ON LINEARLY ACCESSIBLE UNIVALENT FUNCTIONS

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Introduction

A domain D in the plane will be said to be *linearly accessible* if the complement of D can be written as a union of half-lines. Such a domain is clearly simply-connected and therefore, if it is not the whole plane, is the image of the unit disc $\{|z| < 1\}$ under an analytic univalent function $f(z)$. We will call such functions *linearly accessible.* The notion of linear accessibility is due to Biernacki **[1]**, who studied the case when the complements of the domains are unions of mutually disjoint half-lines (except that the endpoint of one half-line can lie on another half-line). As is now well known Lewandowski **[5, 6]** showed that these latter domains coincide with the close-to-convex domains of Kaplan **[4]**. Our problem, then, is to emulate the work of Lewandowski and seek analytic conditions equivalent to the geometrical definition of a linearly accessible function. Certain difficulties arise in this programme and the conditions which we obtain are a good deal more complicated than the conditions for the best known subclasses of schlicht functions. The main difference between these classes and our class is that the former are essentially determined by knowledge of local behaviour (in particular, tangential behaviour). Our conditions appear to indicate the necessity for global knowledge to fix a function in the class (in particular, knowledge of the chordal behaviour). This entails having conditions involving several variables and the univalence has to be carried as an explicit assumption throughout. Despite these drawbacks we are still able to obtain information about linearly accessible functions, and some of this is new even for starlike functions.

1. The first main problem which we must tackle is concerned with the question as to whether $f(z)$ being linearly accessible in $|z| < 1$ implies that $f(z)$ is linearly accessible in $|z| < r$ $(0 < r < 1)$ (or, equivalently, $f(rz)$ linearly accessible). Our approach to this problem is *via* an approximation argument together with an analytical formulation of the geometric condition to which the maximum principle can be applied. We shall require a few simple geometrical lemmas.

(1.1) *A domain D is linearly accessible if corresponding to each accessible boundary point w there is a half-line l with endpoint w which does not meet D.*

Proof. Consider first $w \in \partial D$, the boundary of D. Each open disc of centre w and radius $1/n$ $(n = 1, 2, \ldots)$ contains a point of D and therefore an accessible point w_n of ∂D. By hypothesis we obtain a sequence of half-lines l_n with endpoints w_n such that $l_n \subset D^c$, the complement of D, and there is clearly a limiting half-line $l \subset D^c$ whose endpoint is w. Secondly, we suppose that $w \in D^c - \partial D$ and let L be a line through w. L is divided by w into half-lines L_1 and L_2 each with endpoint w, and we may assume that neither L_1 nor L_2 lies in D^c. There are then points w_1 and w_2 of ∂D lying on L_1 and L_2 respectively such that the open segment $(w_1, w_2) \subset D^c - \partial D$. By the first part there are half-lines l_1 and l_2 in D^c with respective endpoints w_1 and w_2. The plane is

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divided by l_1, l_2 and (w_1, w_2) into either two or three open components, and just one of these contains the whole of D. Since for each such component there is at least one half-line l with endpoint w which does not meet it, the result follows.

(1.2) *If D_1 and D_2 are linearly accessible, then each component of $D_1 \cap D_2$ is linearly accessible.*

Proof. If D is a component of $D_1 \cap D_2$, and if $w \in \partial D$, then $w \in D_1{}^c \cup D_2{}^c \subset D^c$ and the conclusion is immediate, applying (1.1).

(1.3) *Let l_i $(1 \leqslant i \leqslant n)$ be n half-lines in the plane and let $E = \bigcup_{i=1}^{n} l_i$. Then each component of E^c is linearly accessible.*

The proof is a simple induction argument applying (1.2).

The approximation result which we require can now be obtained by an application of the Carathéodory kernel theorem [3; p. 74].

(1.4) *Let $f(z)$ be linearly accessible in $|z| < 1$. Then there is a sequence $\{f_n(z)\}$ of functions linearly accessible in $|z| < 1$ with $f_n(0) = f(0)$ $(n = 1, 2, \ldots)$, which converges locally uniformly in $|z| < 1$ to $f(z)$ and which has the property that each $f_n(z)$ has a continuous extension to the unit circle $|z| = 1$.*

Proof. Let D be the image domain of $f(z)$. The boundary of D is a non-void closed set and therefore, since the plane is separable, we can find a sequence of points $\{w_n\}$ such that the closure of the set $E = \{w_1, w_2, \ldots\}$ is the boundary of D. From each point w_n we can find a half-line l_n not meeting D. Let D_n be that component of the complement of $l_1 \cup l_2 \cup \ldots \cup l_n$ which contains $f(0)$, so that by (1.3) D_n is a linearly accessible domain containing D. Let $D_n{}'$ be that component of $D_n \cap \{|w - f(0)| < n\}$ which contains $f(0)$. Again $D_n{}'$ is linearly accessible by (1.2). We define $f_n(z)$ as the function analytic and univalent in $|z| < 1$ whose image is $D_n{}'$, with $f_n(0) = f(0)$, $\arg f_n{}'(0) = \arg f'(0)$. The density of the sequence $\{w_n\}$ on ∂D clearly implies that $\{D_n{}'\}$ converges to its kernel D, so that $f_n(z)$ converges locally uniformly in $|z| < 1$ to $f(z)$. Finally, since the boundary of each domain $D_n{}'$ is a locally-connected continuum, $f_n(z)$ extends continuously to $|z| = 1$.

