ON BAZILEVIČ FUNCTIONS

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1. WITH the help of the Löwner–Kufarev differential equation Bazilevič (**1**) gave an explicit construction for a class of functions analytic and univalent in the unit disc [see also (**3**)]. His result was as follows.

THEOREM 1. *Let* $g(z)$ *be analytic and starlike univalent in* $|z| < 1$, *with* $g(0) = 0$, *and let* $h(z)$ *be analytic and satisfy* $\mathrm{re}(e^{i\lambda}h(z)) > 0$ *in* $|z| < 1$ *for some real number* λ. *Then if* $\alpha > 0$ *and* β *is real, the function*

$$f(z) = \left\{\int_0^z g^{\alpha}(\zeta)h(\zeta)\zeta^{i\beta-1}\,d\zeta\right\}^{1/(\alpha+i\beta)} \tag{1}$$

is analytic and univalent in $|z| < 1$.

The main purpose of this note is to give a proof of Bazilevič's result based on geometrical ideas and at the same time to provide an 'intrinsic' characterization of this class of functions along the lines of Kaplan's characterization of the class of close-to-convex functions in (**2**) (the case $\alpha = 1$, $\beta = 0$ here). A function $f(z)$ of the form (1) will be called *Bazilevič of type* (α, β). We shall establish

THEOREM 2. *Let* $f(z)$ *be Bazilevič of type* (α,β). *Then, for each* r $(0 < r < 1)$,

$$\int_{\theta_1}^{\theta_2} \left\{1+\mathrm{re}\,\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}+(\alpha-1)\mathrm{re}\,\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}-\beta\,\mathrm{im}\,\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right\} d\theta > -\pi \tag{2}$$

whenever $\theta_2 > \theta_1$. *Conversely, if* $f(z)$ *is analytic in* $|z| < 1$, *with* $f(0) = 0$, $f(z) \neq 0$ $(0 < |z| < 1)$, *and* $f'(z) \neq 0$ *for* $|z| < 1$, *and if* $f(z)$ *satisfies the condition* (2) *for* $0 < r < 1$, *where* $\alpha \geqslant 0$ *and* β *is real, then* $f(z)$ *is univalent in* $|z| < 1$, *and is Bazilevič of type* (α,β) *in the case* $\alpha > 0$.

We must distinguish between two different points of view. The first centres on the condition (1) as a means of constructing a univalent function given $g(z)$, $h(z)$, α and β. The second centres on the condition (2) as defining a class of univalent functions. In the second condition α can be zero, but we will show that given h and β we can still construct a corresponding function f for which (2) holds with $\alpha = 0$ (the one restriction being that $h(0) = 1$ when $\beta = 0$). When this has been accomplished, we can justifiably refer to the class of Bazilevič functions of type (α, β), where $\alpha \geqslant 0$ and β is real, using (2) as our defining condition. We will show that *this latter class is compact* (with the usual normalization $f'(0) = 1$).

2. Let us begin by assuming that g, h, $\alpha > 0$, and β are given as in Theorem 1. We cannot study f given by (1) directly as it is not clear *a priori* that the function is without branch points. We therefore construct f in stages. We first set

$$F(z) = \left(\frac{g(z)}{z}\right)^{\alpha} h(z) = \sum_{n=0}^{\infty} c_n z^n,$$

for a suitable branch of the non-vanishing function $(g(z)/z)^{\alpha}$. Let

$$G(z) = \sum_{n=0}^{\infty} \frac{c_n}{n+\alpha+i\beta} z^n,$$

so that $G(z)$ is analytic in $|z| < 1$ and satisfies the differential equation

$$(\alpha+i\beta)G(z)+zG'(z) = F(z). \tag{3}$$

Notice that $c_0 \neq 0$ and so $G(0) \neq 0$. Let

$$R = \sup\{r : G(z) \neq 0 \text{ for } |z| \leqslant r < 1\}, \tag{4}$$

so that $0 < R \leqslant 1$. A crucial part of our argument will be in showing that $R = 1$. Meanwhile $G(z) \neq 0$ for $|z| < R$. For $|z| < R$ we define $f(z)$ by the equation

