CONSTANTS FOR PLANAR HARMONIC MAPPINGS

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Introduction

The purpose of this article is to discuss in detail certain constants which arise in the theory of harmonic univalent functions [1] defined in the unit disc U. In the class S of normalised analytic univalent functions, the fundamental constant which determines the growth of the functions, the asymptotic coefficient behaviour, the growth of integral means, the distance from the origin to the boundary and much else besides, is the maximum size of the coefficient a_2 in the power-series expansion of the functions. The classical result is, of course, that $|a_2| \leqslant 2$. With the natural analogy to Fourier series, harmonic functions have a two-sided series structure consisting of the 'analytic part', which is a power series in the variable z, and a 'co-analytic part', which is a power series in the variable $\bar{z}$. The analytic part has coefficients labelled with non-negative integers and the co-analytic part has coefficients labelled with negative integers. The class S_H is then the normalised class defined by $a_0 = 0$ and $a_1 = 1$. With these conventions the maximum size of a_2 again plays a crucial role in determining the extremal behaviour of functions in S_H. We shall give a detailed description of how this constant determines the bounds on both the maximum and minimum modulus for functions lying in an affine and linear invariant subclass of S_H. By an affine invariant subclass of S_H we mean a class $L \subset S_H$ for which $f \in L$ implies that $A\bar{f} + Bf \in L$ for all constants A, B maintaining the normalisation in S_H (so $A\bar{a}_{-1}(f) + B = 1$). Linear invariance has its usual meaning as defined by Pommerenke [5]. See also (1.2). In this way we shall show that among all the possible candidates for a harmonic Koebe function (that is, a normalised mapping onto the plane slit along a ray), there is one which stands out as the most likely candidate for an extremal function analogous to the classical Koebe function extremal for functions in S. This is the function $k_0(z)$ defined in (4.3), which maps U onto the plane slit from $-\frac{1}{6}$ to ∞ along the negative real axis.

A more general concept than that of linear invariance is composition invariance; that is, the composition of a univalent harmonic function with an analytic univalent function. An interesting fact emerges here: if we begin with a subclass of S_H, permit arbitrary composition with an analytic univalent function in U followed by a renormalisation, then finally form the closure of the resulting class, we obtain a very much larger class (in general) than the initial class. Nevertheless, bounds obtained in the initial class may determine bounds in the final class. If this sequence of operations is applied to the function $f(z) = z$, we obtain the whole of S: a fact which to some extent lies behind the Loewner method. In a similar manner we may begin with a simple class in S_H—such as the class of harmonic convex mappings—and so generate a subclass of S_H, whose members map onto arbitrarily shaped domains. In this

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subclass we shall obtain the de Branges estimate for the coefficients (though using his result for S, which is contained in this subclass). We also obtain the $\frac{1}{4}$-theorem for those functions in this subclass lying in

$$S_H^0 = \{f \in S_H : a_{-1}(f) = 0\}.$$

Thus our class has a Koebe constant $\frac{1}{4}$. However, this extension process does not generate the whole of S_H. Thus an interesting question arises as to how small a class we may start with and generate by this method all of S_H. We shall show that, if the extension is applied to the harmonic class of close-to-convex functions, we obtain a class with a Koebe constant $\frac{1}{6}$ and sharp coefficient bounds $|a_n| \leqslant \frac{1}{3}(2n^2+1)$ as conjectured in [1]. It seems to us unlikely that the class so constructed is the whole of S_H, but we are not able to establish that it is not.

We conclude the article with an improvement for our estimate for a_2 given in [1]. The conjecture is that $|a_2| < 3$ for $f \in S_H$, but all attempts to prove this by modifying the various classical techniques available in S have failed. We hope that this article will support our belief that this is the central unsolved problem in S_H.

1. *Affine and linear invariant classes*

Let L denote a subfamily of S_H which is both affine and linear invariant: that is, if $f \in L$ $(f = \bar{g} + h)$, then

$$z \longrightarrow \frac{\varepsilon\overline{f(z)} + f(z)}{1 + \varepsilon\bar{a}_{-1}} \in L \quad \text{for } |\varepsilon| < 1; \tag{1.1}$$

$$z \longrightarrow \frac{f((z+z_0)/(1+\bar{z}_0 z)) - f(z_0)}{(1-|z_0|^2)\,h'(z_0)} \in L \quad \text{for } |z_0| < 1. \tag{1.2}$$

We adopt the following notation:

$$L^0 = L \cap S_H^0 = \{f \in L : a_{-1}(f) = 0\}; \tag{1.3}$$

$$m(r,f) = \min_{|z|=r} |f(z)|; \qquad M(r,f) = \max_{|z|=r} |f(z)|; \tag{1.4}$$

$$\alpha = \alpha(L) = \sup\{|a_2(f)| : f \in L\}; \tag{1.5}$$

$$d = d(L^0) = \liminf_{r \to 1}\{m(r,f) : f \in L^0\}. \tag{1.6}$$

It is known that [1]

$$d(S_H^0) \geqslant \tfrac{1}{16}. \tag{1.7}$$

Our first theorem implies that, if our conjecture that $|a_2| \leqslant 3$ is correct, then $d(S_H^0) = \frac{1}{6}$.

