ON CONVEX UNIVALENT FUNCTIONS

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Introduction

Let $g(z) = \sum_1^{\infty} b_n z^n$ be regular and univalent in the unit disc $\{|z| < 1\}$, and suppose that $w = g(z)$ maps the disc onto a convex domain D. Then $g(z)$ is said to be *convex univalent* or simply *convex*. It is well known (see e.g. [2]) that $g(z)$ is convex if, and only if,

$$\operatorname{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\} > 0, \quad (|z| < 1), \tag{1}$$

and $g'(0) \neq 0$. This paper is divided into two sections. In the first section we show (Theorem 1) that for each fixed z_0, $|z_0| < 1$, the function

$$z\left(\frac{g(z)-g(z_0)}{z-z_0}\right)^2$$

is starlike in the variable z for $|z| < 1$. (1) follows easily from this, although that (1) implies this result does not seem to be obvious.

In §2 we assume that $|g(z)| < B$ and prove (Theorem 2) that in this case

$$b_n = o(n^{-1-\delta-1/2B}),$$

where $\delta > 0$ depends only on B. This improves an earlier result due to Pommerenke [4].

Section 1

THEOREM 1. *Let $g(z)$ be convex in $|z| < 1$. If $|z_1| < 1$ and $|z_2| < 1$, then*

$$\operatorname{Re}\left\{\frac{2z_1 g'(z_1)}{g(z_1)-g(z_2)} - \frac{z_1+z_2}{z_1-z_2}\right\} > 0. \tag{2}$$

Remark 1. As $z_1 \to z_2$

$$\frac{2z_1 g'(z_1)}{g(z_1)-g(z_2)} - \frac{z_1+z_2}{z_1-z_2} \to 1+\frac{z_2 g''(z_2)}{g'(z_2)},$$

so the left-hand expression is understood to be equal to the right-hand expression for $z_1 = z_2$.

Remark 2. When $z_2 = 0$ we obtain the known inequality

$$\operatorname{Re}\frac{z_1 g'(z_1)}{g(z_1)} > \tfrac{1}{2}, \tag{3}$$

see [7]. J. Clunie has recently given in an unpublished letter to the author a very simple and elegant proof of (3).

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Remark 3. We can reverse the roles of z_1 and z_2 in (2), and by adding (2) to the resulting inequality we deduce

$$\operatorname{Re}\frac{z_1 g'(z_1)-z_2 g'(z_2)}{g(z_1)-g(z_2)} > 0. \tag{4}$$

This is also known: see [3].

Proof of Theorem 1. Choose $R < 1$ satisfying $|z_1| < R$ and $|z_2| < R$. Let z_0 be a point satisfying $|z_0| = R$. Now $w = g(z)$ maps $|z| = R$ onto a convex curve C_R. If $w \in C_R$ and $w \neq w_0 = g(z_0)$ then we can define the "*chordal angle*" $\arg(w-w_0)$ in the following manner. Let $z_0 = Re^{i\theta_0}$ and $w = g(z)$, where $z = Re^{i\theta}$ and $\theta_0 < \theta < \theta_0+2\pi$. Then, because

$$\frac{z-z_0}{\sqrt{(zz_0)}} = 2i\sin\frac{\theta-\theta_0}{2},$$

the angle is given by

$$\arg\frac{g(z)-g(z_0)}{z-z_0} + \frac{\pi}{2} + \tfrac{1}{2}(\theta+\theta_0), \tag{5}$$

where because g is univalent we can define a single-valued branch of

$$\arg\frac{g(z)-g(z_0)}{z-z_0}$$

for all z and z_0 in the disc, choosing the principal value of $\arg g'(0)$ to determine the argument.