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{\alpha+i\beta} \frac{zG'(z)}{G(z)} = \frac{1}{\alpha+i\beta} \frac{F(z)}{G(z)}. \tag{5}$$

It is easily verified that apart from a constant factor this defines a branch of the formula (1) analytic for $|z| < R$, and conversely that any such branch must satisfy (5). Note also that $f(0) = 0$, $f(z) \neq 0$ for $0 < |z| < R$, and $f'(z) \neq 0$ for $|z| < R$.

3. The proof of Theorem 1 thus reduces to showing that $R = 1$ and that $f(z)$ is univalent for $|z| < R$. The fact that $R = 1$ is actually a consequence of the univalence for $|z| < R$. To see this, suppose that $R < 1$. Then there exists z_0 ($|z_0| = R$) such that $G(z_0) = 0$. Since $F(z) \neq 0$ in $|z| < 1$, $G'(z_0) \neq 0$, and so $G(z) = (z-z_0)H(z)$, where $H(z)$ is analytic for $|z| < 1$ and $H(z_0) \neq 0$. Thus

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{\alpha+i\beta}\left(\frac{z}{z-z_0} + \frac{zH'(z)}{H(z)}\right),$$

$$f(z) = (z-z_0)^{1/(\alpha+i\beta)} K(z)$$

in a region $D = \{|z-z_0| < \delta\} \cap \{|z| < R\}$, where $K(z)$ is analytic and bounded in D. Thus in D

$$|f(z)| = |z-z_0|^{\alpha/(\alpha^2+\beta^2)}|K(z)|\exp\left\{-\beta\frac{\arg(z-z_0)}{\alpha^2+\beta^2}\right\},$$

and hence, since $\alpha > 0$, $|f(z)| \to 0$ as $z \to z_0$ along the path in D for which $\arg(z-z_0)$ is constant. But $f(0) = 0$, and hence if $f(z)$ is univalent in $|z| < R$, then there exists $\epsilon > 0$ such that $|f(z)| \geqslant \epsilon|z|$ for $|z| < R$. This establishes our assertion and means that to prove Theorem 1 we need only show that $f(z)$ is univalent for $|z| < R$.

4. To do this we first establish the first part of Theorem 2 and show that the condition (2) holds for $|z| < R$. We have by (5) for $|z| < R$,

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} = \frac{zF'(z)}{F(z)}-\frac{zG'(z)}{G(z)}$$
$$= \alpha\frac{zg'(z)}{g(z)}-\alpha+\frac{zh'(z)}{h(z)}-(\alpha+i\beta)\left(\frac{zf'(z)}{f(z)}-1\right),$$

and we obtain

$$1+\frac{zf''(z)}{f'(z)}+(\alpha-1+i\beta)\frac{zf'(z)}{f(z)} = \alpha\frac{zg'(z)}{g(z)}+\frac{zh'(z)}{h(z)}+i\beta.$$

Thus, taking real parts,

$$1+\operatorname{re}\frac{zf''(z)}{f'(z)}+(\alpha-1)\operatorname{re}\frac{zf'(z)}{f(z)}-\beta\operatorname{im}\frac{zf'(z)}{f(z)}$$
$$= \alpha\operatorname{re}\frac{zg'(z)}{g(z)}+\operatorname{re}\frac{zh'(z)}{h(z)}.$$

Since g is starlike, $\operatorname{re}(zg'(z)/g(z)) > 0$, and therefore since $\operatorname{re}(e^{i\lambda}h(z)) > 0$, we obtain (2) on integration.

