countable divisible groups were summands of any group containing them as subgroups. In 1940, Baer classified the divisible groups up to isomorphism, and introduced the general notion of an injective module – an idea with widespread ramifications in more recent algebra. Abelian group theory is actually full of such classes of groups, characterized by some universal property, and with a more-or-less satisfactory structure theory. Some of these already have important analogues in other areas of algebra, while others remain unexploited. The authors consider several such classes of modules, in as general a setting as seems appropriate.

Other topics covered include homological properties of modules, the special features of uniserial modules (that is, modules whose submodules form a chain), and the structure of finitely generated and polyserial modules (which is developing in an interesting way, and shows promise). Many results are recent work of the authors and their students. There are particularly satisfactory recent results concerning the structure of finitely generated modules and the invariants for modules analogous to the Ulm invariants for Abelian groups.

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LINEAR PROBLEMS AND CONVEXITY TECHNIQUES IN GEOMETRIC FUNCTION THEORY

(Monographs and Studies in Mathematics 22)

By D. J. HALLENBECK and T. H. MACGREGOR: pp. 182. £26.50. (Pitman Publishing Ltd, 1984.)

Geometric Function Theory has its origins in Riemann's classical theorem on the conformal equivalence of simply-connected plane regions. The modern proof is still essentially that of P. Koebe (1907) – see e.g. Ahlfors's excellent text [1] – and exhibits the mapping of such a region onto the disc as the solution of an extremal problem among a whole family of mappings into the disc. Koebe's work is central to the modern theme of solving extremal problems among a variety of families of mappings. At the centre of attention is the family S of univalent (one-to-one) mappings $f(z) = z + a_2 z^2 + \dots$ of the unit disc. Ideally we should like to know how the analytic properties of such functions relate to the geometry of the image domains. At times the connections are really surprising. For example, if $f(z) = \sum_1^\infty a_n z^n$, $g(z) = \sum_1^\infty b_n z^n$ are univalent mappings of the disc onto convex regions, then the convolution $(f*g)(z) = \sum_1^\infty a_n b_n z^n$ is also a univalent mapping onto a convex region [7]. A number of results of this type, relating the algebra of power series to geometric and analytic properties of classes of mappings, have appeared in the last dozen years. During this time there has also been an intense development of a closely related theme, similarly tied up with linear ideas, namely the study of extreme points and support points for classes of functions.

This is the central topic of the current book of Hallenbeck and MacGregor, which is the first text to give a thoroughgoing development of the basic theory as it has been applied to conformal mapping problems. The idea is to exploit fully the topological and linear structure of the family of power series converging in the unit disc. In short the family has the structure of a complete, metrizable, locally convex linear topological space. Therefore if, for example, we wish to maximize a convex or linear functional on the class S, we need look no further than the set of extreme points of the closed convex hull of S. Crucial to such discussions is the Krein–Milman theorem, which ensures that such points are themselves members of S. For this idea to be useful in the solution of extremal problems, one needs to characterise the set of extreme points by properties which lend themselves to explicit calculations not available in the whole class. In 1970 L. Brickman [3] showed that an extreme point of S maps the disc onto the plane cut along a curve, whose distance from the origin steadily increases as the curve is traversed from its tip. Although this was not a new result, his proof, which appears in the book, attracted considerable attention for its simplicity and elegance.

Like almost all methods introduced into the study of the class S, the hope was that some light might be thrown on the celebrated Bieberbach conjecture: $|a_n| \leqslant n$ $(n = 2, 3, \ldots)$ for any function in S. Equivalently, the conjecture has been that the Koebe function $z/(1-z)^2 = \sum_1^\infty nz^n$, which maps the disc onto the plane cut along the negative real axis from $-\frac{1}{4}$ to ∞, is extremal for maximizing the size of each coefficient. By now it will be well-known to most readers that this problem has been completely solved by the American mathematician Louis de Branges [4]. His proof is quite extraordinary for its subtlety and delicacy and further is remarkable for its brevity and simplicity in comparison with what was previously known. Bieberbach's conjecture has stood as one of the great problems of analysis, not only because of its intrinsic depth and deceptive simplicity, but because of the great wealth of methods and ideas which have been created in the many attempts at a solution.