We can now establish an analytic criterion for $f(z)$ to be linearly accessible, which is adequate for the qualitative information which we require.

(1.5) *A function $f(z)$ analytic and univalent in $|z| < 1$ is linearly accessible if, and only if,*

$$\left|\arg \frac{f(zz_2)-f(z)}{f(zz_1)-f(z)}\right| < 2\pi \qquad (|z| < 1) \tag{1.5.1}$$

for any two points z_1 and z_2 satisfying $|z_1| < 1$ and $|z_2| < 1$.

(1.6) *Remark.* To be precise, the branch of the argument is chosen to be the imaginary part of the logarithm, where

$$\log \frac{f(zz_2)-f(z)}{f(zz_1)-f(z)} = \int_{z_1}^{z_2} \frac{zf'(z\zeta)}{f(z\zeta)-f(z)}\, d\zeta,$$

the integral being taken along any path in $|\zeta| < 1$ joining z_1 to z_2. This function is well-defined and analytic in the three variables z, z_1 and z_2.

Denoting by B the normalized class of linearly accessible functions $f(z)$ satisfying $f(0) = 0, f'(0) = 1$, we deduce

(1.7) THEOREM. *The class B is a linear invariant, compact, normal family. Furthermore $f(z) \in B$ if, and only if, for each r $(0 < r < 1)$ $r^{-1} f(rz) \in B$.*

(1.8) To prove the sufficiency of the condition (1.5.1) we choose an accessible boundary point w of the image domain D of f, and observe that the condition implies that for any two points x and y satisfying $|x| < 1$ and $|y| < 1$, we have

$$\left|\arg \frac{f(x)-w}{f(y)-w}\right| < 2\pi,$$

the inequality being strict by the maximum principle. It follows immediately that there is a half-line with endpoint w which does not meet D. Applying (1.1) we deduce that D is linearly accessible.

(1.9) For the necessity we may assume by (1.4) that f extends continuously to $|z| = 1$. Then for each pair of points z_1 and z_2 in the open unit disc the function

$$\arg \frac{f(zz_2)-f(z)}{f(zz_1)-f(z)}$$

is harmonic for $|z| < 1$ and continuous for $|z| \leqslant 1$. If $|z| = 1$, the inequality (1.5.1) is a clear consequence of the existence of a half-line from $w = f(z)$ which does not meet D. Hence, by the maximum principle, (1.5.1) holds for $|z| < 1$, and the proof is complete.

2. Let $f(z)$ be linearly accessible and choose r $(0 < r < 1)$. As we have seen $f(z)$ maps $|z| = r$ onto a simple closed curve C_r bounding a linearly accessible domain D_r. Thus from each point on C_r we can proceed along a half-line to ∞ without meeting D_r. It is natural to focus attention on the directions of these half-lines as $|z| = r$ is traversed in the positive sense. In the close-to-convex case these directions may be chosen to increase continuously as the circle is traversed, and we are led by straightforward arguments to the Kaplan condition [4]

$$\text{(2.1)} \qquad \operatorname{Re} \frac{f'(z)}{g'(z)} \geqslant 0$$

where $g(z)$ is convex, the function $\theta + \arg g'(re^{i\theta})$ giving the direction at $w = f(re^{i\theta})$ of an appropriate half-line. In the general case we can formulate a similar condition and it is not too difficult to see that in this case the function $g(z)$ will have to be close, to-convex, for we can obtain Kaplan's condition [4]

$$\theta_2 + \arg g'(re^{i\theta_2}) - \theta_1 - \arg g'(re^{i\theta_1}) > -\pi$$

for $\theta_2 > \theta_1$. It is not, however, apparent whether or not one can say much more than this about $g(z)$. Without some additional restriction on g the condition (2.1) is extremely weak and we are compelled to approach the problem from rather a different angle. However, our failure to obtain really sharp results for our class (such as the Bieberbach conjecture) seems to stem from the difficulties encountered at this stage.

The basic result of this paper is an intrinsic characterisation of the linearly accessible functions which comes essentially from a consideration of the behaviour of the chords emanating from a fixed point on C_r.

(2.2) THEOREM. *A function $f(z)$ analytic and univalent in $|z| < 1$ is linearly accessible if, and only if, for each r $(0 < r < 1)$ and each $z_0(|z_0| < 1)$, we have*

$$\tfrac{1}{2}\theta_2 + \arg\frac{f(re^{i\theta_2})-f(z_0)}{re^{i\theta_2}-z_0} - \tfrac{1}{2}\theta_1 - \arg\frac{f(re^{i\theta_1})-f(z_0)}{re^{i\theta_1}-z_0} > -\pi \tag{2.2.1}$$

whenever $\theta_2 > \theta_1$.