5. The proof of Theorem 1 will thus be complete if we establish the second part of Theorem 2, and show that the condition (2) holding for $0 < r < R$, where $\alpha \geqslant 0$ and β is real, implies the univalence of $f(z)$ for $|z| < R$. Let

$$\rho = \sup\{r: f(z) \text{ is univalent for } |z| \leqslant r < R\}.$$

We have to show that $\rho = R$. By hypothesis $f'(z) \neq 0$ and so $\rho > 0$. Suppose that $\rho < R$. Then $f(z)$ is not univalent for $|z| = \rho$, and we can find θ_1 and θ_2 satisfying $\theta_1 < \theta_2 < \theta_1+2\pi$ such that $f(\rho e^{i\theta_1}) = f(\rho e^{i\theta_2})$. We denote by γ_1 the closed curve $\{f(\rho e^{i\theta})\colon \theta_1 \leqslant \theta \leqslant \theta_2\}$ and by γ_2 the closed curve $\{f(\rho e^{i\theta})\colon \theta_2 \leqslant \theta \leqslant \theta_1+2\pi\}$. The winding number of a closed curve γ about a point w will be denoted by $n(\gamma, w)$, and the expression on the left-hand side of (2) by $I_r(\theta_1, \theta_2)$. We have

$$n(\gamma_1, 0) = \frac{1}{2\pi i}\int\limits_{\gamma_1} \frac{dw}{w} = \frac{1}{2\pi}\int\limits_{\theta_1}^{\theta_2} \frac{\rho e^{i\theta}f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}\,d\theta,$$

$$n(\gamma_2, 0) = \frac{1}{2\pi}\int\limits_{\theta_2}^{\theta_1+2\pi} \frac{\rho e^{i\theta}f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}\,d\theta.$$

Thus

$$I_\rho(\theta_1, \theta_2) = \int\limits_{\theta_1}^{\theta_2} \left\{1+\mathrm{re}\,\frac{\rho e^{i\theta}f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\right\}d\theta + 2\pi(\alpha-1)n(\gamma_1, 0),$$

$$I_\rho(\theta_2, \theta_1+2\pi) = \int\limits_{\theta_2}^{\theta_1+2\pi} \left\{1+\mathrm{re}\,\frac{\rho e^{i\theta}f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\right\}d\theta + 2\pi(\alpha-1)n(\gamma_2, 0).$$

Now if Γ is any simple smooth path joining $\rho e^{i\theta_1}$ to $\rho e^{i\theta_2}$ and otherwise lying in $|z| < \rho$, then $f(\Gamma) = \gamma$ is a smooth simple closed path. If σ denotes the arc $\{\rho e^{i\theta}, \theta_1 \leqslant \theta \leqslant \theta_2\}$ and if 0 does not lie inside nor on the closed curve formed by Γ and σ, then since $f'(z)/f(z)$ is analytic in the region bounded by this curve, we can apply Cauchy's theorem to deduce that

$$n(\gamma_1, 0) = n(\gamma, 0).$$

It follows that $n(\gamma_1, 0) = 0$, 1, or -1 and similarly $n(\gamma_2, 0) = 0$, 1, or -1. Also $n(\gamma_1, 0)+n(\gamma_2, 0) = 1$, and hence either (i) $n(\gamma_1, 0) = 0$ and $n(\gamma_2, 0) = 1$, or (ii) $n(\gamma_1, 0) = 1$ and $n(\gamma_2, 0) = 0$. Without loss of generality we may assume that (i) holds. Thus

$$I = I_\rho(\theta_1, \theta_2) = \int\limits_{\theta_1}^{\theta_2} \left\{1+\mathrm{re}\,\frac{\rho e^{i\theta}f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\right\}d\theta.$$

