)}{\cos^2\frac{1}{2}\theta} \, d\phi = \int_0^{\pi/2} 2\sin\phi \, d\phi = 2,$$

as required.

Remarks

1. We conjecture that the constant $1+\log 4$ can be replaced by the constant 2 in (7). We verify that this is the case for the function (25). We have for this function and for $0 < \theta < \pi$,

$$l(r,\theta) = \int_0^r \frac{|1+\rho e^{i\theta}|}{|1-\rho e^{i\theta}|^3} \, d\rho \leqslant \int_0^r \frac{1+\rho}{|1-\rho e^{i\theta}|^3} \, d\rho = I \quad \text{say.}$$

Again substituting $\rho = \sin\phi/\sin(\phi+\theta)$ we deduce that

$$I = \int_0^{\phi_0} \left(1+\frac{\sin\phi}{\sin(\phi+\theta)}\right) \frac{\sin(\phi+\theta)}{\sin^2\theta} \, d\phi$$

$$= \frac{1}{\sin^2\theta} \left\{ \int_0^{\phi_0} \sin(\phi+\theta) \, d\phi + \int_0^{\phi_0} \sin\phi \, d\phi \right\}$$

$$= \frac{1}{\sin^2\theta} \{\cos\theta - \cos(\phi_0+\theta) + 1 - \cos\phi_0\},$$

where

$$0 < \phi_0 < (\pi-\theta)/2.$$

Now

$$|f(re^{i\theta})| = \frac{r}{|1-re^{i\theta}|^2} = \frac{\sin\phi_0}{\sin(\phi_0+\theta)} \cdot \frac{\sin^2(\phi_0+\theta)}{\sin^2\theta} = \frac{\sin\phi_0 \sin(\phi_0+\theta)}{\sin^2\theta}.$$

Hence

$$\frac{l(r,\theta)}{|f(re^{i\theta})|} \leqslant \frac{1+\cos\theta-\cos(\phi_0+\theta)-\cos\phi_0}{\sin\phi_0 \sin(\phi_0+\theta)} = \frac{\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\phi_0 \cos\frac{1}{2}(\phi_0+\theta)}.$$

As we saw earlier this latter expression is smaller than 2, as required.

Proof of second part of Theorem 1. If $f(z)$ is convex we proceed as before using the result

$$\operatorname{Re} F(z) > \tfrac{1}{2}, \quad (|z| < 1).$$

It follows that we can write

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1-ze^{-it}} \, dV(t),$$

where $V(t)$ is increasing, $V(t)-t$ has period 2π, and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dV(t) = 1.$$

Thus it is clear that to prove (9) we only have to show that if $0 < t < \pi$, then

$$I_6 = \int_0^r \left(\frac{1}{|1-\rho e^{it}|} - \mathrm{Re}\frac{1}{1-\rho e^{it}} \right) \frac{d\rho}{\rho} < \log 2.$$

As before we put $\rho = \sin\phi/\sin(\phi+t)$, and then

$$I_6 = \int_0^{\phi_0} \left\{ \frac{\sin(\phi+t)}{\sin t} - \frac{\sin^2(\phi+t)}{\sin^2 t}\left(1 - \frac{\cos t \sin\phi}{\sin(\phi+t)}\right) \right\} \frac{\sin t}{\sin(\phi+t)\sin\phi} d\phi$$

$$= \int_0^{\phi_0} \tan\tfrac{1}{2}\phi \, d\phi = 2\log \frac{1}{\cos\frac{\phi_0}{2}}$$

$$< 2\log \frac{1}{\cos\frac{\pi-t}{4}} = \log 2 + \log\frac{1}{1+\sin\frac{1}{2}t}$$

$$< \log 2 \quad \text{as required.}$$

To prove (10) we note that

$$T(r,\theta) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{ \int_0^r \frac{|\sin(\theta-t)|}{|1-\rho e^{i(\theta-t)}|^2} d\rho \right\} dV(t)$$

$$< \frac{\pi}{2}$$

as in the proof of (8).