The remainder of this section will be devoted to the proof of this theorem. Suppose, then, that $f(z)$ is linearly accessible and choose r $(0 < r < 1)$. Let $z_0 = re^{i\theta_0}, z_1 = re^{i\theta_1}$, $z_2 = re^{i\theta_2}$ where $\theta_0 < \theta_1 < \theta_2 < \theta_0 + 2\pi$ and define

$$C_0 = \{w = f(re^{i\theta}) : \theta_1 \leqslant \theta \leqslant \theta_2\} \tag{2.2.2}$$

$$C_1 = \{w = f(re^{i\theta}) : \theta_2 \leqslant \theta \leqslant \theta_0 + 2\pi\} \tag{2.2.3}$$

$$C_2 = \{w = f(re^{i\theta}) : \theta_0 \leqslant \theta \leqslant \theta_1\}. \tag{2.2.4}$$

Writing $w_i = f(z_i)(i = 0, 1, 2)$ we denote by α_i the change of $\arg(w - w_i)$ as w traverses C_i:

$$\alpha_i = \Delta \arg(w - w_i) \quad \text{along } C_i \quad (i = 0, 1, 2). \tag{2.2.5}$$

(2.3) LEMMA. *We have*

$$-\pi < \alpha_0 < 2\pi. \tag{2.3.1}$$

Proof. We observe firstly that

$$\sum_{i=0}^{2} \alpha_i = \sum_{i=0}^{2} \Delta \arg(w - w_i)$$

$$= \sum_{i=0}^{2} \Delta \arg\frac{f(z)-f(z_i)}{z - z_i} + \sum_{i=0}^{2} \Delta \arg(z - z_i) = S_1 + S_2 \text{ say,}$$

where $\Delta \arg(z - z_i)$ denotes the change in argument along $f^{-1}(C_i)$ on the circle $|z| = r$. It is clear that $S_1 = 0$ and $S_2 = \pi$, and therefore

$$\sum_{i=0}^{2} \alpha_i = \pi. \tag{2.3.2}$$

Assume then that $\alpha_0 \leqslant -\pi$. By (1.5), for each w_i we can find a half-line l_i with endpoint w_i, which lies (apart from w_i) in the exterior of $D_r = f(|z| < r)$. The assumption $\alpha_0 \leqslant -\pi$ implies that the line segment $[w_1, w_2]$ meets l_0. The existence of l_1 means that C_1 cannot enclose w_1, and so α_1 is the angle at w_1 of the triangle T whose vertices are w_0, w_1 and w_2, and we have $0 \leqslant \alpha_1 < \pi$. Similarly, the existence of l_2

implies that α_2 is the angle at w_2 of T. Since α_0 is the exterior angle (measured negatively) at w_0 formed by T, we see that $\alpha_0+\alpha_1+\alpha_2 = -\pi$ contradicting (2.3.2). That $\alpha_0 < 2\pi$ is an immediate consequence of the existence of l_0. This establishes the lemma.

(2.4) We now choose a fixed branch of

$$\arg\frac{f(z)-f(\zeta)}{z-\zeta} \qquad (|z| < 1, |\zeta| < 1)$$

and define

$$h_r(\theta, \phi) = h(\theta, \phi) = \tfrac{1}{2}(\theta+\phi)+\arg\frac{f(re^{i\theta})-f(re^{i\phi})}{re^{i\theta}-re^{i\phi}}. \tag{2.4.1}$$

We will show that for every ϕ we have

$$h(\theta_2, \phi)-h(\theta_1, \phi) > -\pi \tag{2.4.2}$$

whenever $\theta_2 > \theta_1$.

In the case $\phi < \theta_1 < \theta_2 < \phi+2\pi$ the expression $h(\theta_2, \phi)-h(\theta_1, \phi)$ is the change of $\arg(f(re^{i\theta})-f(re^{i\phi}))$ for $\theta_1 \leqslant \theta \leqslant \theta_2$ and therefore, by Lemma 2.3,

$$-\pi < h(\theta_2, \phi)-h(\theta_1, \phi) < 2\pi. \tag{2.4.3}$$

It is easily seen geometrically that (2.4.3) remains valid for $\phi \leqslant \theta_1 < \theta_2 \leqslant \phi+2\pi$. For the general case, suppose that

$$\phi+2m\pi \leqslant \theta_1 \leqslant \phi+2(m+1)\pi$$

$$\phi+2n\pi \leqslant \theta_2 \leqslant \phi+2(n+1)\pi$$

for some integers m and n, where $\theta_1 < \theta_2$. Clearly, for any integers j and k we have

$$h(\theta+2j\pi, \phi+2k\pi) = h(\theta, \phi)+(k+j)\pi. \tag{2.4.4}$$

There are two possibilities; either (i) $\phi+2n\pi \leqslant \theta_1+2(n-m)\pi \leqslant \theta_2 \leqslant \phi+2(n+1)\pi$, or (ii) $\phi+2n\pi \leqslant \theta_2 \leqslant \theta_1+2(n-m)\pi \leqslant \phi+2(n+1)\pi$. In case (i) we apply the left inequality in (2.4.3), replacing ϕ by $\phi+2n\pi$ and θ_1 by $\theta_1+2(n-m)\pi$, and we obtain, with the help of (2.4.4),

$$h(\theta_2, \phi)-h(\theta_1, \phi) > -\pi+(n-m)\pi \geqslant -\pi$$

since in this case $n \geqslant m$. In case (ii) we apply the right inequality in (2.4.3) replacing ϕ by $\phi+2n\pi$, θ_1 by θ_2 and θ_2 by $\theta_1+2(n-m)\pi$, and we obtain using (2.4.4)

$$h(\theta_2, \phi)-h(\theta_1, \phi) > (n-m-1)\pi-\pi \geqslant -\pi$$

since in this case $n \geqslant m+1$. This establishes (2.4.2).