More generally allowing for a continuous variation in the chordal angle except for jumps of π at w_0, we have

$$\arg\left(g(Re^{i\theta})-g(Re^{i\theta_0})\right) = \arg\frac{g(z)-g(z_0)}{z-z_0} + n\pi + \frac{\pi}{2} + \tfrac{1}{2}(\theta+\theta_0), \tag{6}$$

for $\theta_0+2n\pi < \theta < \theta_0+(2n+2)\pi$, and each integer n. Because C_R is convex this chordal angle increases with θ, so if $\theta_0 < \theta_1 < \theta_2 < \theta_0+2\pi$ we deduce that

$$\arg\frac{g(Re^{i\theta_2})-g(Re^{i\theta_0})}{Re^{i\theta_2}-Re^{i\theta_0}} - \arg\frac{g(Re^{i\theta_1})-g(Re^{i\theta_0})}{Re^{i\theta_1}-Re^{i\theta_0}} + \tfrac{1}{2}(\theta_2-\theta_1) > 0,$$

i.e.

$$\arg Re^{i\theta_2}\left(\frac{g(Re^{i\theta_2})-g(Re^{i\theta_0})}{Re^{i\theta_2}-Re^{i\theta_0}}\right)^2 - \arg Re^{i\theta_1}\left(\frac{g(Re^{i\theta_1})-g(Re^{i\theta_0})}{Re^{i\theta_1}-Re^{i\theta_0}}\right)^2 > 0.$$

It is not difficult to prove this for all $\theta_2 > \theta_1$, so we deduce that the function

$$F(z;\, z_0) = z\left(\frac{g(z)-g(z_0)}{z-z_0}\right)^2$$

is starlike for $|z| \leqslant R$. This implies that

$$\operatorname{Re}\frac{zF'(z)}{F(z)} > 0, \quad (|z| \leqslant R),$$

i.e.

$$\operatorname{Re}\left\{1+\frac{2zg'(z)}{g(z)-g(z_0)} - \frac{2z}{z-z_0}\right\} > 0, \quad (|z| \leqslant R).$$

In particular

$$\operatorname{Re}\left\{1+\frac{2z_1 g'(z_1)}{g(z_1)-g(z_0)}-\frac{2z_1}{z_1-z_0}\right\}>0.$$

The inequality holds at all points z_0 on $|z| = R$ and the expression is harmonic in z_0, so the inequality holds replacing z_0 by z_2. The theorem is proved.

Remark 4. The theorem is equivalent to the statement that for each z_0, $|z_0| < 1$, the function

$$F(z;\, z_0)=z\left(\frac{g(z)-g(z_0)}{z-z_0}\right)^2$$

is starlike univalent in $|z| < 1$. Such a function satisfies

$$\operatorname{Re}\left\{\frac{F(z)}{zF'(0)}\right\}^{\frac{1}{2}}>\tfrac{1}{2}, \tag{7}$$

see [5]. Thus we have

$$\operatorname{Re}\left\{\frac{g(z)-g(z_0)}{z-z_0}\cdot\frac{z_0}{g(z_0)}\right\}>\tfrac{1}{2}. \tag{8}$$

If $g'(0) = 1$ we deduce setting $z_0 = 0$ the known result [5]

$$\operatorname{Re}\frac{g(z)}{z}>\tfrac{1}{2},\quad (|z|<1). \tag{9}$$

We also deduce the distortion theorem:

$$\frac{1}{(1+R)^2}\leqslant\left|\frac{g(z_1)-g(z_2)}{z_1-z_2}\right|\leqslant\frac{1}{(1-R)^2}, \tag{10}$$

for $|z_1| \leqslant R < 1$, $|z_2| \leqslant R < 1$; using the usual inequalities for functions with positive real part.