THEOREM 1. *For* $f \in L^0$,

$$\frac{1}{2\alpha}\left(1 - \left(\frac{1-r}{1+r}\right)^{\alpha}\right) \leqslant m(r,f) \leqslant M(r,f) \leqslant \frac{1}{2\alpha}\left(\left(\frac{1+r}{1-r}\right)^{\alpha} - 1\right) \tag{1.8}$$

for $0 < r < 1$, *where* $\alpha = \alpha(L)$;

$$d(L^0) \geqslant \frac{1}{2\alpha(L)}. \tag{1.9}$$

Proof. We consider first $f = \bar{g} + h \in L$, so that if $|z_0| < 1$, the function

$$F(z) = \frac{f((z+z_0)/(1+\bar{z}_0 z)) - f(z_0)}{(1-|z_0|^2)\,h'(z_0)} = \bar{G} + H \in L. \tag{1.10}$$

We have

$$a_2(F) = \tfrac{1}{2}H''(0) = \frac{1}{2}\frac{(1-|z_0|^2)\,h''(z_0)}{h'(z_0)} - \bar{z}_0 \tag{1.11}$$

and so by hypothesis

$$\left|\tfrac{1}{2}(1-|z_0|^2)\frac{h''(z_0)}{h'(z_0)} - \bar{z}_0\right| \leqslant \alpha. \tag{1.12}$$

Thus if $|z_0| = r < 1$,

$$\frac{-2\alpha+2r}{1-r^2} \leqslant \operatorname{Re}\left(\frac{z_0}{|z_0|}\frac{h''(z_0)}{h'(z_0)}\right) \leqslant \frac{2\alpha+2r}{1-r^2}. \tag{1.13}$$

On integrating this inequality from 0 to r we obtain

$$\frac{(1-r)^{\alpha-1}}{(1+r)^{\alpha+1}} \leqslant |h'(z_0)| \leqslant \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}. \tag{1.14}$$

Suppose now that $f = \bar{g} + h \in L^0$. Then for each ε ($|\varepsilon| < 1$) the function

$$\varepsilon\bar{f} + f = (\varepsilon\bar{h} + \bar{g}) + (h + \varepsilon g) \tag{1.15}$$

is in L. Hence for $|z| = r < 1$,

$$\frac{(1-r)^{\alpha-1}}{(1+r)^{\alpha+1}} \leqslant |h'(z) + \varepsilon g'(z)| \leqslant \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}. \tag{1.16}$$

We deduce that

$$|h'(z)| - |g'(z)| \geqslant \frac{(1-r)^{\alpha-1}}{(1+r)^{\alpha+1}}; \tag{1.17}$$

$$|h'(z)| + |g'(z)| \leqslant \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}. \tag{1.18}$$

Because f is univalent, we have

$$\begin{aligned} m(r,f) = |f(z_0)| &= \int_\gamma |dw| = \int_\Gamma \left|\frac{\partial f}{\partial z}dz + \frac{\partial f}{d\bar{z}}d\bar{z}\right| \\ &\geqslant \int_\Gamma (|h'(z)| - |g'(z)|)\,|dz| \\ &\geqslant \int_\Gamma \frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}}|dz| \geqslant \int_0^r \frac{(1-\rho)^{\alpha-1}}{(1+\rho)^{\alpha+1}}d\rho \\ &= \frac{1}{2\alpha}\left(1 - \left(\frac{1-r}{1+r}\right)^\alpha\right), \end{aligned}$$

where z_0 is a point on $\{|z| = r\}$ at which $|f|$ attains its minimum value, γ is the line segment joining 0 to $f(z_0)$ and $\Gamma = f^{-1}(\gamma)$ is a path in U joining 0 to z_0. This proves the left-hand inequality in (1.8). The right-hand inequality follows similarly by integrating (1.18). Finally, we obtain (1.9) by letting $r \to 1$ in the left-hand inequality.

2. *Composition invariance*

For an affine and linear invariant class $L \subset S_H$, we consider the class $\tilde{L}$ formulated as follows: $f \in \tilde{L}$ if there exists $\{f_n\} \subset L$ and a sequence $\{\sigma_n\}$ of analytic univalent functions in U satisfying $|\sigma_n(z)| < 1$ $(z \in U)$ such that

$$f(z) = \lim_{n\to\infty} \frac{f_n(\sigma_n(z)) - f_n(\sigma_n(0))}{\sigma_n'(0)\, h_n'(\sigma_n(0))}, \tag{2.1}$$

where the limit is locally uniform in U.