(2.5) We have thus proved the necessity of the condition (2.2.1) in the case $|z_0| = r$. Since for fixed r, θ_1 and θ_2 the left member of (2.2.1) is harmonic in the variable z_0, (2.2.1) remains valid if $|z_0| \leqslant r$. Let us then choose z_0 arbitrarily ($|z_0| < 1$) and also θ_1 and θ_2 with $\theta_2 > \theta_1$, so that (2.2.1) holds for every r satisfying

$|z_0| \leqslant r < 1$. In particular let $r = |z_0|$, choose ϕ real and arbitrary and set $\zeta = re^{i\phi}$. We then have

$$\tfrac{1}{2}\theta_2 + \arg\frac{f(\zeta e^{i\theta_2}) - f(z_0)}{\zeta e^{i\theta_2} - z_0} - \tfrac{1}{2}\theta_1 - \arg\frac{f(\zeta e^{i\theta_1}) - f(z_0)}{\zeta e^{i\theta_1} - z_0} > -\pi.$$

Since the left member of this inequality is harmonic in ζ and the inequality holds for $|\zeta| = |z_0|$, the inequality also holds for all ζ satisfying $|\zeta| \leqslant |z_0|$. Thus (2.2.1) is proved to hold in all cases.

(2.6) It remains to establish the sufficiency of the condition (2.2.1). If (2.2.1) is satisfied, then it is clear that the condition (2.4.2) holds. For each real ϕ we define

$$K(\phi) = \tfrac{1}{2}\Big(\inf_{\theta \geqslant \phi} h(\theta, \phi) + \sup_{\theta \leqslant \phi} h(\theta, \phi)\Big). \tag{2.6.1}$$

It is clear from (2.4.2) and the continuity and periodicity property of $h(\theta, \phi)$ that the infimum and supremum occurring in (2.6.1) are each attained finitely. Thus, if $\theta > \phi$, then

$$\begin{aligned} h(\theta, \phi) - K(\phi) &= h(\theta, \phi) - \inf_{\theta \geqslant \phi} h(\theta, \phi) + \tfrac{1}{2}\Big(\inf_{\theta \geqslant \phi} h(\theta, \phi) - \sup_{\theta \leqslant \phi} h(\theta, \phi)\Big) \\ &\geqslant \tfrac{1}{2}\Big(\inf_{\theta \geqslant \phi} h(\theta, \phi) - \sup_{\theta \leqslant \phi} h(\theta, \phi)\Big) > -\pi/2 \end{aligned}$$

by (2.4.2), the inequality being strict since the infimum and the supremum are attained. Similarly, if $\theta < \phi$ we obtain

$$h(\theta, \phi) - K(\phi) < \pi/2.$$

It follows that, if $\phi < \theta < \phi + 2\pi$, then

$$K(\phi) < h(\theta, \phi) + (\pi/2) < K(\phi) + 2\pi. \tag{2.6.2}$$

Now $h(\theta, \phi) + \pi/2$ is the chordal direction $\arg\big(f(re^{i\theta}) - f(re^{i\phi})\big)$ as the circle $|z| = r$ is traversed once positively starting at $z = re^{i\phi}$. Hence the inequality (2.6.2) implies that the half-line with endpoint $f(re^{i\phi})$ and direction $K(\phi)$ fails to meet the curve $\{f(re^{i\theta}) : \phi < \theta < \phi + 2\pi\}$. Since ϕ is arbitrary and the circle $|z| = r$ $(0 < r < 1)$ is arbitrary, $f(z)$ is linearly accessible by Theorem 1.7. This completes the proof of Theorem 2.2.

3. In this section we construct a family of auxiliary functions which are "close" to a given linearly accessible function $f(z)$. This is to be contrasted with the close-to-convex case where just one auxiliary function fixes f in the class.

(3.1) THEOREM. *A function $f(z)$ analytic and univalent in $|z| < 1$ is linearly accessible if, and only if, corresponding to each fixed point $z_0 (|z_0| < 1)$ there exists a function $g(z) = g(z; z_0)$ starlike of order $\tfrac{1}{2}$, that is, one which satisfies*

$$\operatorname{Re}\frac{zg'(z)}{g(z)} > \tfrac{1}{2} \qquad (|z| < 1), \tag{3.1.1}$$

for which we have the inequality

$$\text{Re}\left(\frac{f(z)-f(z_0)}{z-z_0}\Big/\frac{g(z)}{z}\right) > 0 \quad (|z| < 1). \tag{3.1.2}$$

Proof. The sufficiency is almost trivial if we observe that $g(z)$ is starlike of order $\frac{1}{2}$ if, and only if, $g^2(z)/z$ is starlike, i.e. if, and only if, for each r $(0 < r < 1)$ we have

$$\arg g(re^{i\theta_2})-\tfrac{1}{2}\theta_2-\arg g(re^{i\theta_1})+\tfrac{1}{2}\theta_1 > 0 \tag{3.1.3}$$

whenever $\theta_2 > \theta_1$. This together with (3.1.2) immediately yields (2.2.1).

