# A NOTE ON THE PARTIAL SUMS OF CONVEX SCHLICHT FUNCTIONS

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Let $f(z) = \sum_{1}^{\infty} a_n z^n$ be regular and convex univalent for $|z| < 1$. For each integer $n \geqslant 1$, we write

$$P_n(z) = \sum_{k=1}^{n} a_k z^k, \tag{1}$$

for the partial sums of $f(z)$. We prove

THEOREM 1. *For each integer* $n \geqslant 1$,

$$\operatorname{Re} \frac{f(z)}{P_n(z)} > \tfrac{1}{2} \quad (|z| < 1). \tag{2}$$

*Remark.* The case $n = 1$ is a well-known result.

If we write

$$Q_n(z) = \int_0^z \frac{P_n(\zeta)}{\zeta} d\zeta = \sum_{k=1}^{n} \frac{a_k}{k} z^k, \tag{3}$$

we can deduce from Theorem 1:

THEOREM 2. *For each* $n \geqslant 1$, $Q_n(z)$ *is a close-to-convex, schlicht function for* $|z| < 1$. *Moreover for* $n \geqslant 3$, $Q_n(z)$ *may not be starlike univalent.*

*Proof of Theorem* 1. We showed in **[2]** that for each fixed $\zeta$, $|\zeta| < 1$, the function

$$z\left(\frac{f(z)-f(\zeta)}{z-\zeta}\right)^2, \tag{4}$$

is starlike univalent for $|z| < 1$. It follows immediately that the function

$$g(z) = g(z, \zeta) = z\,\frac{f(z)-f(\zeta)}{z-\zeta}, \tag{5}$$

is starlike of order $\tfrac{1}{2}$ for $|z| < 1$, which is to say that

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > \tfrac{1}{2} \quad (|z| < 1). \tag{6}$$

Writing

$$g(z) = \sum_{1}^{\infty} b_n z^n, \tag{7}$$

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it is well-known that (6) implies that

$$|b_n| \leqslant |b_1| = \left|\frac{f(\zeta)}{\zeta}\right|, \tag{8}$$

for $n \geqslant 1$. But

$$g(z) = z\sum_{1}^{\infty} a_n \frac{z^n-\zeta^n}{z-\zeta} = \sum_{n=1}^{\infty}\left\{\sum_{k=0}^{\infty} a_{n+k}\zeta^k\right\} z^n. \tag{9}$$

Thus equating coefficients in (7) and (9) we deduce that

$$b_n = \sum_{k=0}^{\infty} a_{n+k}\zeta^k = \frac{f(\zeta)-P_{n-1}(\zeta)}{\zeta^n} \quad (n \geqslant 2). \tag{10}$$

Thus applying (8) we have

$$\left|\frac{f(\zeta)-P_n(\zeta)}{\zeta^{n+1}}\right| \leqslant \left|\frac{f(\zeta)}{\zeta}\right|, \tag{11}$$

valid for $n \geqslant 1$ and for every $\zeta$ such that $|\zeta| < 1$. This can be rewritten

$$\left|1-\frac{P_n(\zeta)}{f(\zeta)}\right| \leqslant |\zeta^n| < 1. \tag{12}$$

Replacing $\zeta$ by $z$, the inequality (2) is an immediate consequence of (12), and we have proved Theorem 1.

*Remark.* It is to be noted that for the convex function

$$f(\zeta) = \frac{\zeta}{1-\zeta} = \sum_{1}^{\infty} \zeta^n,$$

we have

$$\frac{f(\zeta)-P_n(\zeta)}{\zeta^n} \equiv f(\zeta),$$

for every $n$, so there is certainly no possibility of improving (12).

*Proof of Theorem* 2. We write

$$g(z) = \int_0^z \frac{f(\zeta)}{\zeta}\,d\zeta, \tag{13}$$

so that $g(z)$ is convex and $zg'(z) = f(z)$. Since $zQ_n'(z) = P_n(z)$, we deduce from Theorem 1 that

$$\mathrm{Re}\,\frac{g'(z)}{Q_n'(z)} > \tfrac{1}{2}, \quad (|z| < 1) \tag{14}$$

for every $n \geqslant 1$. Hence

$$\mathrm{Re}\,\frac{Q_n'(z)}{g'(z)} > 0,$$

and $Q_n(z)$ is indeed close-to-convex, see **[1]**.