THEOREM 2. *$\tilde{L}$ is an affine and linear invariant class and a compact subset of $\bar{S}_H$. Furthermore,*

$$\alpha(\tilde{L}) = \max(2, \alpha(L)), \tag{2.2}$$

$$d(\tilde{L}^0) \geqslant \min(\tfrac{1}{4}, 1/2\alpha(L)). \tag{2.3}$$

Proof. For $f \in L$ let

$$F(z) = \frac{f(\sigma(z)) - f(\sigma(0))}{\sigma'(0)\, h'(\sigma(0))}, \tag{2.4}$$

where $\sigma(z)$ is analytic and univalent in U with $|\sigma(z)| < 1$. Then

$$P(z) = \frac{F((z+z_0)/(1+\bar{z}_0 z)) - F(z_0)}{(1-|z_0|^2)\, H'(z_0)} = \frac{f(\tau(z)) - f(\tau(0))}{\tau(0)\, h'(\tau(0))}, \tag{2.5}$$

where $\tau(z) = \sigma((z+z_0)/(1+\bar{z}_0 z))$. Hence $P \in \tilde{L}$. Next let

$$Q(z) = \frac{\varepsilon\bar{F} + F}{1+\varepsilon\bar{a}_{-1}(F)} = \frac{\tilde{f}(\sigma(z)) - \tilde{f}(\sigma(0))}{\sigma'(0)\, \tilde{h}'(\sigma(0))}, \tag{2.6}$$

where

$$\tilde{f} = \frac{\eta\bar{f} + f}{1+\eta\bar{a}_{-1}(f)} \quad \text{and} \quad \eta = \varepsilon \frac{\bar{h}'(\sigma(0))\,\sigma'(0)}{h'(\sigma(0))\,\bar{\sigma}'(0)}.$$

Hence $Q \in \tilde{L}$.

Thus the class of functions of the form F is affine and linear invariant. As $\tilde{L}$ is the closure of this class, $\tilde{L}$ is also affine and linear invariant.

Next we note that

$$a_2(F) = \frac{1}{2}\frac{\sigma''(0)}{\sigma'(0)} + \frac{1}{2}\sigma'(0)\frac{h''(\sigma(0))}{h'(\sigma(0))}. \tag{2.7}$$

Also

$$\frac{h''(\sigma(0))}{h'(\sigma(0))} = \frac{2\overline{\sigma(0)}}{1-|\sigma(0)|^2} + \frac{2\alpha\zeta}{1-|\sigma(0)|^2}, \tag{2.8}$$

where $\alpha = \alpha(L)$ and $|\zeta| \leqslant 1$. Hence

$$a_2(F) = \frac{\sigma_2}{\sigma_1} + \sigma_1\left(\frac{\bar{\sigma}_0}{1-|\sigma_0|^2} + \frac{\alpha\zeta}{1-|\sigma_0|^2}\right), \tag{2.9}$$

where $\sigma(z) = \sigma_0 + \sigma_1 z + \sigma_2 z^2 + \ldots$. Let

$$\tau(z) = \frac{\sigma(z) - \sigma_0}{1-\bar{\sigma}_0\sigma(z)}, \qquad \sigma(z) = \frac{\tau(z)+\sigma_0}{1+\bar{\sigma}_0\tau(z)}. \tag{2.10}$$

Then

$$\sigma_1 = \tau_1(1-|\sigma_0|^2); \qquad \sigma_2 = (1-|\sigma_0|^2)(\tau_2-\tau_1^2\bar{\sigma}_0).$$

Note that $\tau(0) = 0$ so $|\tau(z)| \leqslant |z|$. We obtain

$$a_2(F) = \frac{\tau_2}{\tau_1}+\alpha\zeta\tau_1. \tag{2.12}$$

It is well known that the univalence of τ implies that

$$\left|\frac{\tau_2}{\tau_1}\right| \leqslant 2(1-|\tau_1|). \tag{2.13}$$

Hence

$$|a_2(F)| \leqslant 2+(\alpha-2)|\tau_1|. \tag{2.14}$$

If $\alpha \geqslant 2$ we deduce, since $|\tau_1| \leqslant 1$, that

$$|a_2(F)| \leqslant \alpha. \tag{2.15}$$

If $\alpha < 2$ we obtain

$$|a_2(F)| \leqslant 2. \tag{2.16}$$

As we can choose τ as a mapping of U onto U cut along a portion of a radius, we can make $\tau_2/\tau_1 = 2(1-|\tau_1|)$. Then allowing $\tau_1 \to 0$ we can make $a_2(F)$ as near to 2 as we please. Also by choosing $\sigma(z)$ as a Möbius transformation we can make $|a_2(F)|$ as near α as we please. We deduce (2.2). Then (2.3) follows from Theorem 1 and (2.2).