Section 2

Let

$$g(z)=z+\sum_2^\infty b_n z^n,\quad (|z|<1), \tag{11}$$

be convex in $|z| < 1$. In this section we shall study the coefficients b_n. It is well known [2] that the sharp inequality

$$|b_n|\leqslant 1 \tag{12}$$

is always satisfied. If $g(z)$ is bounded in $|z| < 1$, J. Clunie and F. Keogh [1] have shown that

$$b_n=o(n^{-1}) \text{ as } n\to\infty. \tag{13}$$

Pommerenke [4] has further proved that if

$$|g(z)|<B,\quad (|z|<1), \tag{14}$$

then for each $\varepsilon > 0$ there is a constant A depending only on B and ε such that

$$|b_n|<An^{-1-(1-\varepsilon)/\pi B}. \tag{15}$$

We shall prove

THEOREM 2. *Let*

$$g(z) = z + \sum_2^\infty b_n z^n \tag{16}$$

be convex for $|z| < 1$, *and satisfy*

$$|g(z)| < B, \quad (|z| < 1). \tag{17}$$

Then there is $\delta > 0$, *depending only on B, such that*

$$b_n = o(n^{-1-\delta-1/2B}) \text{ as } n \to \infty. \tag{18}$$

Furthermore, given $\varepsilon > 0$, *there is* $B = B(\varepsilon)$ *and a convex function* $g(z)$ *satisfying* (16) *and* (17), *and an increasing sequence* $\{n_k\}$ *of natural numbers such that*

$$n_k^{1+1/(2-\varepsilon)B} |b_{n_k}| \to \infty \text{ as } k \to \infty. \tag{19}$$

Remark 5. (18) essentially reduces Pommerenke's constant π in (15) to the constant 2. (19) shows that the constant 2 cannot be further reduced independently of B.

Remark 6. We shall require the following theorem proved by Pommerenke [**4**] and based on the work of Clunie and Keogh [**1**].

THEOREM A. *Let* $f(z) = \sum_1^\infty a_n z^n$ *be starlike univalent in* $|z| < 1$, *and write*

$$M(r) = \max_{|z|=r} |f(z)|, \quad (0 \leqslant r < 1).$$

If

$$M(r) \leqslant (1-r)^{-\alpha}, \quad (0 \leqslant r < 1), \tag{20}$$

where $0 \leqslant \alpha \leqslant 2$, *then*

$$|a_n| < 2(e/\alpha)^\alpha (n+1)^{\alpha-1}, \quad (n > 1). \tag{21}$$

Proof of Theorem 2. We write

$$f(z) = zg'(z) = z + \sum_1^\infty a_n z^n, \quad (|z| < 1), \tag{22}$$

so that $f(z)$ is starlike for $|z| < 1$, and

$$nb_n = a_n, \quad n = 2, 3, \dots . \tag{23}$$

We have to show that

$$|a_n| = o(n^{-1/(2B)-\delta}),$$

and by Theorem A, to prove this it will be sufficient to show that

$$\alpha = \lim_{r\to 1} \frac{\log M(r)}{\log 1/(1-r)} < 1 - 1/2B. \tag{24}$$

The limit in (24) exists, as shown by Pommerenke [**4**], and satisfies $0 \leqslant \alpha \leqslant 2$. By the Schwarz lemma, $B \geqslant 1$, so if $\alpha = 0$ there is nothing to prove. If $\alpha > 0$, then the largest sector of the form

$$\{w/\lambda_1 < \arg w < \lambda_2\}$$

contained in the image domain of $w = f(z)$ has angle $\pi\alpha$, and there is κ, $|\kappa| = 1$, such that

$$f(z) = \frac{h(z)}{(1-\kappa z)^{\alpha}}, \quad (|z| < 1), \tag{25}$$

where

$$\operatorname{Re}\frac{zh'(z)}{h(z)} \geqslant \frac{\alpha}{2}, \quad (|z| < 1). \tag{26}$$

From this we easily deduce that

$$|f(z)| \geqslant \frac{|z|}{(1+|z|)^{2-\alpha}|1-\kappa z|^{\alpha}}, \quad (|z| < 1). \tag{27}$$