Now I represents the change in tangent as γ_1 is traversed. Since $\arg f'(z)$ is single-valued for $|z| < R$ and since the mapping is conformal, this change in tangent will remain the same for the simple smooth curve γ mentioned previously, provided that Γ has the same initial and final tangential directions as the arc σ, namely $\theta_1+\frac{1}{2}\pi$ and $\theta_2+\frac{1}{2}\pi$. However, γ is a simple closed curve described negatively: indeed, if z_0 lies inside the region bounded by Γ and σ, and if $w_0 = f(z_0)$, then

$$n(\gamma, w_0)+n(\gamma_2, w_0) = \frac{1}{2\pi i}\int\limits_{\Gamma+\tau} \frac{f'(z)}{f(z)-w_0}\,dz = 0,$$

where

$$\tau = f^{-1}(\gamma_2) = \{\rho e^{i\theta}\colon \theta_2 \leqslant \theta \leqslant \theta_1+2\pi\}.$$

Also $$n(\gamma_2, w_0) = n(\gamma_2, 0) = 1,$$

and so $n(\gamma, w_0) = -1$. We show that this implies that the change in tangent as γ is traversed is $-\pi$, giving $I = -\pi$, which contradicts the hypothesis (2). Let $w = k(\zeta)$ be a 1–1 conformal mapping of $|\zeta| < 1$ on to the region bounded by γ. Then $k(\zeta)$ extends continuously and univalently to $|\zeta| \leqslant 1$, mapping $|\zeta| = 1$ onto γ, with $\zeta = 1$ say corresponding to $W = f(\rho e^{i\theta_1}) = f(\rho e^{i\theta_2})$. Since γ is smooth, $\arg k'(\zeta)$ exists and is continuous on $|\zeta| = 1$, except for a jump at $\zeta = 1$. The value of this jump is clearly π, in view of the conformality of f, and since the total change in tangent of the boundary curve for $|\zeta| = 1$ described positively is 2π by the argument principle, our assertion follows. The contradiction which we obtain implies that $\rho = R$ and so the proof of Theorem 1 is complete.

6. To complete the proof of Theorem 2 we must show that the condition (2) holding for $\alpha > 0$ implies that $f(z)$ is Bazilevič of type (α, β). The proof of this is an adaptation of Kaplan's proof in (**2**) of the same result in the close-to-convex case. We choose a branch of $\log(f(z)/z)$ and of $\log f'(z)$ and put

$$t(\theta) = t_r(\theta) = \alpha\theta + \arg f'(re^{i\theta}) + (\alpha-1)\arg\frac{f(re^{i\theta})}{re^{i\theta}} + \beta\log\left|\frac{f(re^{i\theta})}{re^{i\theta}}\right|.$$

Then condition (2) can be written

$$t(\theta_2) - t(\theta_1) > -\pi \quad \text{whenever} \quad \theta_2 > \theta_1. \tag{6}$$

Notice also that $$t(\theta+2\pi) = t(\theta) + 2\pi\alpha.$$

We define $$s(\theta) = \tfrac{1}{2}\pi + \inf_{\theta' > \theta} t(\theta'),$$

and observe that $s(\theta)$ is increasing with θ, $s(\theta+2\pi) = s(\theta) + 2\pi\alpha$, and

$$|t(\theta) - s(\theta)| \leqslant \tfrac{1}{2}\pi. \tag{7}$$

We next set $p(\theta) = (1/\alpha)s(\theta) - \theta$, so that $p(\theta)$ is periodic with period 2π and $\theta + p(\theta)$ is increasing with θ. We can construct a function $g_r(z)$ analytic and starlike univalent for $|z| < r$ with $|g_r'(0)| = |f'(0)|$ such that

$$\arg\frac{g_r(re^{i\theta})}{re^{i\theta}} = p(\theta)$$

(e.g. by means of the Integral Representation formula), and applying (7) together with the maximum principle observe that

$$\left|\arg f'(\rho e^{i\theta}) + (\alpha-1)\arg\frac{f(\rho e^{i\theta})}{\rho e^{i\theta}} + \beta\log\left|\frac{f(\rho e^{i\theta})}{\rho e^{i\theta}}\right| - \alpha\arg\frac{g_r(\rho e^{i\theta})}{\rho e^{i\theta}}\right| \leqslant \tfrac{1}{2}\pi$$