Conversely, if $f(z)$ is linearly accessible we construct $g(z)$ by a method similar to Kaplan's construction [4] of a convex function from his intrinsic characterisation of the close-to-convex functions. We merely sketch the argument. Putting

$$F(z) = \left(\frac{f(z)-f(z_0)}{z-z_0}\right)^2$$

we have, by (2.2.1),

$$\theta_2+\arg F(re^{i\theta_2})-\theta_1-\arg F(re^{i\theta_1}) > -2\pi$$

whenever $\theta_2 > \theta_1$. We set

$$t_r(\theta) = \inf_{\theta' \geqslant \theta}\left(\theta'+\arg F(re^{i\theta'})\right)+\pi.$$

Then $t_r(\theta)$ is increasing with θ, $t_r(\theta+2\pi) = t_r(\theta)+2\pi$ and

$$|t_r(\theta)-\arg F(re^{i\theta})| \leqslant \pi.$$

We can find a function $G_r(z)$ starlike for $|z| \leqslant r$ with $|G_r'(0)| = |F'(0)|$ such that

$$t_r(\theta) = \arg G_r(re^{i\theta}).$$

We then have

$$|\arg F(z)-\arg G_r(z)| \leqslant \pi \qquad (|z| \leqslant r).$$

Choosing a sequence of values of r tending to 1 we obtain a starlike function $G(z)$ such that

$$\left|\arg\frac{F(z)}{G(z)}\right| \leqslant \pi \qquad (|z| < 1).$$

Putting $g(z) = z\left(G(z)/z\right)^{\frac{1}{2}}$ it follows that

$$\text{Re}\left(\frac{f(z)-f(z_0)}{z-z_0}\Big/\frac{g(z)}{z}\right) \geqslant 0 \qquad (|z| < 1)$$

where $g(z)$ is starlike of order $\frac{1}{2}$. If equality occurs here, we have

$$\frac{f(z)-f(z_0)}{z-z_0}\Big/\frac{g(z)}{z} = c$$

by the maximum principle, where $c \neq 0$ depends only on z_0 and is purely imaginary. We then obtain (3.1.2) with $cg(z)$ replacing $g(z)$.

(3.2) The condition (3.1.2) can be formulated in a way that makes its relation to the close-to-convexity condition clearer. If we set

(3.2.1)
$$\phi(z) = z_0 g'(0) + (z - z_0)\frac{g(z)}{z}$$

then we have

(3.2.2)
$$z\frac{\phi(z)-\phi(z_0)}{z-z_0} = g(z)$$

and

(3.2.3)
$$\operatorname{Re}\frac{f(z)-f(z_0)}{\phi(z)-\phi(z_0)} > 0 \qquad (|z| < 1).$$

If $\phi(z)$ is convex then [7] the left-hand member of (3.2.2) is always starlike of order $\frac{1}{2}$ and therefore if $\phi(z)$ were independent of z_0 the relation (3.2.3) would imply that $f(z)$ was close-to-convex. On the other hand, if $f(z)$ is close-to-convex, it is well known that (3.2.3) holds for a convex ϕ and therefore (3.1.2) holds. In any case it is not difficult to show that the function $\phi(z)$ defined by (3.2.1) maps the unit disc onto a domain starlike with respect to the point $\phi(z_0)$, and in particular that $\phi(z)$ is univalent for $|z| < 1$.

4. A more general class of functions than the close-to-convex functions can be obtained by assuming that the function $\phi(z)$ of (3.2.1) has the form

(4.1)
$$\phi(z) = c(z_0)\,\Phi(z)$$

where $\Phi(z)$ is convex and independent of z_0 and $c(z_0)$ is a function of z_0 alone. Again $g(z)$ is starlike of order $\frac{1}{2}$, so that $f(z)$ satisfying (3.2.3) is linearly accessible. The condition (3.2.3) can in this case be formulated as follows.

(4.2)
$$\left|\arg\frac{f(z_2)-f(z_0)}{\Phi(z_2)-\Phi(z_0)} - \arg\frac{f(z_1)-f(z_0)}{\Phi(z_1)-\Phi(z_0)}\right| < \pi$$

for any three points z_0, z_1 and z_2 in the open unit disc. This is precisely the condition for $f(z)$ to be a function " of convex type ", a notion which we have studied elsewhere [**8**]. We showed there by other means that such a function was linearly accessible, and we also showed that these functions are not necessarily close-to-convex. We conjectured that every linearly accessible function was of convex type. However this is not the case and we have the following result.

(4.3) THEOREM. *$f(z)$ is linearly accessible in $|z| < 1$ if, and only if, for each function $\phi(z)$ convex univalent in $|z| < 1$, we have*

(4.3.1)
$$\left|\arg\frac{f(z_2)-f(z_0)}{\phi(z_2)-\phi(z_0)} - \arg\frac{f(z_1)-f(z_0)}{\phi(z_1)-\phi(z_0)}\right| < 2\pi$$

for any three points z_0, z_1 and z_2 in the open unit disc. Furthermore, there exists a *linearly accessible function $f(z)$ such that*

$$\sup_{z_0, z_1, z_2} \left| \arg \frac{f(z_2)-f(z_0)}{\phi(z_2)-\phi(z_0)} - \arg \frac{f(z_1)-f(z_0)}{\phi(z_1)-\phi(z_0)} \right| = 2\pi \tag{4.3.2}$$

for every convex univalent function $\phi(z)$.