These results can be found in [6]. Without loss of generality we may take $\kappa = 1$, so that

$$|g'(z)| \geqslant \frac{1}{(1+|z|)^{2-\alpha}|1-z|^{\alpha}}, \quad (|z| < 1). \tag{28}$$

Let D be the convex domain onto which the unit disc is mapped by the function $g(z)$. Let

$$\zeta = \lim_{r\to 1} g(r).$$

This limit exists for any bounded starlike function, r tending to 1 through real values less than 1. ζ is a boundary point of D, and since D is convex the line segment Γ joining the origin to ζ lies, except for ζ, in D. Let γ be the inverse image under g of Γ, so that γ is an arc in the closed disc $|z| \leqslant 1$ joining the origin to the point $z = 1$. Furthermore, γ meets each circle $|z| = r$, $0 \leqslant r \leqslant 1$, once and only once. Thus γ is given by

$$\gamma = \{z | z = z(r),\ 0 \leqslant r \leqslant 1\},$$

where $z(r)$ is differentiable for $0 \leqslant r < 1$. The length of Γ is given by

$$l(\Gamma) = \int_{\Gamma} |dw| = \int_{\gamma} |g'(z)|\,|dz|.$$

Thus

$$B \geqslant l(\Gamma) \geqslant \int_{\gamma} \frac{|dz|}{(1+|z|)^{2-\alpha}|1-z|^{\alpha}} = I, \tag{29}$$

applying (28). We show that $I = I(\gamma)$ is minimised if γ is the line segment joining $z = 0$ to $z = 1$. To do this we integrate I by parts as follows. Let

$$u_1(r) = \frac{1}{(1+r)^{2-\alpha}},$$

$$v_1(r) = \int_0^r \frac{|z'(\rho)|\,d\rho}{|1-z(\rho)|^{\alpha}},$$

Then

$$\begin{aligned} I &= \int_0^1 \frac{|z'(r)|\,dr}{(1+r)^{2-\alpha}|1-z(r)|^{\alpha}} \\ &= \frac{2^{\alpha}}{4} v_1(1) + (2-\alpha)\int_0^1 \frac{v_1(r)}{(1+r)^{3-\alpha}}\,dr \\ &= I_1 + J_1 \text{ say.} \end{aligned}$$

We also write

$$I' = \int_0^1 \frac{dr}{(1+r)^{2-\alpha}(1-r)^\alpha}$$

$$= \frac{2^\alpha}{4}\int_0^1 \frac{d\rho}{(1-\rho)^\alpha} + (2-\alpha)\int_0^1 \left\{ \int_0^r \frac{d\rho}{(1-\rho)^\alpha} \right\} \frac{dr}{(1+r)^{3-\alpha}}$$

$$= I_1' + J_1' \text{ say.}$$

We note that

$$I_1 \geqslant \left| \frac{2^\alpha}{4} \int_\gamma \frac{dz}{(1-z)^\alpha} \right| = I_1'.$$

We now integrate J_1 and J_1' by parts similarly, as follows. Let

$$u_2(r) = \frac{1}{(1+r)^{3-\alpha}},$$

$$v_2(r) = \int_0^r v_1(R)\,dR.$$

Then

$$J_1 = (2-\alpha)\frac{2^\alpha}{8} v_2(1) + (2-\alpha)(3-\alpha)\int_0^1 \frac{v_2(r)\,dr}{(1+r)^{4-\alpha}}$$

$$= I_2 + J_2 \text{ say.}$$

Similarly $J_1' = I_2' + J_2'$ where the expression $|z'(\rho)|\,|1-z(\rho)|^{-\alpha}\,d\rho$ is replaced by $(1-\rho)^{-\alpha}\,d\rho$.