3. *Convex composition*

The class $K_H \subset S_H$ of harmonic mappings onto convex regions is clearly affine and linear invariant. Moreover, $\alpha(K_H) = 2$ [**1**]. Hence we have the following.

THEOREM 3. *If* $f \in \tilde{K}_H^0$, *then for* $|z| = r < 1$,

$$\frac{r}{(1+r)^2} \leqslant |f(z)| \leqslant \frac{r}{(1-r)^2} \tag{3.1}$$

and

$$d(\tilde{K}_H^0) = \tfrac{1}{4}. \tag{3.2}$$

REMARK 1. The class $\tilde{K}_H^0$ contains the class S of all normalised, analytic univalent functions. Indeed if $f \in S$ then, for $0 < \rho < 1$,

$$\left|\frac{1}{\rho}f(\rho z)\right| \leqslant \frac{1}{(1-\rho)^2} \qquad (|z| < 1) \tag{3.3}$$

and so

$$\frac{1}{\rho}f(\rho z) = \frac{\omega_\rho(z)}{(1-\rho)^2} = \frac{\omega_\rho(z)}{\omega_\rho'(0)}, \tag{3.4}$$

where $|\omega_\rho(z)| < 1$ and ω_ρ is analytic univalent. Thus

$$\frac{1}{\rho}f(\rho z) = \frac{z \circ \omega_\rho(z)}{\omega_\rho'(0)} \tag{3.5}$$

and since $z \in K \subset K_H$, we see that $(1/\rho)f(\rho z) \in \tilde{K}_H^0$.

Letting $\rho \to 1$, $f \in \tilde{K}_H^0$. This shows that Theorem 3 is sharp.

THEOREM 4. *If* $f(z) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \in \tilde{K}_H$ $(z = re^{i\theta})$, *then*

$$|a_n| \leqslant |n| \qquad (n = \pm 1, \pm 2, \ldots). \tag{3.6}$$

Proof. Since the analytic part of a function in K_H is close-to-convex and therefore univalent [1], the analytic part of $f \in \tilde{K}_H$ is in S. Therefore (3.6) holds for $n \geqslant 1$ by de Branges's theorem [2]. On the other hand $\tilde{K}_H^0$ is a compact class, so as shown in [1, Theorem 2.5] the extreme points of $\tilde{K}_H$ lie on $\partial S_H = \bar{S}_H - S_H$. This implies that the coefficients of the co-analytic part have the same bounds as those of the analytic part. Thus the result holds for $n \leqslant 1$.

REMARK 2. Note that, as shown in [1], $d(K_H^0) = \frac{1}{2}$, whereas Theorem 1 gives $d(K_H^0) \geqslant \frac{1}{4}$. Thus it is not necessarily true that $d(L^0) = 1/2\alpha(L)$. However the estimate $1/2\alpha(L)$ is correct for $d(\tilde{K}_H^0)$. Therefore we may conjecture that

$$d(\tilde{L}^0) = \min\left(\tfrac{1}{4}, 1/2\alpha(L)\right)$$

for any affine, linear invariant class L. If this is true, then the $\frac{1}{16}$ estimate for $d(S_H^0)$ would imply that

$$\alpha(S_H) \leqslant 8.$$

In fact we believe that $\alpha(S_H) = 3$, $d(S_H^0) = \frac{1}{6}$.

Our evidence for this is in the next section.

4. *Close-to-convex composition*

The class $C_H \subset S_H$ of harmonic mappings onto close-to-convex regions is affine and linear invariant. Moreover $\alpha(C_H) = 3$ [1]. We have the following, therefore.

THEOREM 5. *If* $f \in \tilde{C}_H^0$, *then for* $|z| = r < 1$

$$\frac{r + \frac{1}{3}r^3}{(1+r)^3} \leqslant |f(z)| \leqslant \frac{r + \frac{1}{3}r^3}{(1-r)^3} \tag{4.1}$$

and

$$d(\tilde{C}_H^0) = \tfrac{1}{6}. \tag{4.2}$$

REMARK 3. As C_H^0 contains all possible Koebe mappings, that is, mappings in S_H^0 onto the plane cut along a half-line, radial or otherwise, we see that the mapping

$$k_0(z) = \operatorname{Re}\left(\frac{z + \frac{1}{3}z^3}{(1-z)^3}\right) + i \operatorname{Im}\left(\frac{z}{(1-z)^2}\right), \tag{4.3}$$

which is in C_H^0 and maps U onto the plane cut along the negative real axis from $-\frac{1}{6}$ to ∞ [1], gives the extremal distance $\frac{1}{6}$ for the nearest boundary point to 0. Thus $\frac{1}{6}$ is certainly correct for S_H^0 if the extremal mapping is a Koebe mapping—which seems quite likely. Note also that (4.1) is sharp for $k_0(z)$.

REMARK 4. It is not known whether or not $\bar{S}_H = \tilde{C}_H$.