To complete the proof of the theorem we consider the function

$$f(z) = \frac{z}{1-z}, \quad (|z| < 1), \tag{26}$$

which maps the unit disc onto the half-plane $\mathrm{Re}\, w > -\frac{1}{2}$. If $0 < \theta < \pi$,

$$\lim_{r\to 1} T(r,\theta) \to \frac{\pi}{2} \quad \text{as } \theta \to 0.$$

Also

$$\lim_{r\to 1} \frac{l(r,\theta)}{|f(re^{i\theta})|} = \int_0^1 \frac{|1-e^{i\theta}|}{|1-\rho e^{i\theta}|^2} d\rho$$

$$= \int_0^{(\pi-\theta)/2} 2\sin\tfrac{1}{2}\theta \frac{\sin^2(\phi+\theta)}{\sin^2\theta} \frac{\sin\theta}{\sin^2(\phi+\theta)} d\phi$$

$$= \int_0^{(\pi-\theta)/2} \frac{d\phi}{\cos\frac{1}{2}\theta} = \frac{\pi-\theta}{2\cos\frac{1}{2}\theta} \to \pi/2 \quad \text{as } \theta \to 0. \tag{27}$$

This completes the proof of Theorem 1.

Remarks

2. We note further from (27) that

$$\frac{l(r,\theta)}{|f(re^{i\theta})|} = \frac{\phi_0}{\cos\frac{1}{2}\theta} < \frac{\frac{1}{2}(\pi-\theta)}{\cos\frac{1}{2}\theta}$$

$$= \frac{\lambda}{\sin\lambda} < \frac{\pi}{2}, \tag{28}$$

since $\lambda = \frac{1}{2}(\pi-\theta)$ satisfies $0 < \lambda < \pi/2$. We therefore conjecture that the constant $1+\log 2$ in (9) can be replaced by the constant $\pi/2$.

3. If $f(z)$ is starlike, Theorem 1 shows that, for fixed θ, $\arg f(re^{i\theta})$ has bounded variation, $0 < r < 1$, and so

$$\lim_{r\to 1} \arg f(re^{i\theta}) \tag{29}$$

exists, which Pommerenke **[2]** has previously proved.

Section 2

Proof of Theorem 2. Let $g(w)$ be the inverse map from D onto $|z| < 1$. Then, if $w = f(z)$,

$$\frac{wg'(w)}{g(w)} = 1\Big/\frac{zf'(z)}{f(z)}. \tag{30}$$

Hence

$$\operatorname{Re}\frac{wg'(w)}{g(w)} > 0, \quad w \in D. \tag{31}$$

Let $w_0 = R_0 e^{i\phi_0}$, $z_0 = r_0 e^{i\theta_0}$. Then

$$l(\gamma) = \int_\gamma |dz| = \int_L |g'(w)|\,|dw|$$

$$= \int_0^{R_0} |g'(\mathrm{R}e^{i\phi_0})|\,dR$$

$$\leqslant \int_0^{R_0} \operatorname{Re}\frac{\mathrm{R}e^{i\phi_0} g'(\mathrm{R}e^{i\phi_0})}{g(\mathrm{R}e^{i\phi_0})}\,|g(\mathrm{R}e^{i\phi_0})|\,\frac{dR}{R}$$

$$+ \int_0^{R_0} \left|\operatorname{Im}\frac{\mathrm{R}e^{i\phi_0} g'(\mathrm{R}e^{i\phi_0})}{g(\mathrm{R}e^{i\phi_0})}\right| |g(\mathrm{R}e^{i\phi_0})|\,\frac{dR}{R}$$

$$= I_1 + I_2$$

say. Now

$$I_1 = \int_0^{R_0} \frac{\partial}{\partial R}|g(\mathrm{R}e^{i\phi_0})|\,dR = |g(R_0 e^{i\phi_0})| = |z_0|.$$