We again note that

$$I_2 \geqslant I_2'.$$

We continue this process on J_2 and J_2', and in general we set

$$u_n(r) = \frac{1}{(1+r)^{n+1-\alpha}},$$

$$v_n(r) = \int_0^r v_{n-1}(\rho)\,d\rho,$$

obtaining

$$I = \sum_{k=1}^n I_k + J_n,$$

$$I' = \sum_{k=1}^n I_k' + J_n'.$$

and

$$I_k \geqslant I_k', \quad k = 1, \ldots, n.$$

We now show that

$$J_n' \to 0 \text{ as } n \to \infty. \tag{30}$$

We have

$$J_n' = \prod_{k=1}^n (k+1-\alpha)\int_0^1 \left\{ \int_0^{r_1} \left\{ \int_0^{r_2} \cdots \left\{ \int_0^{r_n} \frac{d\rho}{(1-\rho)^\alpha} \right\} dr_n \right\} \cdots \frac{dr_1}{(1+r_1)^{n+2-\alpha}} \right\}.$$

Now
$$\int_0^{r_n} \frac{d\rho}{(1-\rho)^\alpha} \leqslant \int_0^1 \frac{d\rho}{(1-\rho)^\alpha} = \frac{1}{1-\alpha}.$$

Hence
$$J_n' \leqslant \frac{\prod_{k=1}^{n}(k+1-\alpha)}{1-\alpha} \int_0^1 \left\{ \int_0^{r_1} \left\{ \int_0^{r_2} \cdots \left\{ \int_0^{r_{n-1}} dr_n \right\} dr_{n-1} \right\} \cdots \frac{dr_1}{(1+r_1)^{n+2-\alpha}} \right\}$$
$$\leqslant \frac{(n+1)!}{(1-\alpha)(n-1)!} \int_0^1 \frac{r_1^{\,n-1}\,dr_1}{(1+r_1)^{n+2-\alpha}}$$
$$\leqslant \frac{n(n+1)}{1-\alpha} \int_0^1 \frac{r_1^{\,n-1}\,dr_1}{(1+r_1)^{n+1}}$$
$$= \frac{n(n+1)}{1-\alpha} \int_0^{\frac{1}{2}} u^{n-1}\,du, \text{ where } u = \frac{r_1}{1+r_1},$$
$$= \frac{n+1}{2^n}\cdot\frac{1}{1-\alpha} \to 0 \text{ as } n \to \infty.$$

It follows that
$$I' = \sum_{k=1}^{\infty} I_k',$$

and so
$$I \geqslant \sum_{k=1}^{\infty} I_k \geqslant \sum_{k=1}^{\infty} I_k' = I'.$$

Hence
$$B \geqslant \int_0^1 \frac{dr}{(1+r)^{2-\alpha}(1-r)^\alpha}$$
$$= \tfrac{1}{2} \int_0^1 (1-u)^{-\alpha}\,du, \text{ where } 1-u = \frac{1-r}{1+r},$$
$$= \tfrac{1}{2}(1-\alpha)^{-1}.$$

We therefore obtain
$$\alpha \leqslant 1 - \frac{1}{2B}. \tag{31}$$

To prove (18) we need to show that equality in (31) cannot occur. For suppose that
$$B = \tfrac{1}{2}(1-\alpha)^{-1}.$$

Then
$$B = l(\Gamma) = I = I_1 + J_1 = I_1' + J_1'.$$

By our earlier argument we easily obtain
$$J_1 \geqslant J_1'$$

Hence
$$I_1' \geqslant I_1 \geqslant I_1',$$

so
$$I_1 = I_1'.$$

Thus
$$\int_\gamma \frac{|dz|}{|1-z|^\alpha} = \int_0^1 \frac{d\rho}{(1-\rho)^\alpha}. \tag{32}$$