THEOREM 6. *If* $f(z) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \in \tilde{C}_H$ $(z = re^{i\theta})$, *then*

$$|a_n| \leqslant \tfrac{1}{3}(2n^2 + 1) \qquad (n = \pm 1, \pm 2, \ldots). \tag{4.4}$$

REMARK 5. This result is sharp for the function

$$k(z) = 2\,\mathrm{Re}\,k_0(z) = 2\,\mathrm{Re}\left(\frac{z+\frac{1}{3}z^3}{(1-z)^3}\right) \tag{4.5}$$

which is in $\partial C_H = \bar{C}_H - C_H$.

Proof. As in the proof of Theorem 4 it is sufficient to prove the theorem for $n \geqslant 2$. If h denotes the analytic part of f, then we can assume that

$$h(z) = \frac{H(\sigma(z)) - H(\sigma(0))}{\sigma'(0)\,H'(\sigma(0))}, \tag{4.6}$$

where H is the analytic part of a function F in C_H and σ is analytic univalent in U with $|\sigma(z)| < 1$ in U. It was shown in [**1**] that H belongs to the Kaplan class $K(2, 4)$. This means that we can write $zH'(z) = P^2(z)\,S(z)$, where $\mathrm{Re}\,P(z) > 0$ in U and where $S(z)$ is starlike univalent in U. Then the function $P(z)\,S(z)/z$ is the derivative $K'(z)$ of a function $K(z)$ close-to-convex in U. Hence we can write

$$H'(z) = P(z)\,K'(z), \tag{4.7}$$

where $\mathrm{Re}\,P(z) > 0$ in U and where $K(z)$ is univalent in U. We then have

$$h'(z) = \frac{P(\sigma(z))\,\sigma'(z)\,K'(\sigma(z))}{P(\sigma(0))\,\sigma'(0)\,K'(\sigma(0))} = Q(z)\,L'(z), \tag{4.8}$$

where

$$Q(z) = \frac{P(\sigma(z))}{P(\sigma(0))} \quad \text{and} \quad L(z) = \frac{K(\sigma(z))}{\sigma'(0)\,K'(\sigma(0))}.$$

Thus there is some real λ such that $\mathrm{Re}\,(e^{i\lambda}Q(z)) > 0$ with $Q(0) = 1$ and $L(z) \in S$. Applying de Branges's theorem we deduce that

$$h'(z) \ll \frac{1+z}{1-z}\frac{1+z}{(1-z)^3} = \frac{(1+z)^2}{(1-z)^4}, \tag{4.9}$$

and this gives (4.3).

5. *A generalisation of the Bieberbach conjecture*

The general coefficient bound for functions $f \in S_H$ is conjectured to be

$$|a_n| < \tfrac{1}{3}(2n^2+1) \qquad (n = \pm 2, \pm 3, \ldots). \tag{5.1}$$

This is proved for all functions $f \in S_H$ with real coefficients and all functions $f \in S_H$ for which either $f(U)$ is starlike with respect to the origin or $f(U)$ is convex in one direction [**1**]. The slightly weaker

$$|a_n| \leqslant \tfrac{1}{3}(2n^2+1) \tag{5.2}$$

is proved for $\tilde{C}_H$. For functions $f \in S_H^0$ it is a remarkable fact that

$$||a_n| - |a_{-n}|| \leqslant n \qquad (n = 2, 3, \ldots) \tag{5.3}$$

when the coefficients a_n are real. We conjecture that this holds for all $f \in S_H^0$. We give a proof in the following two cases.

THEOREM 7. *Let $f \in S_H^0$. If either*
(a) *$f(U)$ is starlike with respect to the origin, or*
(b) *$f(U)$ is convex in one direction,*
then

$$||a_n| - |a_{-n}|| \leqslant n \qquad (n = 2, 3, \ldots) \tag{5.4}$$

and

$$|a_n| \leqslant \tfrac{1}{6}|(n+1)(2n+1)| \qquad (n = \pm 1, \pm 2, \ldots). \tag{5.5}$$

Equality occurs for $f = k_0$.

Proof. (a) Using the approximation method of [**1**, Theorem 3.7] we may assume that f extends continuously and smoothly to $\bar{U}$ and gives a 1:1 sense preserving mapping of ∂U onto a curve starlike with respect to 0. We define

$$F(re^{i\theta}) = \sum_{n \neq 0} \frac{a_n}{n} r^{|n|} e^{in\theta} \tag{5.6}$$

for $0 \leqslant r < 1$, $0 \leqslant \theta \leqslant 2\pi$. Then the starlikeness of $f(e^{i\theta})$ implies that

$$\frac{d}{d\theta} \arg\left(\frac{d}{d\theta} F(e^{i\theta})\right) = \frac{d}{d\theta} \arg f(e^{i\theta}) \geqslant 0, \tag{5.7}$$