Also in view of (31), $|g(\mathrm{R}e^{i\phi_0})|$ is increasing with R. Hence

$$I_2 \leqslant |z_0| \int_0^{R_0} \left|\operatorname{Im}\frac{e^{i\phi_0} g'(\mathrm{R}e^{i\phi_0})}{g(\mathrm{R}e^{i\phi_0})}\right| dR.$$

Thus to prove (11) it will be enough to show that

$$I_3 = \int_0^{R_0} \left|\operatorname{Im} \frac{e^{i\phi_0} g'(\mathrm{R}e^{i\phi_0})}{g(\mathrm{R}e^{i\phi_0})}\right| dR < \pi.$$

Let $G(w) = wg'(w)/g(w)$ and choose r, $|z_0| < r < 1$. If C is the image of $|z| = r$, then $w = \mathrm{R}e^{i\phi_0}$. $0 \leqslant R \leqslant R_0$, lies inside the simple closed path C. Thus by Cauchy s integral formula

$$G(w) = \frac{1}{2\pi i} \int_C \frac{G(\zeta)\,d\zeta}{\zeta - w}$$

$$= \frac{1}{2\pi i} \int_{|z|=r} \frac{G(\zeta(z))\,\zeta'(z)\,dz}{\zeta(z) - w}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{f(re^{i\theta}) - w}\, d\theta.$$

Thus

$$I_3 = \int_0^{R_0} \left| \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Im} \frac{f(re^{i\theta})}{f(re^{i\theta}) - \mathrm{R}e^{i\phi_0}}\, d\theta \right| \frac{dR}{R}$$

$$\leqslant \frac{1}{2\pi} \int_0^{2\pi} \left\{ \int_0^{R_0} \left| \operatorname{Im} \frac{f(re^{i\theta})}{f(re^{i\theta}) - \mathrm{R}e^{i\phi_0}} \right| \frac{dR}{R} \right\} d\theta.$$

Thus it is enough to show that

$$I_4 = \int_0^{R_0} \left| \operatorname{Im} \frac{f(re^{i\theta})}{f(re^{i\theta}) - \mathrm{R}e^{i\phi_0}} \right| \frac{dR}{R} < \pi$$

for each θ, $0 \leqslant \theta \leqslant 2\pi$. Now

$$I_4 = \left| \operatorname{Im} \int_0^{R_0} \frac{e^{i\phi_0}}{f(re^{i\theta}) - \mathrm{R}e^{i\phi_0}}\, dR \right|$$

$$= \left| \operatorname{Im} \left[\log \frac{f(re^{i\theta})}{f(re^{i\theta}) - \mathrm{R}e^{i\phi_0}} \right]_0^{R_0} \right|$$

$$= \left| \left[\arg \frac{f(re^{i\theta})}{f(re^{i\theta}) - \mathrm{R}e^{i\phi_0}} \right]_0^{R_0} \right|. \tag{32}$$

The curve C does not meet the line segment L, and thus if $0 \leqslant R \leqslant R_0$ the expression

$$\frac{f(re^{i\theta})}{f(re^{i\theta}) - \mathrm{R}e^{i\phi_0}}$$

is never a negative real number. We deduce immediately that

$$I_4 < \pi$$

as required. This proves (11). To show that $1+\pi$ is best possible we consider the function

$$f(z) = \frac{z}{(1+z)^2}, \quad (|z| < 1), \tag{33}$$

which maps the disc onto the whole plane cut along the positive real axis from $\frac{1}{4}$ to $+\infty$. Consider the half-line $L = \{w/w = \mathrm{R}e^{i\phi},\ R \geqslant 0\}$, where ϕ is small. Let $\gamma = g(L)$. Then clearly γ is an arc joining $z = 0$ to $z = -1$. γ meets each circle $|z| = r$, $0 \leqslant r \leqslant 1$, once and only once. Furthermore if ϕ is sufficiently small γ will possess points which lie in the disc $|z-1| < \varepsilon$. Thus it is clear that

$$l(\gamma) \geqslant 1+\pi-o(\phi).$$

This completes the proof of the first part of Theorem 2.