Consider the mapping
$$W = q(z) = \frac{1}{1-\alpha}\{1-(1-z)^{1-\alpha}\},$$
so
$$zq'(z) = \frac{z}{(1-z)^\alpha}.$$
This latter function is starlike, so $q(z)$ is convex. The image under $q(z)$ of the line segment joining $z = 0$ to $z = 1$ is the line segment joining $W = 0$ to $W = (1-\alpha)^{-1}$. γ is mapped onto an arc from $W = 0$ to $W = (1-\alpha)^{-1}$ whose length is given by (32); i.e. the length of the image of γ is $(1-\alpha)^{-1}$. Hence γ is the line segment joining $z = 0$ to $z = 1$. It follows that
$$\int_0^1 |g'(x)|\,dx = \int_0^1 \frac{dx}{(1+x)^{2-\alpha}(1-x)^\alpha}.$$
But from (28)
$$|g'(x)| \geqslant \frac{1}{(1+x)^{2-\alpha}(1-x)^\alpha}, \quad 0 \leqslant x < 1.$$
Hence
$$|g'(x)| = \frac{1}{(1+x)^{2-\alpha}(1-x)^\alpha}, \quad 0 \leqslant x < 1.$$
Therefore
$$g'(x) = \frac{e^{i\phi(x)}}{(1+x)^{2-\alpha}(1-x)^\alpha}, \quad 0 \leqslant x < 1,$$
where $\phi(x)$ is real and analytic for $0 \leqslant x < 1$.

By analytic continuation we deduce that
$$g'(-x) = \frac{e^{i\phi(-x)}}{(1-x)^{2-\alpha}(1+x)^\alpha},$$
where $\phi(-x)$ is analytic and real for $0 \leqslant x < 1$. Thus
$$|g'(-x)| = \frac{1}{(1-x)^{2-\alpha}(1+x)^\alpha}.$$

It follows that the length of the image of the line segment joining $z = 0$ to $z = -1$ is given by
$$\int_0^1 \frac{dx}{(1-x)^{2-\alpha}(1+x)^\alpha} = \infty,$$
since $\alpha < 1$. This contradicts the boundedness of $|g(z)|$, for as we will show in a forthcoming paper, the length of the image of any ray is smaller than $(1+\log 2)\,B$. (18) now follows for some $\delta > 0$, which may depend on g. That δ can be taken to depend only on B is shown in Remark 7.

To complete the proof of Theorem 2 we consider the function
$$g_\alpha(z) = \int_0^z \frac{dt}{(1-t^2)^\alpha}, \quad (|z| < 1), \tag{33}$$

where $0 < \alpha < 1$. The coefficients a_n of $zg_\alpha'(z)$ are given by

$$a_{2n+3} = \frac{\alpha(\alpha+1)\dots(\alpha+n)}{(n+1)!}$$
$$> Kn^{\alpha-1}, \tag{34}$$

where $K > 0$ depends only on α and is bounded away from zero for $\alpha > \frac{1}{2}$ say. The function is clearly convex, and its upper bound is easily seen to be

$$B_\alpha = \int_0^1 \frac{dt}{(1-t^2)^\alpha}$$
$$= \tfrac{1}{2}\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\alpha}\,du, \text{ where } t^2 = u.$$

This latter integral (see [8], page 253) is given by

$$B_\alpha = \tfrac{1}{2}\frac{\Gamma(\frac{1}{2})\,\Gamma(1-\alpha)}{\Gamma(\frac{1}{2}+1-\alpha)}.$$

Thus

$$(1-\alpha)\,B_\alpha = \tfrac{1}{2}\frac{\Gamma(\frac{1}{2})\,\Gamma(2-\alpha)}{\Gamma(\frac{1}{2}+1-\alpha)}$$
$$\to \tfrac{1}{2} \text{ as } \alpha \to 1.$$

Thus given $\varepsilon > 0$ there is $B = B(\varepsilon)$ and a convex function $g(z)$, $|g(z)| < B$, such that

$$\alpha > 1 - \frac{1}{(2-\varepsilon)\,B}.$$

Applying (34), the proof of Theorem 2 is complete.