Proof. Suppose first that $f(z)$ is linearly accessible and $\phi(z)$ convex. Let z_0, z_1 and z_2 be three points in $|z| < 1$ and choose $r < 1$ so that z_0, z_1 and z_2 lie in $|z| < r$. Then, if $\theta_0 < \theta_1 < \theta_2 < \theta_0 + 2\pi$, we have

$$\arg \frac{f(re^{i\theta_2})-f(re^{i\theta_0})}{\phi(re^{i\theta_2})-\phi(re^{i\theta_0})} - \arg \frac{f(re^{i\theta_1})-f(re^{i\theta_0})}{\phi(re^{i\theta_1})-\phi(re^{i\theta_0})}$$

$$= \left\{ \tfrac{1}{2}(\theta_2-\theta_1) + \arg \frac{f(re^{i\theta_2})-f(re^{i\theta_0})}{re^{i\theta_2}-re^{i\theta_0}} - \arg \frac{f(re^{i\theta_1})-f(re^{i\theta_0})}{re^{i\theta_1}-re^{i\theta_0}} \right\}$$

$$- \left\{ \tfrac{1}{2}(\theta_2-\theta_1) + \arg \frac{\phi(re^{i\theta_2})-\phi(re^{i\theta_0})}{re^{i\theta_2}-re^{i\theta_0}} - \arg \frac{\phi(re^{i\theta_1})-\phi(re^{i\theta_0})}{re^{i\theta_1}-re^{i\theta_0}} \right\}.$$

The expression in the first bracket lies between $-\pi$ and 2π and the expression in the second bracket lies between 0 and π. Thus

$$\left| \arg \frac{f(re^{i\theta_2})-f(re^{i\theta_0})}{\phi(re^{i\theta_2})-\phi(re^{i\theta_0})} - \arg \frac{f(re^{i\theta_1})-f(re^{i\theta_0})}{\phi(re^{i\theta_1})-\phi(re^{i\theta_0})} \right| < 2\pi.$$

By periodicity this must hold for all real values of θ_0, θ_1 and θ_2 and therefore (4.3.1) holds by the maximum principle.

(4.4) Conversely, assume that (4.3.1) holds for every convex function $\phi(z)$. In particular, for an arbitrary real α, (4.3.1) holds for the function

$$\phi(z) = \frac{z}{1-ze^{-i\alpha}}.$$

Thus for $0 < r < 1$ and $\theta_0 < \theta_1 < \theta_2 < \theta_0 + 2\pi$, we obtain with the notation of section 2,

$$-2\pi < h_r(\theta_2, \theta_0) - h_r(\theta_1, \theta_0) + \left\{ \arg \frac{1-re^{i(\theta_2-\alpha)}}{1-re^{i(\theta_1-\alpha)}} - \tfrac{1}{2}(\theta_2-\theta_1) \right\} < 2\pi.$$

Now

$$\lim_{r\to 1} \left\{ \arg \frac{1-re^{i(\theta_2-\alpha)}}{1-re^{i(\theta_1-\alpha)}} - \tfrac{1}{2}(\theta_2-\theta_1) \right\} = \begin{cases} 0 \text{ if } \alpha = \theta_0 \\ -\pi \text{ if } \alpha = \tfrac{1}{2}(\theta_1+\theta_2). \end{cases} \tag{4.4.1}$$

Thus, given $\varepsilon > 0$, we can find R ($r < R < 1$) so that

$$-\pi - \varepsilon < h_R(\theta_2, \theta_0) - h_R(\theta_1, \theta_0) < 2\pi + \varepsilon. \tag{4.4.2}$$

Now (4.4.2) remains valid for the same ε and the same R if we replace θ_0, θ_1 and θ_2 by $\theta_0+\beta$, $\theta_1+\beta$ and $\theta_2+\beta$, where β is arbitrary; for the left member of (4.4.1) will remain the same by putting $\alpha = \theta_0+\beta$ and $\alpha = \frac{1}{2}(\theta_1+\theta_2)+\beta$. Thus we can employ the maximum principle to replace R by r in (4.4.2); then, since $\varepsilon > 0$ is arbitrary, we deduce that

$$-\pi < h_r(\theta_2, \theta_0)-h_r(\theta_1, \theta_0) < 2\pi$$

for $\theta_0 \leqslant \theta_1 < \theta_2 \leqslant \theta_0+2\pi$, the strictness of the inequalities following from the maximum principle. It follows that $f(z)$ is linearly accessible, as required.

(4.5) It remains to construct a linearly accessible function $f(z)$ for which (4.3.2) holds for every convex ϕ. We define the image domain D of $f(z)$ as the upper half-plane $H = \{\operatorname{Im} w > 0\}$ cut along a countable set $\{\lambda_n\}$ of line segments with one endpoint on the real axis R and the other in H, $\lambda_n = [a_n, b_n]$, say, where $a_n \in R$, $b_n \in H$. The construction of each segment λ_n will depend only on the previous λ_k $(1 \leqslant k \leqslant n-1)$ and will have the following properties.