for $0 \leqslant \rho \leqslant r$. We then choose a sequence $\{r_n\}$ of values of r increasing to 1 and apply Montel's theorem to $\{g_{r_n}(z)\}$ to obtain a starlike function $g(z)$ in $|z| < 1$ for which we have the inequality

$$\left| \arg f'(z) + (\alpha - 1)\arg\frac{f(z)}{z} + \beta \log\left|\frac{f(z)}{z}\right| - \alpha \arg\frac{g(z)}{z}\right| \leqslant \tfrac{1}{2}\pi$$

for $|z| < 1$. In other words

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta} = h(z) \qquad (|z| < 1),$$

where $\operatorname{re} h(z) \geqslant 0$, and the conclusion follows. This completes the proof of Theorem 2.

7. We consider now the case $\alpha = 0$. Suppose that (2) holds for $\alpha = 0$. The function

$$t(\theta) = \arg f'(re^{i\theta}) - \arg\frac{f(re^{i\theta})}{re^{i\theta}} + \beta \log\left|\frac{f(re^{i\theta})}{re^{i\theta}}\right|$$

satisfies (6) and is periodic with period 2π, and hence we have

$$|t(\theta_1) - t(\theta_2)| < \pi$$

for any values of θ_1 and θ_2. In other words for some real λ we have

$$\left| \arg f'(z) - \arg\frac{f(z)}{z} + \beta \log\left|\frac{f(z)}{z}\right| - \lambda\right| < \tfrac{1}{2}\pi$$

in $|z| < 1$, and this is equivalent to the statement

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{i\beta} = h(z) \qquad (|z| < 1), \tag{8}$$

where $\operatorname{re}(e^{-i\lambda}h(z)) > 0$. In the case $\beta = 0$ we obtain the class of spiral-like functions of Špaček [see (4)]. On the other hand, provided that $h(0) = 1$, we can integrate the equation

$$\frac{zf'(z)}{f(z)} = h(z)$$

to obtain a function $f(z)$ analytic and univalent in $|z| < 1$, which satisfies (2) with $\alpha = 0, \beta = 0$. It is also possible in the case $\beta \neq 0$ to integrate the differential equation (8) (where β and h are assumed given) and the solution $f(z)$ will then be univalent and satisy (2) (with $\alpha = 0$) for $|z| < 1$. To prove this we choose a sequence $\{\alpha_n\}$ of positive numbers tending to 0 and choose $f_n(z)$ to satisfy

$$\frac{zf_n'(z)}{f_n(z)}\left(\frac{f_n(z)}{z}\right)^{\alpha_n + i\beta} = h(z).$$

This is the case $g(z) = z$ of Theorem 1, so this equation can be solved and the functions $f_n(z)$ are analytic and univalent in $|z| < 1$. Provided that $|f_n'(0)|$ does not tend to 0 or ∞, we can find a convergent subsequence of $\{f_n\}$ tending to a function $f(z)$ analytic and univalent in $|z| < 1$. This function clearly satisfies (8). It remains to consider $|f_n'(0)|$. Clearly

$$\{f_n'(0)\}^{\alpha_n+i\beta} = h(0),$$

and we easily obtain $\log|f_n'(0)| \to \beta^{-1}\arg h(0)$ as $n \to \infty$. Our assertion follows.

8. We henceforth include among the class of Bazilevič functions those arising from the case $\alpha = 0$. We will show that with the usual normalization $f'(0) = 1$, the class is compact.