Remark 7. If we take B fixed and write $\mathscr{C}(B)$ for the class of convex functions $g(z) = z + \sum_2^\infty b_n z^n$ satisfying $|g(z)| < B$, then the essence of the coefficient problem is to maximize the number

$$\alpha = \lim_{r\to 1} \frac{\log M(r, zg')}{\log 1/(1-r)}, \quad g \in \mathscr{C}(B).$$

As Pommerenke [4] points out, this is equivalent to minimizing the angle $\pi\beta = \pi(1-\alpha)$, where $\pi\beta$ is the smallest angle made by the boundary curve of the image D of g. Since $\mathscr{C}(B)$ is a compact, normal family there will be an extremal function $g^*(z)$ maximizing $\pi\alpha$. So there is a definite function $\tau(B)$ such that

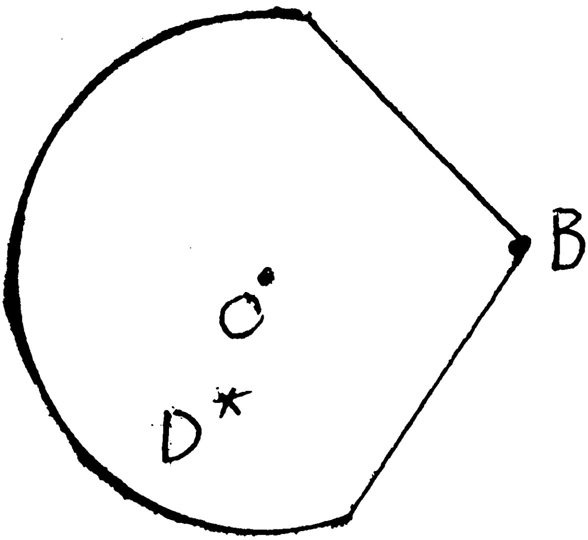
$$\alpha \leqslant \tau(B),$$

where equality occurs for g^*. We have shown that

$$\tau(B) < 1 - \frac{1}{2B},$$

and we deduce that the δ of Theorem 2 depends only on B. Let $\beta^* = 1-\tau(B)$. We conjecture that the image of g^* may be taken to be

$$D^* = \{|w| < B\} \cap \left\{|\arg(w-B)-\pi| < \frac{\pi\beta^*}{2}\right\}.$$



$\tau(B)$ is clearly increasing with B. If we assume that $\tau(B)$ is strictly increasing and that g^* can be taken to be symmetric about the real axis, then D^* is indeed the required extremal domain. For if E^* is the extremal domain with extremal angle at B, then $E^* \subset D^*$. If h^* is the convex map onto D^*, then g^* is subordinate to h^*. Hence

$$h^{*\prime}(o) = \rho > 1,$$

unless $h^* = g^*$. But if $\rho > 1$, then $\rho^{-1} h^* \in \mathscr{C}(\rho^{-1} B)$ and has the extremal angle of $\mathscr{C}(B)$, contradicting $\tau(B)$ being strictly increasing.

References

1. J. Clunie and F. R. Keogh, " On starlike and convex schlicht functions ", *J. Lond. Math. Soc.*, 35 (1960), 229–233.
2. W. K. Hayman, *Multivalent functions* (Cambridge, 1958).
3. Ch. Pommerenke, " On close-to-convex analytic functions ", *Trans. Amer. Math. Soc.*, 114 (1965).
4. ———, " On starlike and convex functions ", *J. London Math. Soc.*, 37 (1962), 209–224.
5. M. S. Robertson, " The theory of univalent functions ", *Ann. of Math.*, 37, 2 (1936).
6. T. Sheil-Small, " Starlike univalent functions ". To be published.
7. E. Strohhacker, " Beitrag zur Theorie der schlichten Funktionen ", *Math. Z.* 37 Band, 3 heft (1933), S.362.
8. E. T. Whittaker and G. N. Watson, *A course of modern analysis* (Cambridge, 1963).

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