(4.5.1) $$a_n > a_{n-1};$$

(4.5.2) the angles α_n which λ_n makes with the positive axis are given by

$$\alpha_{2n-1} = \frac{\pi}{2^n}, \; \alpha_{2n} = \frac{2^{n+1}-1}{2^{n+1}}\pi \qquad (n = 1, 2, \ldots);$$

(4.5.3) the λ_n are disjoint and are chosen so that

$$|b_{2n-1}-a_{2n-1}| = 1 \quad \text{and} \quad \arg(b_{2n}-b_{2n-1}) = \frac{2^n-1}{2^n}\pi \; (n = 1, 2, \ldots).$$

The domain D is clearly linearly accessible. Suppose now that $\phi(z)$ is a convex function for which the left member of (4.3.2) has the value $2\pi(1-\delta)$, where $\delta > 0$. Roughly speaking, the idea is to show that there are too many large chordal swings in a forward direction for this to be possible. To be precise, choose k_0, an integer, such that $\delta > 2^{-k_0}$ and choose an integer $p > 2/\delta$. We can find real numbers

$$u_1 < x_1 < y_1 < u_2 < x_2 < y_2 < \ldots < u_1+2\pi$$

such that

$$f(e^{iu_k}) = b_{2k-1}, \qquad f(e^{ix_k}) = a_{2k-1}, \qquad f(e^{iy_k}) = b_{2k}.$$

The forward chordal swings we are concerned with are given by

$$\arg\left(f(e^{iy_k})-f(e^{iu_k})\right)-\arg\left(f(e^{ix_k})-f(e^{iu_k})\right) = 2\pi-\pi/2^{k-1}$$

and with the notation of section 2 we can find a value of r $(0 < r < 1)$ such that

$$h_r(y_k, u_k)-h_r(x_k, u_k) \geqslant 2\pi-\pi/2^{k-2}$$

for $1 \leqslant k \leqslant k_0+p$. If $t_r(\theta_1, \theta_2)$ is the corresponding function for $\phi(z)$, then since ϕ is convex, $t_r(\theta_1, \theta_2)$ increases in θ_1 for fixed θ_2. It is also symmetric in the two variables. We thus have for $k_0 < k \leqslant k_0+p$

$$\begin{aligned} t_r(y_k, u_k)-t_r(x_k, u_k) &= \left(t_r(y_k, u_k)-h_r(y_k, u_k)\right)-\left(t_r(x_k, u_k)-h_r(x_k, u_k)\right) \\ &\quad +\left(h_r(y_k, u_k)-h_r(x_k, u_k)\right) \\ &\geqslant -2\pi(1-\delta)+2\pi-\frac{\pi}{2^{k-2}} = 2\pi\left(\delta-\frac{1}{2^k}\right) > \pi\delta. \end{aligned}$$

Hence

$$
\begin{aligned}
p\pi\delta &< \sum_{k=k_0+1}^{k_0+p} \big(t_r(y_k, u_k) - t_r(x_k, u_k)\big) \\
&\leqslant \sum_{k=k_0+1}^{k_0+p} (t_r(u_{k+1}, u_k) - t_r(u_k, u_k) \\
&\leqslant \sum_{k=k_0+1}^{k_0+p} \big(t_r(u_{k+1}, u_{k+1}) - t_r(u_k, u_k)\big) \\
&= t_r(u_{k_0+p+1}, u_{k_0+p+1}) - t_r(u_{k_0+1}, u_{k_0+1}) \\
&\leqslant 2\pi.
\end{aligned}
$$

This gives $p < 2/\delta$, a contradiction. This completes the proof.

5. We conclude by establishing a few miscellaneous results.

(5.1) *If* $f(z)$ *is linearly accessible in* $|z| < 1$, *then for each* z_0 ($|z_0| < 1$) *the function*

$$\int_0^z \frac{f(\zeta)-f(z_0)}{\zeta - z_0}\, d\zeta$$

is close-to-convex in $|z| < 1$.

This is an immediate consequence of (3.1.2).

(5.2) THEOREM. *If* $f(z)$ *is linearly accessible in* $|z| < 1$, *then*

$$\text{(5.2.1)} \qquad \left|\arg\left(\frac{f(z)-f(\zeta)}{z-\zeta} \cdot \frac{\zeta}{f(\zeta)}\right)\right| < \frac{3\pi}{2} \qquad (|z| < 1, |\zeta| < 1).$$

In particular, for $f \in B$ *we have*

$$\text{(5.2.2)} \qquad \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{3\pi}{2} \qquad (|z| < 1);$$

$$\text{(5.2.3)} \qquad \left|\arg \frac{f(z)}{z}\right| < \frac{3\pi}{2} \qquad (|z| < 1).$$

Proof. We consider the principal branch of $\arg(1-t)$ for $|t| < 1$ and observe that it extends continuously to $|t| = 1$, $t \neq 1$, to give

$$\arg(1-e^{i\theta}) = -\frac{\pi}{2} + \frac{\theta}{2} \qquad (0 < \theta < 2\pi).$$

Writing $z = re^{i\phi}$ we have

$$
\begin{aligned}
&\arg\left((1-e^{i\theta})\frac{f(ze^{i\theta})-f(\zeta)}{ze^{i\theta}-\zeta} \cdot \frac{z-\zeta}{f(z)-f(\zeta)}\right) \\
&= -\frac{\pi}{2} + \tfrac{1}{2}\theta + \arg\frac{f(re^{i(\phi+\theta)})-f(\zeta)}{re^{i(\phi+\theta)}-\zeta} - \arg\frac{f(re^{i\phi})-f(\zeta)}{re^{i\phi}-\zeta}
\end{aligned}
$$

and from Theorem 2.2 we see that this does not exceed $3\pi/2$ in absolute value for $0 < \theta < 2\pi$. Thus

$$\mathrm{Re}\left((1-t)\frac{f(zt)-f(\zeta)}{zt-\zeta}\,\frac{z-\zeta}{f(z)-f(\zeta)}\right)^{1/3} \geqslant 0$$

for $|t| = 1$, and therefore by the maximum principle this inequality holds strictly for $|t| < 1$. Putting $t = 0$ we obtain (5.2.1). The remaining inequalities follow immediately.