and so $F(e^{i\theta})$ describes a convex curve. By the Choquet–Kneser–Rado theorem $F(z)$ is convex univalent in U. Hence as shown in [**1**, Lemma 5.11] there exist real λ, μ such that

$$\operatorname{Re}\left[(e^{-i\mu} H'(z) + e^{i\mu} G'(z))(e^{i\lambda} - e^{-i\lambda} z^2)\right] \geqslant 0 \tag{5.8}$$

for $z \in U$, where $F = \bar{G} + H$. But then

$$g(z) = -zG'(z), \qquad h(z) = zH'(z), \tag{5.9}$$

and so for $z \in U$

$$\operatorname{Re}\left[\left(e^{-i\mu}\frac{h(z)}{z} - e^{i\mu}\frac{g(z)}{z}\right)(e^{i\lambda} - e^{i\lambda} z^2)\right] \geqslant 0. \tag{5.10}$$

We therefore have

$$\sum_{n=1}^{\infty} (e^{-i\mu} a_n - e^{i\mu} \bar{a}_{-n}) z^n = \frac{zP(z)}{e^{i\lambda} - e^{-i\lambda} z^2}, \tag{5.11}$$

where $|P(0)| = 1$ and $\operatorname{Re} P(z) > 0$ for $z \in U$. We obtain

$$|e^{-i\mu} a_n - e^{i\mu} \bar{a}_{-n}| \leqslant \left(\frac{z}{1-z^2}\frac{1+z}{1-z}\right)_n = n \tag{5.12}$$

for $n = 1, 2, 3, \ldots$. We deduce (5.4). Furthermore, we have

$$g'(z) = \omega(z)\, h'(z) \qquad (z \in U), \tag{5.13}$$

where $|\omega(z)| \leqslant |z|$ by Schwarz's lemma. From (5.11) we obtain

$$g'(z) = \frac{\omega(z)}{e^{-i\mu} - e^{i\mu}\omega(z)} \frac{d}{dz}\left(\frac{z}{e^{i\lambda} - e^{-i\lambda} z^2} P(z)\right) \ll \frac{z}{1-z} \frac{1+z}{(1-z)^3} = \frac{z(1+z)}{(1-z)^4} \tag{5.14}$$

and

$$h'(z) = \frac{1}{e^{-i\mu} - e^{i\mu}\omega(z)} \frac{d}{dz}\left(\frac{z}{e^{i\lambda} - e^{-i\lambda} z^2} P(z)\right) \ll \frac{1}{1-z} \frac{1+z}{(1-z)^3} = \frac{1+z}{(1-z)^4}, \tag{5.15}$$

These inequalities give (5.5).

(b) As shown in [**1**, Theorem 5.3], $f = \bar{g}+h$ is convex in the direction ϕ $(0 \leqslant \phi < \pi)$ if and only if the analytic function $h - e^{2i\phi}g$ is convex in the direction ϕ. Hence for such a function f in S_H^0 we obtain

$$|a_n - e^{2i\phi}\bar{a}_{-n}| \leqslant n \qquad (n = 2, 3, \ldots) \tag{5.16}$$

and this proves (5.4). Also, writing $g' = \omega h'$ we have

$$\begin{aligned} g'(z) &= \frac{\omega(z)}{1-e^{2i\phi}\omega(z)}(h'(z)-e^{2i\phi}g'(z)) \\ &\ll \frac{z(1+z)}{(1-z)^4} \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} h'(z) &= \frac{1}{1-e^{2i\phi}\omega(z)}(h'(z)-e^{2i\phi}g'(z)) \\ &\ll \frac{1+z}{(1-z)^4}, \end{aligned} \tag{5.18}$$

and hence (5.5) follows.

REMARK 6. As both the cases (a) and (b) consist of functions in C_H^0, we would certainly expect the conjecture (5.3) to hold for C_H^0. However, we have no proof. Notice that case (b) includes all functions in S_H^0 where $f(U)$ is the plane slit along a ray, radial or non-radial; that is, all Koebe mappings.

REMARK 7. We also note that in proving case (a) we have shown that if

$$f(z) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \qquad (z = re^{i\theta})$$

is starlike, then

$$F(z) = \sum_{n \neq 0} \frac{a_n}{n} r^{|n|} e^{in\theta}$$

is convex. This is a familiar result in the analytic case. However, in the analytic case the converse result is also valid—but this is not true in the harmonic case. Because we cannot apply the Choquet–Kneser–Rado theorem to a starlike curve, the converse argument is certainly invalid and in fact the example

$$\begin{aligned} f(z) &= \frac{z}{(1-z)^3} - \frac{\bar{z}^2}{(1-\bar{z})^3} \\ &= \sum_{n=1}^{\infty} \tfrac{1}{2}n(n+1)\,z^n - \sum_{n=1}^{\infty} \tfrac{1}{2}n(n-1)\,\bar{z}^n \end{aligned} \tag{5.19}$$

is not starlike univalent, since $a_2 = 3$ and from (7.2) we should have $|a_2| \leqslant 2\tfrac{1}{2}$. But

$$\sum_{n=1}^{\infty} \tfrac{1}{2}(n+1)\,z^n + \sum_{n=1}^{\infty} \tfrac{1}{2}(n-1)\,\bar{z}^n = l_0(z), \tag{5.20}$$

which is convex in U [**1**].