Regrettably, in the present context, the proof owes nothing to extreme point theory and purely linear methods do not appear to be sufficiently powerful to handle significant problems in a class with such complex structure as S. (The actual proof has as its main punch ideas from Topological Dynamics, namely Löwner's beautiful geometric construction of a flow of mappings parametrically connected by means of a differential equation [5]: plus, one must add, some delicate power series inequalities due to Lebedev and Milin [6] and some deep work by Askey and Gasper [2] on special functions.) Nevertheless, extreme point theory has had some notable successes in various sub-classes of S and other classes of a related nature. A number of interesting classes of functions can be shown to be very closely related to analytic functions whose real part is positive. The best known example is the sub-class of S whose members map the disc onto a domain starlike with respect to the origin. The characterization here is the inequality

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0.$$

The set of extreme points turns out to consist of the functions $z/(1-xz)^2$ $(|x| = 1)$. With such information, maximizing a linear functional becomes a mere formal calculation. Hallenbeck and MacGregor give a substantially complete development for the extreme point and support point theory for such classes and solve many linear extremal problems with elegance and clarity. They include a very full discussion of certain types of linear operator – particularly subordination and majorization. However they stop short of a general discussion of such operators, which would surely entail a development of the convolution theory mentioned earlier. This receives little attention in this book.

In the final two chapters the book moves into a decidedly higher gear. In chapter 8 the basic problem considered is as follows: for a given class A of functions analytic in the disc denote by $s(A)$ the family of functions subordinate to a member of A; find the set of extreme points of $s(A)$. If A consists of a single function F, univalent and mapping the disc onto a Jordan domain, then every such extreme point has the form $F \circ \phi$, where ϕ is an extreme point of B_0 (class of analytic functions in the disc bounded by 1 and vanishing at the origin). Some knowledge of H^p theory is needed for this chapter. Chapter 9 studies variability regions and coefficient bodies. Much of the work here is recent and due to the authors. Indeed the authors are among a small number responsible for the dissemination and development of these linear ideas in Geometric Function Theory and their book is an excellent source of reference to the current state of knowledge and to the existing literature.

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THE RIEMANN ZETA-FUNCTION
The Theory of the Riemann Zeta-Function with Applications

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Before the appearance of the monograph under review, there were only two books in literature devoted solely to Riemann's zeta-function, namely the classical treatise [3] of Titchmarsh and the scholarly work [1] of Edwards tracing the history of the study starting from Riemann's memoir and his mathematical inheritance. For decades, Titchmarsh has been the standard reference, though of course the zeta-function has been touched upon in several more recent textbooks and monographs on analytic number theory. Therefore, I guess, experts of analytic number theory have faced the challenge of writing a modern comprehensive text on the zeta-function with a similar awe as a Christian urged to rewrite the Bible.

But there are also more concrete problems in such an undertaking. The literature on the zeta-function was vast in Titchmarsh's time, and nowadays it is much vaster still. Moreover, a motivated treatment of the zeta-function necessitates a discussion of number-theoretic applications as well. The word 'application' is actually not quite adequate here, because analysis and number theory are just different aspects of the theory, like the two sides of a coin.

The author was led to the heroic project of writing an account of the zeta-function via his notes [2] containing a unified discussion of topics like the theory of exponent pairs, the saddle-point method, approximate functional equations, mean value theorems (including the important formula of Atkinson on the mean value of $|\zeta(\frac{1}{2}+it)|^2$), zero-density estimates, and the divisor problem. The present volume is an expanded version of [2]. What is new is primarily the introductory material in Chapter 1, and Chapter 12 on the distribution of primes, though there are several other