(5.3) THEOREM. *Let $f(z) = \sum_1^\infty a_n z^n$ be linearly accessible in $|z| < 1$ and for each $n = 1, 2, \ldots$ let*

$$P_n(z) = \sum_{k=1}^{n} a_k z^k \tag{5.3.1}$$

denote the n^{th} partial sum. We then have

$$\left|1 - \frac{P_n(z)}{f(z)}\right| \leqslant (2n+1)|z|^n \qquad (|z| < 1) \tag{5.3.2}$$

with equality only when $z = 0$. We deduce that

$$|a_n| \leqslant (2n+1)|f(z)| + (2n-1)\left|\frac{f(z)}{z}\right| \tag{5.3.3}$$

which gives

$$|a_n| < (2n-1)|a_1| \qquad (n = 2, 3, \ldots), \tag{5.3.4}$$

$$|a_n| \leqslant 4dn \qquad (n = 1, 2, \ldots), \tag{5.3.5}$$

where d is the distance from the origin of the complement of the image domain of f.

Proof. By Theorem 3.1, for each ζ ($|\zeta| < 1$) we can write

$$z\frac{f(z)-f(\zeta)}{z-\zeta} = g(z)\,F(z) \qquad (|z| < 1),$$

where $g(z) = \sum_1^\infty b_n z^n$ is starlike of order $\frac{1}{2}$ and $F(z) = \sum_0^\infty c_n z^n$ has positive real part. This implies that

$$|b_n| \leqslant |b_1|; \qquad |c_n| \leqslant 2|c_0| \tag{5.3.6}$$

for $n = 1, 2, \ldots$. Now it is easily verified that

$$z\frac{f(z)-f(\zeta)}{z-\zeta} = \sum_{n=1}^{\infty}\left(\frac{f(\zeta)-P_{n-1}(\zeta)}{\zeta^n}\right)z^n$$

where we have set $P_0(\zeta) = 0$. We therefore obtain

$$\frac{f(\zeta)-P_n(\zeta)}{\zeta^{n+1}} = b_{n+1}c_0+b_n c_1+\dots+b_1 c_n$$

and hence

$$\left|\frac{f(\zeta)-P_n(\zeta)}{\zeta^{n+1}}\right| \leqslant (2n+1)|b_1 c_0| = (2n+1)\left|\frac{f(\zeta)}{\zeta}\right|.$$

We immediately deduce (5.3.2). If for some n and $z \neq 0$ we have equality, then by the maximum principle

$$1-\frac{P_n(z)}{f(z)} = (2n+1)tz^n \qquad (|z|<1)$$

where $|t| = 1$. Simple manipulations then yield the relation

$$a_{n+k} = (2n+1)ta_k \qquad (k = 1, 2, \dots)$$

and in particular

$$|a_{2n+1}| = (2n+1)^2|a_1|.$$

This contradicts the inequality $|a_k| \leqslant (2k-1)|a_1|$ for the case $k = 2n+1$. The remaining relations are all easily seen.

(5.4) *Remark.* It is worth noting that for the Koebe function $f(z) = z(1-z)^{-2}$ we obtain the sharp inequalities

$$\left|1-\frac{P_n(z)}{f(z)}\right| \leqslant n|z|^{n+1}+(n+1)|z|^n$$

and so the factor $2n+1$ cannot be improved, and it appears that only a slight improvement in (5.3.2) is needed to give the Bieberbach conjecture.

(5.5) Finally we state a conjecture whose validity would yield solutions to many linear extremal problems for B and would give, in particular, the Bieberbach conjecture for the class. Our conjecture is that the extreme points of the closed convex hull of the family B are close-to-convex. From what is known about extreme points of classes of schlicht functions it is natural to expect in our case for these to be linearly accessible functions whose image domains have empty exteriors. Clearly such functions are close-to-convex. Since the extreme points of the close-to-convex functions are known **[2]**, our conjecture is equivalent to asserting for each $f \in B$ the existence of a positive measure μ of mass 1 on the torus $T^2 = \{|x| = 1\}\times\{|y| = 1\}$ such that

$$f(z) = \int_{T^2} \frac{z-\frac{1}{2}(x+y)z^2}{(1-xz)^2}\,d\mu. \tag{5.5.1}$$

This representation would, for example, be valid if there were a function $\phi(z)$ satisfying $\mathrm{Re}\,\big(\phi(z)/(z\phi'(0))\big) > \frac{1}{2}$ such that $\mathrm{Re}\,\big(f'(z)/\phi'(z)\big) > 0$.

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