6. *A lower bound for the inner mapping radius*

For $f \in S_H$ we denote by $\rho(f)$ the inner mapping radius of the domain $f(U)$. This value is $F'(0)$, where $F(z)$ is the conformal mapping of U onto $f(U)$ satisfying $F(0) = 0$, $F'(0) > 0$. Now $\inf\{\rho(f): f \in S_H\} = 0$, since

$$\bar{z}+z = \lim_{\varepsilon \to 1} \varepsilon\bar{z}+z \tag{6.1}$$

is in ∂S_H. However, we have the following.

THEOREM 8. *For $f \in S_H^0$, $\rho(f) \geqslant \frac{1}{4}$.*

Proof. It was shown in [1] that $f(U)$ contains the disc $\{|w| < \frac{1}{16}\}$. However, a glance at the proof shows that this disc is contained in $p(U)$ for any 1:1 mapping $p(z)$ of U satisfying $p(0) = 0$, $p'(0) = 1$ and

$$\left|\frac{\partial p}{\partial \bar{z}}\right| \leqslant |z| \left|\frac{\partial p}{\partial z}\right| \tag{6.2}$$

for $|z| < 1$. In other words for a function satisfying (6.2) and satisfying $p(z) \sim z$ as $z \to 0$. For $f \in S_H^0$ the function

$$p(z) = K(f(z)), \tag{6.3}$$

where $K(w) = w + \dots$ is the conformal mapping of $f(U)$ onto the plane cut along the negative real axis from $-t$ to $-\infty$, satisfies (6.2). The undetermined value t must therefore satisfy $t \geqslant \frac{1}{16}$. Now if $F(z)$ is the conformal mapping associated with $f(U)$, then $K(f(z))$ maps U onto the plane cut from $-t$ to $-\infty$, and so

$$K(F(z)) = \frac{4tz}{(1-z)^2}. \tag{6.4}$$

Thus

$$\rho(f) = F'(0) = 4t \geqslant \tfrac{1}{4}. \tag{6.5}$$

REMARK 8. The lowest known value for $\rho(f)$, when $f \in S_H^0$, is $\rho(k_0) = \frac{2}{3}$.

REMARK 9. R. R. Hall [4] has shown recently that if $f \in S_H$, then f omits a value w satisfying $|w| \leqslant \frac{1}{2}\pi$, and this is the best constant. From the Koebe $\frac{1}{4}$-theorem it follows that $\rho(f) \leqslant 2\pi$ for $f \in S_H$. The highest known value for $\rho(f)$ when $f \in \bar{S}_H$ is $\rho(f) = \frac{1}{2}\pi$. Here f is a limit of mappings of U onto concentric discs, the limiting radius being $\frac{1}{2}\pi$. Now f itself, the extremal function for Hall's result, is not in S_H, but is a real-valued harmonic function. It seems likely that the correct upper bound for $\rho(f)$ over S_H is $\frac{1}{2}\pi$.

7. *Radius of convexity*

THEOREM 9. *Let L be an affine and linear invariant subclass of S_H and let $\alpha = \alpha(L)$. Then each $f \in L$ maps the disc $\{|z| < \alpha - \sqrt{(\alpha^2 - 1)}\}$ onto a convex domain.*

Proof. If $f = \bar{g} + h \in L$ we have from (1.12) that

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) \geqslant \frac{1-2\alpha|z|+|z|^2}{1-|z|^2} > 0 \tag{7.1}$$

if $|z| < \alpha - \sqrt{(\alpha^2-1)}$; (observe that necessarily $\alpha \geqslant 1$). Hence h is convex for $|z| < \alpha - \sqrt{(\alpha^2-1)}$. By the affine invariance it follows that $h + \varepsilon g$ is convex for $|z| < \alpha - \sqrt{(\alpha^2-1)}$, when $|\varepsilon| < 1$. This remains true, therefore, when $|\varepsilon| = 1$. Applying the criterion for convexity given in [**1**, Theorem 5.7], we deduce that f is convex for $|z| < \alpha - \sqrt{(\alpha^2-1)}$.

COROLLARY. (a) *If $f \in C_H$, then f is convex for $|z| < 3 - \sqrt{8}$.*
(b) *If $f \in K_H$, then f is convex for $|z| < 2 - \sqrt{3}$.*

REMARK 10. It is not possible in general to improve on the radius $\alpha - \sqrt{(\alpha^2-1)}$. For example the constant $3-\sqrt{8}$ is sharp for C_H, as can be seen by considering

$$\frac{d}{dt}\arg\left(\frac{d}{dt}k_0(re^{it})\right) < 0 \tag{7.2}$$

for $r > 3 - \sqrt{8}$ and $t = \pi$.