Proof of second part of Theorem 2. If f is convex we argue as before and it will be enough to show that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\left[\arg\frac{f(re^{i\theta})}{f(re^{i\theta})-\mathrm{R}e^{i\phi_0}}\right]_0^{R_0}\right| d\theta < \frac{3\pi}{4},$$

i.e.
$$\frac{1}{2\pi}\int_0^{2\pi}\left|\arg\frac{f(re^{i\theta})}{f(re^{i\theta})-R_0\, e^{i\phi_0}}\right| d\theta < \frac{3\pi}{4}.$$

Now we have shown in another paper [3] that, if $f(z)$ is convex, then for each fixed ζ the function

$$F(z,\zeta) = z\left(\frac{f(z)-f(\zeta)}{z-\zeta}\right)^2$$

is a starlike function of z. It follows that

$$\mathrm{Re}\,\frac{f(z)-f(\zeta)}{z-\zeta}\,\frac{\zeta}{f(\zeta)} > 0, \quad (|z| < 1,\ |\zeta| < 1).$$

Thus, writing $z = re^{i\theta}$,

$$\left|\arg\frac{f(z)}{f(z)-f(z_0)}\right| = \left|\arg\frac{f(z)-f(z_0)}{z-z_0}\,\frac{z}{f(z)}\cdot\frac{z-z_0}{z}\right| < \frac{\pi}{2} + |\arg(1-\rho e^{i(\theta_0-\theta)})|,$$

where $\rho = |z_0|/|z| < 1$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\arg\frac{f(re^{i\theta})}{f(re^{i\theta})-f(z_0)}\right| d\theta < \frac{\pi}{2} + \frac{1}{2\pi}\int_0^{2\pi}|\arg(1-\rho e^{i(\theta_0-\theta)})|\,d\theta.$$

Since the integrand has period 2π it only remains to show that

$$I_5 = \frac{1}{2\pi}\int_0^{2\pi}|\arg(1-\rho e^{it})|\,dt \leqslant \frac{\pi}{4}.$$

Now
$$I_5 = \frac{1}{\pi}\int_0^{\pi}\arg\frac{1}{1-\rho e^{it}}\,dt.$$

For $0 < t < \pi$, $\arg(1/1-\rho e^{it})$ increases with ρ in the range $0 \leqslant \rho \leqslant 1$. Hence

$$I_5 < \frac{1}{\pi}\int_0^\pi \arg\frac{1}{1-e^{it}}\,dt$$

$$= \frac{1}{2\pi}\int_0^\pi (\pi - t)\,dt$$

$$= \frac{\pi}{4},$$

as required. This proves (12). To complete the proof we consider the function

$$f(z) = \frac{z}{1+z}, \quad (|z| < 1),$$

which maps $|z| < 1$ onto $\mathrm{Re}\, w < \frac{1}{2}$. Let L be the positive imaginary axis in the w-plane, so

$$L = \{w = iv/0 \leqslant v < \infty\}.$$

If γ is the inverse image of L then if $z \in \gamma$

$$z = \frac{iv}{1-iv}, \quad 0 \leqslant v < \infty.$$

Hence

$$|z+\tfrac{1}{2}| = \tfrac{1}{2}\left|\frac{1+iv}{1-iv}\right| = \tfrac{1}{2}.$$

Thus γ is a semi-circle of centre $z = -\frac{1}{2}$, radius $\frac{1}{2}$. Hence

$$l(\gamma) = \frac{\pi}{2}.$$

This completes the proof of Theorem 2.

Remarks

4. The total variation of the argument function along the curve γ is at most π for any starlike function (see proof of Theorem 2), and at most $3\pi/4$ for convex functions. We conjecture that in the convex case the total variation is at most $\pi/2$. In this case $l(\gamma) < 1+\pi/2$. The fact that this total variation is absolutely bounded above implies that the arc γ approaches a definite point on the unit circle as the ray L in the w-plane approaches the boundary of D.

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