Suppose then that $\{f_n(z)\}$ is a sequence of normalized Bazilevič functions of type (α_n, β_n) converging locally uniformly to a univalent function $f(z)$. Choose r $(0 < r < 1)$, θ_1 and θ_2 with $\theta_2 > \theta_1$. Then

$$\int_{\theta_1}^{\theta_2} \left\{1+\operatorname{re}\frac{re^{i\theta}f_n''(re^{i\theta})}{f_n'(re^{i\theta})}+(\alpha_n-1)\operatorname{re}\frac{re^{i\theta}f_n'(re^{i\theta})}{f_n(re^{i\theta})}-\right.$$
$$\left.-\beta_n \operatorname{im}\frac{re^{i\theta}f_n'(re^{i\theta})}{f_n(re^{i\theta})}\right\} d\theta > -\pi \qquad (9)$$

for $n = 1, 2,\ldots$. Without loss of generality we may assume that either $\{\alpha_n+i\beta_n\}$ converges to a finite limit $\alpha+i\beta$, where $\alpha \geqslant 0$, or that it tends to ∞. In the first case it is immediate that (2) is satisfied by f for these values of α and β, the inequality remaining strict by the maximum principle, and therefore $f(z)$ is Bazilevič of type (α, β). In the second case $|\alpha_n+i\beta_n| \to +\infty$ as $n \to \infty$. We note that the sequences $\{(\alpha_n-1)/|\alpha_n+i\beta_n|\}$ and $\{\beta_n/|\alpha_n+i\beta_n|\}$ remain bounded as $n \to \infty$, and so we can pick out a subsequence $\{n_k\}$ such that both sequences converge as $k \to \infty$ to finite limits A and B respectively. Note that $A \geqslant 0$ and that A and B are not both zero. Thus if we divide (9) by $|\alpha_n+i\beta_n|$ and let $k \to \infty$, we obtain

$$\int_{\theta_1}^{\theta_2} \left\{A \operatorname{re}\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})} - B \operatorname{im}\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right\} d\theta \geqslant 0.$$

Since A and B are independent of r, θ_1 and θ_2, it follows that

$$A \operatorname{re}\frac{zf'(z)}{f(z)} - B \operatorname{im}\frac{zf'(z)}{f(z)} \geqslant 0 \qquad (|z| < 1),$$

or equivalently

$$\operatorname{re}\left\{(A+iB)\frac{zf'(z)}{f(z)}\right\} \geqslant 0 \qquad (|z| < 1).$$

Thus $f(z)$ is spiral-like and so Bazilevič of type $(0,0)$, and the result follows.

9. The condition (2) has the following geometrical interpretation. Let $f(z) = \sum_1^\infty a_n z^n$ be analytic and univalent in $|z| < 1$ and let C be a positively described arc of the simple closed curve $f(|z| = r)$, where $0 < r < 1$. $f(z)$ is Bazilevič of type (α, β) if and only if for each such arc

$$\Delta_C(\arg dw + (\alpha-1)\arg w + \beta \log|w|) > -\pi, \tag{10}$$

where, if C joins w_1 to w_2, $\Delta_C(u(w)) = u(w_2) - u(w_1)$. This relation makes it easy to construct non-Bazilevič univalent functions. For example, let D be the plane cut along an infinite Jordan curve γ with endpoint P, and suppose that γ is smooth except at a point $Q \neq P$, and that at Q γ has one-sided tangents. Let C consist of QP along γ followed by PQ along γ. Then by suitable choice of the appropriate one-sided tangents at Q, the first member of (10) will be less than $-\pi$. Therefore if $f(z)$ is a schlicht mapping of $|z| < 1$ on to D, and if α and β are fixed, we can find r $(0 < r < 1)$ and an arc C' of $f(|z| = r)$ such that

$$\Delta_{C'}(\arg dw + (\alpha-1)\arg w + \beta \log|w|) < -\pi$$

by obvious continuity considerations. Thus $f(z)$ is not a Bazilevič function for any values of α and β.

10. To conclude we mention two results which are easy consequences of Theorem 2.

(i) *Let $f(z)$ be a Bazilevič function. The set of points (α, β) in the Euclidean plane for which $f(z)$ is Bazilevič of type (α, β) is closed and convex.*

(ii) *Let $n \geqslant 2$ be an integer. $f(z)$ is Bazilevič of type (α, β) if and only if $(f(z^n))^{1/n}$ is Bazilevič of type $(n\alpha, n\beta)$.*

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