We also observe that the bound $2-\sqrt{3}$ will be valid for $f \in \tilde{K}_H \cap S_H$, in particular for $f \in S$. It is well known that $2-\sqrt{3}$ is the correct bound for S attained by the Koebe function $z/(1-z)^2$.

However we believe that the bound for K_H can be improved to $\sqrt{2}-1$. The function $l_0(z)$ shows that this would be best possible. In particular, for harmonic functions convexity is not an 'hereditary' property as it is in the analytic case.

REMARK 11. It would be interesting to find the correct bound for the radius of univalence of the analytic part h of a function $f = \bar{g} + h \in S_H$. The analytic part of $k(z)$ is

$$h(z) = (z + \tfrac{1}{3}z^3)/(1-z^3)$$

and we have $h(i/\sqrt{3}) = h(-i/\sqrt{3})$, so that the radius on S_H is at most $1/\sqrt{3}$. In fact $h(z)$ displayed above is univalent for $|z| < 1\sqrt{3}$. To see this put $w = (1+z)/(1-z)$. Then

$$\frac{z+\frac{1}{3}z^3}{(1-z)^3} = \tfrac{1}{6}(w^3-1). \tag{7.3}$$

Also, if $|z| < 1/\sqrt{3} = \tan\frac{1}{6}\pi$, then $|\arg w| < \frac{1}{3}\pi$; for it is easily shown that in general we have

$$\left|\arg\frac{1+z}{1-z}\right| < \arg\frac{1+ir}{1-ir} = 2\arctan r \tag{7.4}$$

for $|z| < r \leqslant 1$. Since w^3 is univalent in the sector $\{|\arg w| < \frac{1}{3}\pi\}$, h is univalent for $|z| < 1/\sqrt{3}$.

8. *A bound for a_2*

THEOREM 10. *Let $f \in S_H$. Then*

$$|a_2(f)| \leqslant \frac{96\pi}{\sqrt{27}} - 1 < 57.05. \tag{8.1}$$

Proof. Assume first that $f=\bar{g}+h\in S_H^0$. The domain $f(U)$ contains a disc with centre 0 and radius d, where $d\geqslant\frac{1}{16}$. Therefore there exists $\omega(z)$ analytic and univalent in U with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in U)$ such that $f(\omega(U))=\{|w|<d\}$. Then $(1/d)f(\omega(z))$ is a univalent harmonic mapping $U\to U$. Writing

$$\omega(z)=\sum_{n=1}^{\infty}\omega_n z^n,\qquad f(re^{i\theta})=\sum_{-\infty}^{\infty}a_n r^{|n|}e^{in\theta},$$

we have

$$\begin{aligned} f(\omega(z)) &= +\ldots+a_{-2}\bar{\omega}^2(z)+\omega(z)+a_2\,\omega^2(z)+\ldots \\ &= +\ldots+a_{-2}\bar{\omega}_1^2\bar{z}^2+\omega_1 z+(\omega_2+a_2\,\omega_1^2)\,z^2+\ldots . \end{aligned} \tag{8.2}$$

Since this is a convex harmonic mapping, we have [**1**, Theorem 5.10]

$$\big||a_{-2}\bar{\omega}_1^2|-|\omega_2+a_2\,\omega_1^2|\big|\leqslant|\omega_1| \tag{8.3}$$

and hence

$$|a_2|\leqslant\frac{1}{|\omega_1|}+\frac{|\omega_2|}{|\omega_1|^2}+|a_{-2}|. \tag{8.4}$$

By Schwarz's lemma, $|a_{-2}|\leqslant\frac{1}{2}$, since $|g'(z)|\leqslant|zh'(z)|$. Also the univalence and boundedness of $\omega(z)$ imply that

$$|\omega_2|\leqslant 2|\omega_1|\,(1-|\omega_1|). \tag{8.5}$$

We obtain

$$|a_2|\leqslant\frac{3}{|\omega_1|}-\frac{3}{2}. \tag{8.6}$$

As $(1/d)(f\circ\omega)$ maps U onto U, we have by R. R. Hall's inequality [**3**] that

$$|\omega_1|\geqslant d\sqrt{27}/2\pi. \tag{8.7}$$

Since the bound for a_2 on S_H exceeds that on S_H^0 by at most $\frac{1}{2}$,

$$|a_2(f)|\leqslant\frac{6\pi}{d\sqrt{27}}-1\leqslant\frac{96\pi}{\sqrt{27}}-1. \tag{8.8}$$

Note added in proof. The bound on p. 247, line 17 has now been proved [**6**].

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