To complete the proof, we show that the polynomials

$$Q_n(z) = \sum_{k=1}^{n} \frac{z^k}{k}, \tag{15}$$

are not starlike univalent for $|z| < 1$ if $n \geqslant 3$.

We note that $Q_n(z)$ is regular for $|z| \leqslant 1$ and univalent for $|z| < 1$, so that on $|z| = 1$, $|Q_n(z)| \geqslant \frac{1}{4}$. In particular $Q_n$ does not vanish for $0 < |z| \leqslant 1$. It follows that

$$F(z) = \frac{zQ_n'(z)}{Q_n(z)} = 1 + \sum_{1}^{\infty} c_k z^k, \tag{16}$$

is regular for $|z| \leqslant 1$. Moreover the coefficients $c_k$ are real. To prove that $Q_n$ is not starlike for $|z| < 1$, it will be sufficient to prove that there is a point $\zeta$, $|\zeta| = 1$, such that

$$\operatorname{Re} F(\zeta) < 0. \tag{17}$$

Since

$$Q_n'(z) = \sum_{k=1}^{n} z^{k-1} = \frac{1-z^n}{1-z},$$

we see that $Q_n'(\omega) = 0$, where $\omega = e^{i\alpha}$ and $\alpha = 2\pi/n$. Thus

$$\operatorname{Re} F(\omega) = 0.$$

Writing $z = e^{i\theta}$ we have

$$\operatorname{Re} F(e^{i\theta}) = 1 + \sum_{k=1}^{\infty} c_k \cos k\theta, \tag{18}$$

and this function of $\theta$ is differentiable and vanishes at $\theta = \alpha$. If $Q_n$ is starlike for $|z| < 1$, then the derivative with respect to $\theta$ of (18) must vanish at $\theta = \alpha$, i.e.

$$\operatorname{Im} e^{i\alpha} F'(e^{i\alpha}) = \sum_{1}^{\infty} kc_k \sin k\alpha = 0. \tag{19}$$

Taking into account the fact that $Q_n'(e^{i\alpha}) = 0$, we have

$$e^{i\alpha} F'(e^{i\alpha}) = \frac{e^{2i\alpha} Q_n''(e^{i\alpha})}{Q_n(e^{i\alpha})} = n \Big/ \left\{(1-e^{-i\alpha})\left(\sum_{k=1}^{n} e^{ki\alpha}/k\right)\right\}.$$

Thus from (19) we deduce that

$$\operatorname{Im}\left\{(1-e^{-i\alpha})\left(\sum_{k=1}^{n} \frac{e^{ki\alpha}}{k}\right)\right\} = 0,$$

i.e.

$$\sum_{1}^{n} \frac{\sin k\alpha}{k} - \sum_{1}^{n} \frac{\sin (k-1)\alpha}{k} = 0. \tag{20}$$

To complete the proof, we show that (20) is false for $n \geqslant 3$. We write

$$S = \sum_{1}^{n} \frac{\sin k\alpha}{k} - \sum_{1}^{n} \frac{\sin (k-1)\alpha}{k}.$$

Then since $n\alpha = 2\pi$, we have

$$S = \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) \sin k\alpha$$

$$= \sum_{k=1}^{m} \left(\frac{1}{k} - \frac{1}{k+1} - \frac{1}{n-k} + \frac{1}{n-k+1}\right) \sin k\alpha,$$

where $m = [\frac{1}{2}n]$, denotes the smallest integer $\leqslant \frac{1}{2}n$. We observe that $\sin k\alpha > 0$ for every $k < \frac{1}{2}n$, and $\sin \frac{1}{2}n\alpha = 0$ when $n$ is even. Furthermore for $k < \frac{1}{2}n$,

$$\frac{1}{k} - \frac{1}{k+1} - \frac{1}{n-k} + \frac{1}{n-k+1} > 0.$$

We thus see that $S > 0$, and the proof is complete.

*Remark.* For any convex function $f$, $Q_1$ is trivially convex and $Q_2$ is starlike, as is easily seen using the inequality $|a_2| \leqslant |a_1|$.

## *References*

1. W. Kaplan, " Close-to-convex Schlicht functions ", *Michigan Math. J.*, 1 (1953), 169–185.
2. T. Sheil-Small, " On convex univalent functions ", *J. London Math. Soc.* (2), 1 (1969), 483–492.

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